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Solvabiltiy for a typical system of boundary value problems by fixed point theory

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Abstract: In this paper we study the existence of positive solutions for a class of boundary value problems. We introduce a cone and completely continuous operator. We provide sufficient conditions under which this system according to the fixed point index theorem, has positive solution.

Keywords: boundary value problem; positive solution; fixed point index; Jensen's inequality
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1 Introduction

This paper is concerned with a multi-point boundary value problem system. We study the existence of positive solutions for a class of boundary problems:

$$\begin{cases} \triangle_{p_1} u = -f_1(u)g_1(v) \\ \triangle_{p_2} v = -f_2(u)g_2(v) \end{cases} \qquad (1.1)$$

$$\begin{cases} u(0) = \sum_{i=1}^n a_i u(\xi_i) \quad , \quad u(1) = \sum_{i=1}^n a_i u(\eta_i) \\ v(0) = \sum_{i=1}^n b_i v(\xi_i) \quad , \quad v(1) = \sum_{i=1}^n b_i v(\eta_i) \end{cases} \qquad (1.2)$$

where

$$\triangle_{p_i} s = \phi_{p_i}(s'), \phi_{p_i}(s) = |s|^{p_i-2} s, p_i > 1, \phi_{q_i} = (\phi_{p_i})^{-1}, \frac{1}{p_i} + \frac{1}{q_i} = 1, a_i \geq 0, b_i \geq 0, 0 \leq \sum_{i=1}^n a_i < 1,$$

$$0 \leq \sum_{i=1}^n b_i < 1, 0 < \xi_1 < \xi_2 < ... < \xi_n < \frac{1}{2}, \xi_i + \eta_i = 1, i = 1, 2, ..., n.$$

and $f_i, g_i \in C([0, +\infty), [0, +\infty))$.
In recent years, the existence and multiplicity of positive solutions to boundary value problems have been studied by many authors [1, 3, 4, 6, 10].
In [5] author, studied the problem

$$\begin{cases} \varphi_{p_1}(u_1') + h_1(t) f_1(u_1, u_2) = 0 \\ \varphi_{p_2}(u_2') + h_2(t) f_2(u_1, u_2) = 0 \\ u_1(0) = u_1(1) = u_2(0) = u_2(1) = 0 \end{cases}$$

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By introducing a cone and completely continuous operator, the author proves the existence of positive solutions for this system.
In [8] the author consider the boundary value problem

$$(\varphi_p(x'(t))' + q(t)f(t, x(t), x'(t)) = 0, t \in (0,1)$$

$$x(0) = 0 = x(1)$$
$$x'(0) = 0 = x'(1).$$

The author define a cone and completely continuous operator and then by using the fixed point theorems in cone, they prove the existence of solutions to this problem .
In [10] authors, investigated the existence of solution to the problem

$$x''(t) + a(t)x'(t) + b(t)x(t) + f(t, x(t), x'(t)) = 0, \quad t \in (0,1)$$

$$x(0) = 0, x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i)$$

$$x'(0) = 0, x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i).$$

A completely continuous operator is defined and by using the fixed point theorem in cones, the existence of multiple solutions is proved.

2 The preliminary lemmas

Definitions [7]. Let E be a real Banach space. A nonempty convex closed set $K \subset E$ is said to be a cone provided that
i) $au \in K$ for all $u \in K$ and $a \geq 0$ and
ii) $u, -u \in K$ implies that $u = 0$.

Let $E := C([0,1], \mathbf{R})$ and

$$K := \{u \in E : u(t) \geqslant 0, t \in [0,1]\}, \|u\| := max\{|u(t)| : t \in [0,1]\}$$

and $\|(u,v)\| := max\{\|u\|, \|v\|\}, (u,v) \in E \times E$
$B_r = \{(u,v) \in E^2 : \|(u,v)\| < r\}$ for $r > 0$. Then $(E, \| \, . \, \|)$ is a real Banach space and K, K^2 are cones.

We define the following operators:

$$L(u,v) = (L_1(u,v)(t), L_2(u,v)(t))$$

such that

$$L_1(u,v)(t) := \frac{1}{1 - \sum_{i=1}^n a_i} \sum_{i=1}^n a_i \int_0^t \varphi_{p_1}^{-1}\left(\int_s^1 (f_1(u(r))g_1(v(r)))dr\right)ds$$

$$L_2(u,v)(t) := \frac{1}{1 - \sum_{i=1}^n b_i} \sum_{i=1}^n b_i \int_0^t \varphi_{p_2}^{-1}\left(\int_s^1 (f_2(u(r))g_2(v(r)))dr\right)ds$$

Hence $L : K^2 \to K^2$ is an completely continuous operator.
The main tool of this paper is the following theorem.

