Vol 1(2), 2020, pp:1-16
DOI:10.30511/mcs.2021.141498.1012
ISSN:2717-2708
http://mcs.qut.ac.ir/

Newton-Krylov generalized minimal residual algorithm in solving Volterra-Fredholm-Hammerstein integral equations

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Abstract : In this paper, Galerkin and collocation methods have been applied on nonlinear Volterra-Fredholm-Hammerstein (VFH) integral equations, these methods are besed on shifted Legendre polynomials, then methods transfer the finding solution of a nonlinear integral equation to finding the solution of nonlinear algebraic equations, in order to solve these nonlinear algebraic equations we use Newton method composed by generalized minimal residual (NGMRes) method, the iteration number and running time for implementation of NGMRes method are important parameters that have been considered to solve this type of integral equations. These methods are applied on several various nonlinear VFH integral equations that confirm accuracy and efficiency of the methods.

Keywords : Volterra-Fredholm-Hammerstein integral equations; Collocation method; Galerkin method; Spectral methods; Newton-Krylove GMRes; Shifted Legendre polynomials.
2020 Mathematics Subject Classification: 00A69; 64F12
Receive: 25 November 2020, **Accepted:** 15 January 2021

1 Introduction

Integral equations have found their applications in various fields of mathematics, sciences and technologies that have motivated a large amount of research works in recent years see [3, 13, 16, 20, 22, 32, 37–39]. VFH integral equations are one of the well-known nonlinear integral equations that arise in many fields including electricity and magnetism, communication theory, antenna synthesis problem, mathematical economics, population genetics, radiation, the particle transport problems of astrophysics and reactor theory, fluid mechanics, etc. The nonlinear VFH integral equations have been introduced in general form [28]

$$y(x) = f(x) + \lambda_1 \int_0^x k_1(x,t)G_1(t,y(t))dt + \lambda_2 \int_0^1 k_2(x,t)G_2(t,y(t))dt \quad 0 \leqslant x \leqslant 1, \tag{1.1}$$

where λ_1 and λ_2 are constants and $f(x)$ and kernels $k_1(x,t)$ and $k_2(x,t)$ are given functions assumed to have nth derivatives on intervals $0 \leqslant x, t \leqslant 1$, also $G_1(t, y(t))$ and $G_2(t, y(t))$ are given continuous functions which are nonlinear with respect to $y(t)$ and t. In [17] the existence of solution to nonlinear Hammerstein equations has been discussed.
Recently, many researchers have tried to solve this type of equations with several numerical methods and algorithms, for example, a variation of the Nystrom method proposed in [19]. In [21] the nonlinear VFH integral equations have been solved via Hybrid of block-pulse functions. In [36] Haar wavelet collocation

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method for nonlinear VFH integral equations was proposed. In [10] Tau-collocation method has been applied on this type of integral equations and convergence of method has been investigated. In [40] Legendre Wavelet method have been applied. for more references see [1, 11, 12, 18].
Spectral methods are well-known methods that successfully applied in two past decades. These powerful tools are used to solve linear and nonlinear equations in many fields such as fluid dynamics, quantum dynamics and etc. Parand et al. in [26] discussed nonlinear Volterra-Fredholm Integro-Differential equations of Multi-Arbitrary Order. In [27] was solved Volterra's population growth model of arbitrary order using the generalized fractional order of the Chebyshev functions. In [29] was presented a new numerical algorithm based on the first kind of modified Bessel function to solve population growth in a closed system see [25, 31]. By applying spectral methods on nonlinear integral equations the problems are reduced to a system of nonlinear algebraic equations, so speed and accuracy of solving this nonlinear system have key role. Many works have been done to improve solving the nonlinear systems, Newton-Krylov algorithm is one of the most important and popular method that has been considered in many works such as [2, 7, 15].

We organized this paper as follows, in section 2, Legendre and shifted Legendre polynomials are explained, in section 3, Newton-Krylov GMRes algorithm will be discussed, in section 4, the methods are implemented on problem, section 5, will be devoted to convergence analysis, in section 6, the numerical results are reported, and at last section we will have a conclusion of our study.

2 Preliminary and notations

Properties of Legendre and shifted Legendre polynomials

Assuming that the Legendre polynomial of degree k is denoted by $P_k(z)$. The recurrence formulae of $P_k(z)$ is

$$P_{k+1}(z) = \frac{2k+1}{k+1} z P_k(z) - \frac{k}{k+1} P_{k-1}(z), \quad k = 1, 2, ...,$$
$$P_0(z) = 1, \qquad P_1(z) = z.$$

Substituting $z = 2x - 1$, Legendre polynomials are defined on the interval $[0, 1]$ that may be called shifted Legendre polynomials.The explicit analytical form of the shifted Legendre polynomial $P_k^*(x)$ of degree k is

$$P_k^*(x) = \sum_{j=0}^{k} (-1)^{j+k} \frac{(j+k)! x^j}{(k-j)!(k!)^2},$$

and this in turn, enables one to get

$$P_k^*(0) = (-1)^k, \qquad P_k^*(1) = 1.$$

The orthogonality relation is

$$\int_0^1 P_{\mathrm{i}}^*(x) P_{\mathrm{j}}^*(x) dx = \begin{cases} \frac{1}{2j+1}, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases}$$

Any square integrable function $y(x)$ defined on the interval $[0, 1]$, may be expressed in terms of shifted Legendre polynomials as

$$y(x) = \sum_{k=0}^{\infty} a_k P_k^*(x), \quad \text{where, } a_k = (2k+1) \int_0^1 y(x) P_{\mathrm{k}}^*(x) dx, \quad k = 0, 1, ...$$

3 Newton-Krylov algorithm

By applying spectral methods on nonliear integral equations a system of nonlinear algebraic equations is obtained. Consider $F(x) = 0$ where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a function $F(x) = (f_1(x), f_2(x), ..., f_n(x))^T$ and $x \in \mathbb{R}^n$ is a vector. By applying the well-known Newton's iterative method:

$$F(x_{n+1}) = F(x_n) + (x_n - x_{n+1})F'(x_n),$$

where, $F'(x_n) = J(x_n)$ is the $n \times n$ Jacobian matrix. therefore

$$x_{n+1} = x_n - J(x_n)^{-1}F(x_n).$$

It is obvious in each iteration, a linear system must be solved, by increasing complexity of equations solving this linear system mostly could be time-consuming process. one of the best ideas to overcome this problem is to use NGMRes.

A framework for GMRes implementation

Begin
1. $r = b - Ax,\ v_1 = r/||r||_2,\ \rho = ||r||_2,\ \beta = \rho,\ k = 0.$
2. ***while*** $\rho > \epsilon||b||_2$ and $k < k_{max}$ ***do***
   - **2.a** $k = k + 1.$
   - **2.b** Apply Arnoldi to obtain H_k and V_{k+1} from V_k and H_{k-1}.
   - **2.c** $e_1 = (1, 0, ..., 0)^T \in R^{k+1}$.
   - **2.d** Solve min $||\beta e_1 - H_k y_k||_{R^{k+1}}$ for $y_k \in R^k$.
   - **2.e** $\rho = ||\beta e_1 - H_k y_k||_{R^{k+1}}$.
   - **2.f** $x_k = x_0 + V_k y_k$

   end while

End

For more details see[8, 9, 33].

4 Proposed methods

Spectral methods belong to the family of weighted residual methods (WRMs). WRMs approximate the solution $y(x)$ of the integral equation (1.1) by a finite sum

$$y(x) \simeq y_N(x) = \sum_{k=0}^{N} a_k \phi_k(x),$$

that $\phi_k(x)$, is trial function, and a_k, $\quad k = 0..N$ are unknown coefficients, by substitution $y_N(x)$ in Eq. (1.1), leads us to residual function

$$Res_N(x) = f(x) + \lambda_1 \int_0^x k_1(x,t)G_1(t, y_N(t))dt + \lambda_2 \int_0^1 k_2(x,t)G_2(t, y_N(t)) - y_N(x)dt. \tag{4.1}$$

The notion of the WRMs is to force the residual to zero in suitable norm,

$$\langle Res_N(x), \Psi_j \rangle_\omega = \int_\Omega Res_N(x)\Psi_j(x)\omega(x)dx = 0, \qquad 0 \leqslant j \leqslant N \tag{4.2}$$

which Ψ_j is the test function, and ω is a weight function. Definition of shifted Legendre collocation method and shifted Legendre Galerkin method is as follows in table 1:

Table 1: Definition of shifted Legendre collocation method and shifted Legendre Galerkin method.

Method	Weight function	Trial function	Test function
Shifted Legendre collocation method	1	$P_i^*(x)$	$\delta(x-x_j)$
Shifted Legendre Galerkin method	1	$P_i^*(x)$	$P_j^*(x)$

where $P_i^*(x)$, is shifted Legendre of degree i, and $\delta(x-x_j)$ is Dirac delta function and also x_j is jth zero of shifted Legendre of degree N+1. To calculate the Eq. (4.2) we use the Gauss-Legendre integration rule [5]. Finally we generate $N+1$ nonlinear equations with $N+1$ unknown coefficients, to solve this system of nonlinear algebraic equations and obtaining coefficients $a_k, k = 0,..,N$ we have used NGMRes method.

5 Error estimate

In this section, the theorem on convergence analysis and error estimation of the proposed is discussed. Suppose $y(x)$ is sufficiently smooth function on [0,1], and $p_n(x)$ is the interpolating polynomial to y at x_i points, where $x_i, i = 0, ..., N$ are the roots of chebyshev polynomial of degree $N+1$ in $[0,1]$, and then

$$y(x) - p_n(x) = \frac{y^{(N+1)}(\xi)}{(N+1)!}\prod_{i=0}^{N}(x-x_i) \quad \xi \in [0,1], \tag{5.1}$$

then

$$|y(x) - p_n(x)| \leq \frac{M_N (1)^{N+1}}{2^{2N+1}(N+1)!}, \tag{5.2}$$

where, $M_N = max_{0\leq x\leq 1}\left|y^{(N+1)}(x)\right|$.

Theorem 5.1. *Suppose that $y_N(x) = A^T P(x)$ be the shifted Legendre polynomials expansion of the exact solution, y(x), where*

$$A = [a_0, a_1, ..., a_N]^T,$$

and

$$P(x) = [P_0(x), P_1(x), ..., P_N(x)]^T.$$

Let $\overline{y}_N(x) = \sum_{i=0}^{N} \overline{a}_i P_i(x)$ be the approximate solution obtained by the method proposed in Section 4 and $M_N = max_{0\leq x\leq 1}|y^{N+1}(x)|$, then, there exist real numbers α and β_N such that

$$||y(x) - \overline{y}_N(x)||_2 \leq \alpha\frac{M_N (1)^{N+1}}{2^{2N+1}(N+1)!} + \beta_N ||A - \overline{A}||_2, \tag{5.3}$$

where

$$\overline{A} = [\overline{a}_0, \overline{a}_1, ..., \overline{a}_N]^T,$$

also the norm on the right hand side is the Euclidian norm for vectors.

Proof. Let $R_n[x]$ be the space of all real-valued polynomials of degree $\leq n$. Using the definition, $y_n(x)$ and $\overline{y}_n(x)$ are in $R_n[x]$ and $y_n(x)$ is the best approximation of $y(x)$ in $R_n[x]$. It's clear that

$$||y(x) - \overline{y}_N(x)||_2 \leq ||y(x) - y_N(x)||_2 + ||y_N(x) - \overline{y}_N(x)||_2. \tag{5.4}$$

By using Eq. (5.2) we have

$$\begin{aligned} ||y(x) - y_N(x)||_2 &= \left(\int_0^l |y(x) - y_N(x)|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq \left(\int_0^l \left[\frac{M_N(1)^{N+1}}{2^{2N+1}(N+1)!} \right]^2 \, dx \right)^{\frac{1}{2}} \\ &= \sqrt{l} \frac{M_N(1)^{N+1}}{2^{2N+1}(N+1)!}. \end{aligned} \tag{5.5}$$

Also, we have

$$\begin{aligned} ||y_N(x) - \overline{y}_N(x)||_2 &= \left(\int_0^l \left[\sum_{i=0}^{N} (a_i - \overline{a}_i) P_i(x) \right]^2 \, dx \right)^{\frac{1}{2}} \\ &\leq \left(\int_0^l \left[\sum_{i=0}^{N} |a_i - \overline{a}_i|^2 \right] \left[\sum_{i=0}^{N} |P_i(x)|^2 \right] \, dx \right)^{\frac{1}{2}} \\ &= \left[\sum_{i=0}^{N} |a_i - \overline{a}_i|^2 \right]^{\frac{1}{2}} \left(\sum_{i=0}^{N} \int_0^l |P_i(x)|^2 \, dx \right)^{\frac{1}{2}} \\ &= ||A - \overline{A}||_2 \left(l \sum_{i=0}^{N} \frac{1}{2i+1} \right)^{\frac{1}{2}}. \end{aligned} \tag{5.6}$$

Finally from Eqs. (5.4)-(5.6) we conclude that Eq. (5.3) is valid with

$$\alpha = \sqrt{l}, \qquad \beta_N = \sqrt{l \left(\sum_{i=0}^{N} \frac{1}{2i+1} \right)}.$$

□

6 Illustrative Examples

In this section, In order to illustrate the performance of presented methods in solving VFH integral equations and justify the efficiency of the methods, we considered some various examples. To study the convergence behaviour of the methods, the maximum and mean errors have been used with the following definition, respectively

$$\|e_{(N)}\|_\infty = \max{}_{(x)\in[0,1]} |u_{ex}(x) - u_N(x)|$$
$$\|e_{(N)}\|_2 = \left(\int_0^1 (u_{ex}(x) - u_N(x))^2 dx \right)^{\frac{1}{2}}.$$

The numerical implementation is carried out in Microsoft.maple.15, with hardware configuration: desktop 32-bit Intel core 2 Due CPU, 4 GB of RAM, 32-bit operation system.

Example 6.1. *Consider nonlinear VFH integral equation given in [28], with the exact solution of y(x)=cos(x):*

$$y(x) = 2\cos(x) - 2 + 3\int_0^x \sin(x-t)y^2(t)\,dt + \frac{6}{7-6\cos(1)}\int_0^1 (1-t)\cos^2(x)(t+y(t))\,dt, \qquad (6.1)$$

we solve the Eq. (6.1), Table 2 shows the $\|e\|_2$, $\|e\|_\infty$, running time and number of iteration for implementation of Newton-Krylov algorithm, in Table 3 and Table 4, the absolute error for collocation and Galerkin methods at some point for different number of N is shown, respectively, Figure 1 displays the absolute error for $N = 15$.

Table 2: Some numerical results for example 6.1

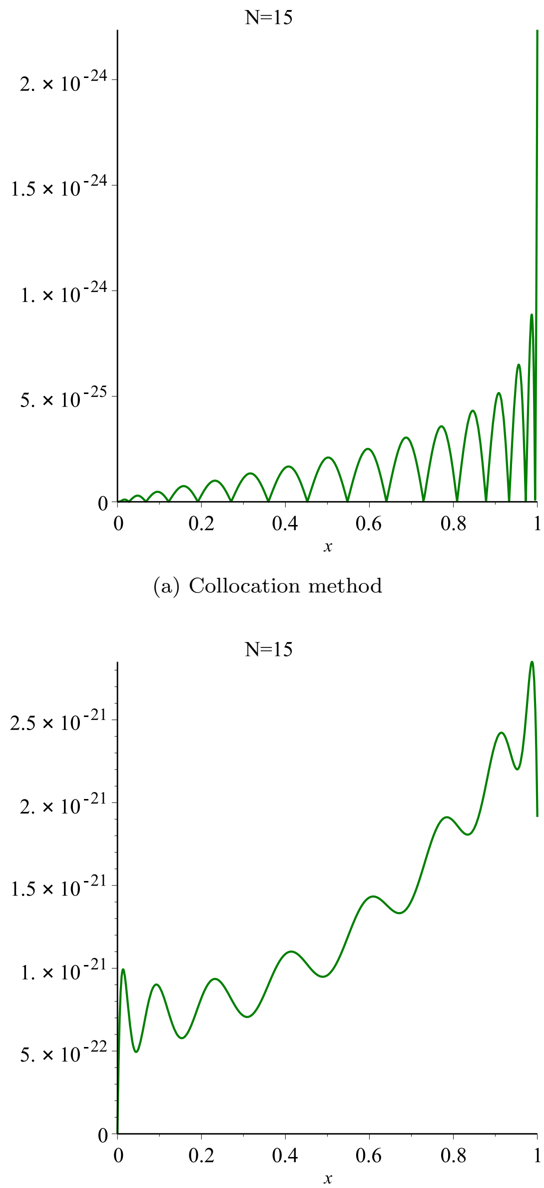
N	Collocation method				Galerkin Method			
	$\|\|e\|\|_2$	$\|\|e\|\|_\infty$	Time (s)	Iteration	$\|\|e\|\|_2$	$\|\|e\|\|_\infty$	Time (s)	Iteration
2	3.71×10^{-3}	1.05×10^{-2}	1.21	3	4.82×10^{-3}	9.19×10^{-3}	3.26	2
4	2.53×10^{-6}	4.82×10^{-6}	1.74	5	2.12×10^{-6}	4.81×10^{-6}	5.43	3
6	3.67×10^{-9}	6.12×10^{-9}	1.86	5	3.92×10^{-9}	6.56×10^{-9}	6.07	3
8	2.29×10^{-12}	4.49×10^{-12}	2.15	5	2.74×10^{-12}	4.95×10^{-12}	7.18	5
10	1.57×10^{-15}	2.75×10^{-15}	3.01	5	6.89×10^{-14}	1.52×10^{-13}	7.93	5

Table 3: Absolute error of collocation method for example 6.1

X	N=2	N=4	N=6	N=8	N=10
0.0	1.02×10^{-4}	7.18×10^{-7}	5.06×10^{-10}	7.41×10^{-14}	3.17×10^{-17}
0.2	2.35×10^{-4}	7.80×10^{-7}	8.99×10^{-11}	8.22×10^{-14}	5.65×10^{-17}
0.4	8.86×10^{-4}	8.21×10^{-7}	6.45×10^{-10}	9.35×10^{-13}	1.27×10^{-16}
0.6	1.17×10^{-3}	9.00×10^{-7}	6.90×10^{-10}	1.00×10^{-12}	2.10×10^{-16}
0.8	1.04×10^{-3}	2.15×10^{-6}	9.32×10^{-10}	1.09×10^{-12}	2.93×10^{-16}
1.0	1.13×10^{-3}	2.80×10^{-6}	1.20×10^{-9}	1.31×10^{-12}	3.68×10^{-16}
$\|\|e\|\|_2$	3.71×10^{-3}	2.53×10^{-6}	3.67×10^{-9}	2.29×10^{-12}	1.57×10^{-15}

Table 4: Absolute error of Galerkin method for example 6.1

X	N=2	N=4	N=6	N=8	N=10
0.0	2.22×10^{-4}	8.40×10^{-8}	6.61×10^{-10}	4.37×10^{-13}	2.93×10^{-14}
0.2	5.81×10^{-4}	2.66×10^{-7}	8.47×10^{-10}	8.75×10^{-13}	4.98×10^{-14}
0.4	7.36×10^{-4}	8.87×10^{-7}	2.28×10^{-10}	1.04×10^{-12}	1.11×10^{-15}
0.6	9.25×10^{-4}	1.16×10^{-6}	9.75×10^{-10}	1.91×10^{-12}	2.30×10^{-15}
0.8	2.07×10^{-3}	9.98×10^{-7}	1.03×10^{-9}	1.40×10^{-12}	2.06×10^{-14}
1.0	3.36×10^{-3}	2.43×10^{-6}	2.05×10^{-9}	2.58×10^{-12}	1.32×10^{-14}
$\|\|e\|\|_2$	4.82×10^{-3}	2.12×10^{-6}	3.92×10^{-9}	2.74×10^{-12}	6.89×10^{-14}



(a) Collocation method

(b) Galerkin method

Figure 1: Absolute error of methods for Example 6.1

Example 6.2. *Consider nonlinear VFH integral equation given in [28], with the exact solution of*

$$y(x) = \ln(\frac{\theta^2}{2\cos^2(\frac{\theta}{2}(x-\frac{1}{2}))}), \theta = 4\cos(\frac{\theta}{4}) :$$

$$y(x) = \int_0^1 K(x,t)e^{y(t)}\,dt, \tag{6.2}$$

where

$$K(x,t) = \begin{cases} -t(1-x), & 0 \le t \le x; \\ -x(1-t), & x \le t \le 1. \end{cases} \tag{6.3}$$

we solve the Eq. (6.2)*, Table 5 shows the* $\|e\|_2$, $\|e\|_\infty$*, running time and number of iteration for implementation of Newton-Krylov algorithm, in Table 6 and Table 7, the absolute error for collocation and Galerkin methods at some point for different number of N is shown, respectively. Figure 2 displays the absolute error of collocation and Galerkin methods for* $N = 25$ *and* $N = 20$*, respectively.*

Table 5: Some numerical results for example 6.2

N	**Collocation method**				**Galerkin Method**			
	$\|\|e\|\|_2$	$\|\|e\|\|_\infty$	Time (s)	Iteration	$\|\|e\|\|_2$	$\|\|e\|\|_\infty$	Time (s)	Iteration
5	3.12×10^{-5}	5.45×10^{-5}	1.16	16	5.27×10^{-9}	9.29×10^{-9}	3.58	5
10	4.96×10^{-8}	1.20×10^{-8}	1.53	17	4.80×10^{-14}	7.17×10^{-14}	26.10	3
15	1.22×10^{-12}	1.81×10^{-11}	1.28	23	9.39×10^{-15}	1.86×10^{-14}	32.78	3
20	3.17×10^{-14}	6.69×10^{-14}	3.40	30	8.11×10^{-26}	1.42×10^{-25}	52.69	5
25	6.18×10^{-18}	1.75×10^{-18}	4.41	35	-	-	-	-

Table 6: Absolute error of collocation method for example 6.2

X	N=5	N=10	N=15	N=20	N=25
0.0	1.91×10^{-6}	7.52×10^{-9}	8.78×10^{-13}	4.82×10^{-15}	1.52×10^{-19}
0.2	2.85×10^{-6}	4.99×10^{-9}	9.83×10^{-13}	9.07×10^{-15}	5.33×10^{-19}
0.4	6.62×10^{-6}	7.55×10^{-9}	9.14×10^{-13}	2.94×10^{-15}	1.27×10^{-16}
0.6	9.66×10^{-6}	3.49×10^{-9}	8.10×10^{-13}	5.29×10^{-15}	3.84×10^{-19}
0.8	4.53×10^{-6}	9.14×10^{-9}	1.13×10^{-12}	2.38×10^{-15}	2.57×10^{-19}
1.0	7.98×10^{-6}	1.02×10^{-8}	1.27×10^{-12}	8.75×10^{-15}	4.25×10^{-19}
$\|\|e\|\|_2$	3.12×10^{-5}	1.27×10^{-8}	1.22×10^{-12}	3.17×10^{-14}	6.18×10^{-18}

Example 6.3. *Consider nonlinear Volterra-Hammerstein integral equation given in [36], with initial conditions y(0)=1. which has the exact solution y(x)=exp(-x):*

$$y(x) = \frac{3}{2} - \frac{1}{2}exp(-2x) - \int_0^x (y(t)^2 + y(t))\,dt, \tag{6.4}$$

we solve the Eq. (6.4)*, Table 8 shows the* $\|e\|_2$, $\|e\|_\infty$*, running time and number of iteration for implementation of Newton-Krylov algorithm, in Table 9 and Table 10, the absolute error for collocation and*

Table 7: Absolute error of Galerkin method for example6.2

X	N=5	N=10	N=15	N=20
0.0	3.90×10^{-10}	6.27×10^{-15}	6.45×10^{-17}	4.13×10^{-27}
0.2	3.22×10^{-10}	9.07×10^{-15}	5.39×10^{-16}	8.67×10^{-27}
0.4	9.39×10^{-10}	9.68×10^{-15}	9.74×10^{-17}	7.77×10^{-27}
0.6	4.32×10^{-10}	9.55×10^{-15}	9.45×10^{-17}	4.01×10^{-27}
0.8	2.49×10^{-10}	1.53×10^{-15}	3.97×10^{-16}	1.38×10^{-26}
1.0	1.28×10^{-10}	2.70×10^{-15}	4.49×10^{-16}	6.40×10^{-27}
$\|\|e\|\|_2$	5.27×10^{-9}	4.80×10^{-14}	9.39×10^{-15}	8.11×10^{-26}

Galerkin methods at some point for different number of N is shown, respectively, Figure 3 displays the absolute error for $N = 15$.

Table 8: Some numerical results for example 6.3

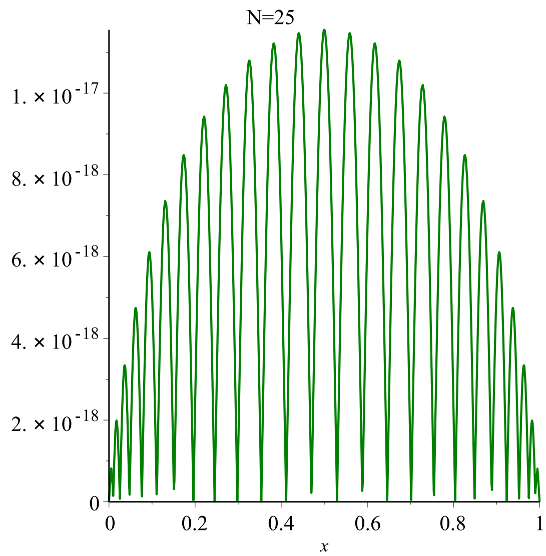
N	**Collocation method**				**Galerkin Method**			
	$\|\|e\|\|_2$	$\|\|e\|\|_\infty$	Time (s)	Iteration	$\|\|e\|\|_2$	$\|\|e\|\|_\infty$	Time (s)	Iteration
2	6.49×10^{-4}	1.67×10^{-3}	0.52	2	4.17×10^{-4}	8.59×10^{-4}	1.26	3
4	2.72×10^{-6}	4.84×10^{-6}	1.04	3	1.10×10^{-6}	2.22×10^{-6}	1.60	5
6	2.03×10^{-9}	4.61×10^{-9}	1.29	3	1.30×10^{-9}	2.55×10^{-9}	1.56	3
8	2.18×10^{-11}	4.22×10^{-11}	1.37	3	8.03×10^{-13}	1.87×10^{-12}	1.92	5
10	8.96×10^{-16}	1.81×10^{-15}	1.42	5	3.18×10^{-16}	9.46×10^{-16}	2.17	5

Table 9: Absolute error of collocation method for example 6.3

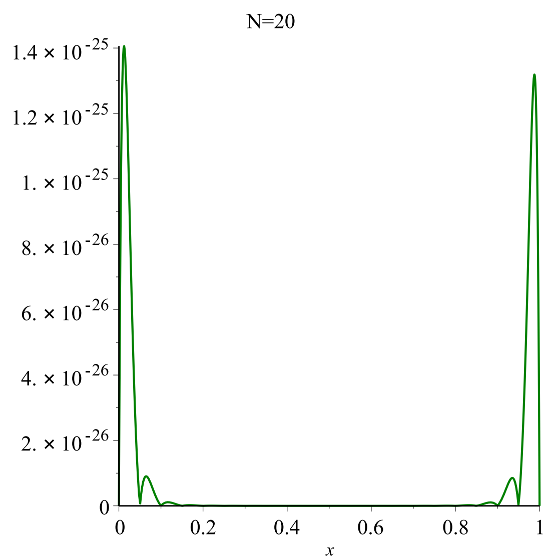
X	N=2	N=4	N=6	N=8	N=10
0.0	4.04×10^{-5}	3.15×10^{-7}	5.49×10^{-10}	9.20×10^{-13}	7.08×10^{-17}
0.2	8.11×10^{-5}	2.89×10^{-6}	8.68×10^{-10}	8.39×10^{-12}	7.88×10^{-17}
0.4	8.86×10^{-5}	9.95×10^{-7}	1.12×10^{-9}	4.25×10^{-12}	9.16×10^{-17}
0.6	6.81×10^{-5}	1.84×10^{-6}	8.49×10^{-10}	8.14×10^{-12}	8.11×10^{-17}
0.8	4.39×10^{-5}	9.32×10^{-7}	7.51×10^{-10}	9.15×10^{-12}	9.97×10^{-17}
1.0	9.66×10^{-5}	7.20×10^{-7}	6.31×10^{-10}	5.48×10^{-12}	9.56×10^{-17}
$\|\|e\|\|_2$	6.49×10^{-4}	2.72×10^{-6}	2.03×10^{-9}	2.18×10^{-11}	8.96×10^{-16}

Example 6.4. *Consider nonlinear Volterra-Hammerstein integral equation given in [12], with the exact solution of y(x)=x:*

$$y(x) = x + \frac{1}{x+1}(e^{-(x+1)} - 1) + \int_0^x \frac{1}{x} e^{-\frac{t}{x}} e^{-y(t)}\, dt. \tag{6.5}$$



(a) Collocation method



(b) Galerkin method

Figure 2: Absolute error of methods for Example 5.2.

we solve the Eq. (6.5)*, Table 11 shows the* $\|e\|_2$, $\|e\|_\infty$*, running time and number of iteration for implementation of Newton-Krylov algorithm, in Table 12 and Table 13, the absolute error for collocation and Galerkin methods at some point for different number of* N *is shown, respectively, Figure 4 displays the*

Table 10: Absolute error of Galerkin method for example 6.3

X	N=2	N=4	N=6	N=8	N=10
0.0	8.71×10^{-5}	1.35×10^{-7}	9.75×10^{-11}	6.08×10^{-15}	5.41×10^{-17}
0.2	7.66×10^{-5}	9.27×10^{-8}	1.33×10^{-10}	1.05×10^{-14}	1.78×10^{-17}
0.4	8.51×10^{-5}	8.61×10^{-8}	6.05×10^{-10}	1.28×10^{-14}	2.42×10^{-17}
0.6	6.45×10^{-5}	2.47×10^{-7}	9.75×10^{-10}	2.47×10^{-14}	1.38×10^{-17}
0.8	8.17×10^{-5}	7.75×10^{-7}	8.30×10^{-10}	5.15×10^{-14}	1.25×10^{-17}
1.0	6.76×10^{-5}	6.15×10^{-7}	5.72×10^{-10}	4.96×10^{-14}	2.07×10^{-17}
$\|\|e\|\|_2$	4.17×10^{-4}	1.10×10^{-6}	1.30×10^{-9}	8.03×10^{-13}	3.18×10^{-16}

absolute error for $N = 15$.

Table 11: Some numerical results for example 6.4

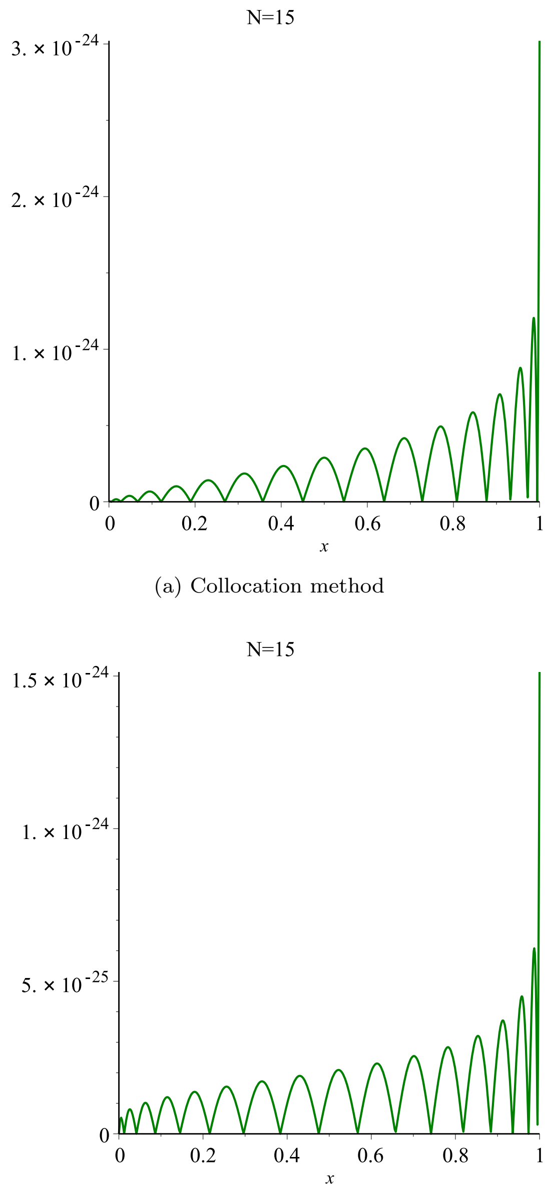
N	Collocation method				Galerkin Method			
	$\|\|e\|\|_2$	$\|\|e\|\|_\infty$	Time (s)	Iteration	$\|\|e\|\|_2$	$\|\|e\|\|_\infty$	Time (s)	Iteration
2	8.03×10^{-20}	1.59×10^{-19}	0.29	6	6.50×10^{-18}	1.16×10^{-17}	0.21	6
4	1.09×10^{-24}	2.33×10^{-24}	0.31	9	2.03×10^{-28}	2.42×10^{-28}	0.27	7
6	8.73×10^{-42}	1.63×10^{-41}	0.47	17	2.03×10^{-43}	2.19×10^{-43}	0.36	7
8	1.12×10^{-58}	2.28×10^{-58}	0.52	23	7.65×10^{-48}	1.05×10^{-47}	0.42	7
10	1.27×10^{-63}	2.42×10^{-63}	1.07	27	3.07×10^{-56}	6.01×10^{-56}	0.44	8

Table 12: Absolute error of collocation method for example 6.4

X	N=2	N=4	N=6	N=8	N=10
0.0	6.08×10^{-22}	6.12×10^{-25}	4.98×10^{-43}	3.05×10^{-59}	8.59×10^{-65}
0.2	7.71×10^{-22}	3.52×10^{-25}	5.09×10^{-43}	2.31×10^{-59}	9.10×10^{-65}
0.4	8.23×10^{-21}	6.19×10^{-25}	5.36×10^{-43}	3.81×10^{-59}	2.46×10^{-64}
0.6	3.35×10^{-21}	8.72×10^{-25}	3.61×10^{-43}	4.76×10^{-59}	4.85×10^{-64}
0.8	1.91×10^{-20}	9.12×10^{-25}	6.41×10^{-43}	2.69×10^{-59}	3.35×10^{-64}
1.0	1.87×10^{-20}	9.90×10^{-25}	4.23×10^{-43}	7.88×10^{-59}	5.07×10^{-64}
$\|\|e\|\|_2$	8.03×10^{-20}	1.09×10^{-24}	8.73×10^{-42}	1.12×10^{-58}	1.27×10^{-63}

7 Conclusion

In this paper, shifted Legendre collocation method and shifted Legendre Galerkin method were used to solve nonlinear VFH integral equations. By applying these two methods on nonlinear VFH integral equations the problems are reduced to a system of nonlinear algebraic equations, therefore, running time

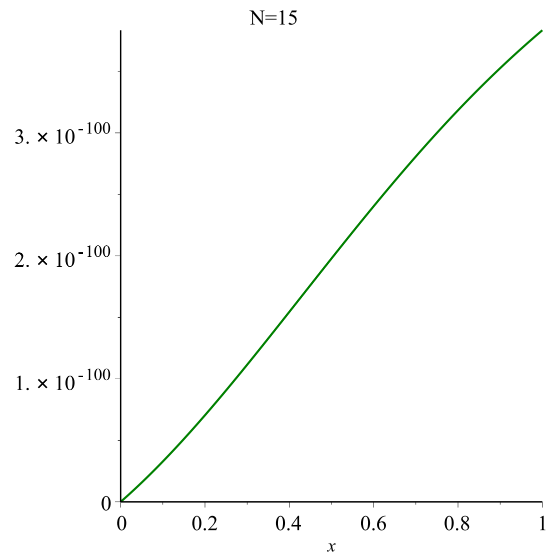


(a) Collocation method

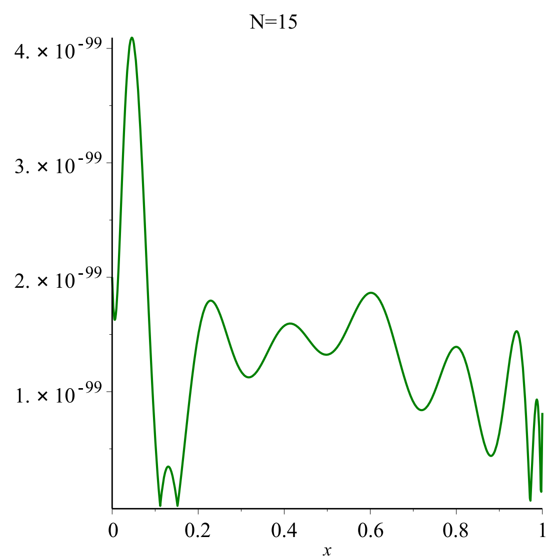
(b) Galerkin method

Figure 3: Absolute error of methods for Example 6.3

and accuracy in solving this system is an important part to approximate nonlinear VFH integral equations. In this work NGMRes method was used to solve obtained nonlinear algebraic equations system . The comparison of the obtained results of proposed methods and exact solutions shows that present methods



(a) Collocation method



(b) Galerkin method

Figure 4: Absolute error of methods for Example 6.4

Table 13: Absolute error of Galerkin method for example 6.4

X	N=2	N=4	N=6	N=8	N=10
0.0	4.20×10^{-19}	9.14×10^{-30}	8.52×10^{-45}	8.02×10^{-50}	9.93×10^{-58}
0.2	7.02×10^{-19}	9.96×10^{-30}	8.68×10^{-45}	7.20×10^{-50}	2.49×10^{-57}
0.4	4.17×10^{-19}	2.40×10^{-29}	9.92×10^{-45}	1.23×10^{-49}	2.53×10^{-57}
0.6	3.52×10^{-19}	2.37×10^{-29}	1.38×10^{-44}	1.17×10^{-49}	3.48×10^{-57}
0.8	9.44×10^{-19}	4.29×10^{-29}	2.42×10^{-44}	1.28×10^{-49}	2.28×10^{-57}
1.0	7.84×10^{-19}	5.50×10^{-29}	5.49×10^{-44}	3.43×10^{-49}	4.12×10^{-57}
$\|\|e\|\|_2$	6.50×10^{-18}	2.03×10^{-28}	2.03×10^{-43}	7.65×10^{-48}	3.07×10^{-56}

are powerful tools to find the numerical solutions of such equations.

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