Theorem 2.1. *[2]. Let E be a real Banach space and $K \subset E$ a cone. Suppose that $\Omega \subset E$ is a bounded open set and $T : \overline{\Omega} \bigcap K \to K$ is a completely continuous operator. Let $x_0 \in K \setminus \{0\}$*
(I) If $x - Tx \neq \eta x_0$ for $\eta \geq 0, x \in \delta\Omega \cap K$, then $i(T, \Omega \bigcap k, k) = 0$, where i indicates the fixed point index on K.
(II) Let E be a real Banach space and K a cone in E. Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and $T : \overline{\Omega} \bigcap K \to K$ is a completely continuous operator. If $x - \eta Tx \neq \eta x_0$ for $\eta \in [0, 1], x \in \delta\Omega \cap K$, then $i(T, \Omega \bigcap k, k) = 1$.

Remark 2.2. *Suppose that $x \in K$ is concave on [0,1] , $\| x \| = x(1)$. then $\| x \| \leq \frac{\pi^2}{4} \int_0^1 x(t) sin \frac{\pi}{2} t dt$.*

Lemma 2.3. *(Jensen's Integral Inequality for Nonnegative Concave Functions) Suppose that $u \in C([a, b], \boldsymbol{R}), \phi \in C(R^+, R^+)$. If ϕ is cocave, then*

$$\phi(\frac{1}{b-a} \int_a^b u(t)dt) \geqslant \frac{1}{b-a} \int_a^b \phi(u(t))dt.$$

In particular, if $b - a \leqslant 1$, then we have $(\int_a^b u(t)dt)^\alpha \geqslant (b-a)^{\alpha-1} \int_a^b u^\alpha(t)dt \geqslant \int_a^b u^\alpha(t)dt$ for $0 < \alpha \leq 1$. Let C is a cone in a Banach space $(K, \| . \|)$, the Bonsall cone spectral radius of T is defined by $R_C(T) := lim_{m\to\infty} \| T^m \|^{\frac{1}{m}} = inf_{m \geqslant 1} \| T^m \|^{\frac{1}{m}}$

Lemma 2.4. *[9], Let C is a cone in a Banach space $(K, \| . \|)$, and $T : C \to C$ is a countiuous homogeneous. If $R_C(T) < 1, u, u_0 \in C$ satisfy $u \leq Tu + u_0$, then $u \leq (I - T)_C^{-1} u_0$, where the Bonsall cone spectral radius of T is $R_C(T) := lim_{m\to\infty} \| T^m \|^{\frac{1}{m}} = inf_{m\geqslant 1} \| T^m \|^{\frac{1}{m}}$ and $(I - T)_C^{-1}$ is the inverse operator of $I - T$ on C.*

Let $\Theta > 1$, we define : $\Theta' := min\{2, \Theta\}, \Theta'' := max\{2, \Theta\}, \Theta_. = \frac{\Theta'-1}{\Theta-1}, \Theta^{\cdot} = \frac{\Theta''-1}{\Theta-1}$

3 Main Results

Suppose that f, g satisfy:
$A_1) p_1, p_2 > 1, f_i, g_i \in C([0, +\infty), [0, +\infty))$
A_2) There are two constants $\alpha > \frac{\pi^4}{16}, d > 0$ and two nonnegative functions $m_1, n_1 \in C([0, +\infty), [0, +\infty))$ such that
i) $m_1^{p_{1\cdot}}$ is concave on $[0, +\infty)$
ii) $f_1(u)g_1(v) \geqslant m_1(v^{p_2'-1}) - c, f_2(u)g_2(v) \geqslant n_1(u^{p_1'-1}) - c$ for all $u, v \in [0, +\infty)$,
iii) $m_1^{p_{1\cdot}}(n_1^{p_{2\cdot}}(w)) \geqslant \iota w - d$ for all $w \in [0, +\infty)$
A_3)There are nonnegative constants $\alpha_1, \beta_1, \gamma_1, \delta_1, R$ such that $R_{K^2}(T_1) < 1$ and

$$f_1(u)g_1(v) \leq \alpha_1 u^{p_1-1} + \beta_1 v^{p_1-1}, f_2(u)g_2(v) \leq \gamma_1 u^{p_2-1} + \delta_1 v^{p_2-1}$$

for $u, v \in [0, R), t \in [0, 1]$ we define $T_1 : K^2 \to K^2$ by $T_1(u, v)(t) =$

$$(\frac{1}{1 - \sum_{i=1}^n a_i} \sum_{i=1}^n a_i \int_0^t \varphi_{p_1}^{-1}(\int_s^1 (\alpha_1 u^{p_1-1}(r) + \beta_1 v^{p_1-1}(r)dr)ds,$$

$$\frac{1}{1 - \sum_{i=1}^n b_i} \sum_{i=1}^n b_i \int_0^t \varphi_{p_2}^{-1}(\int_s^1 (\gamma_1 u^{p_2-1}(r) + \delta_1 v^{p_2-1}(r)dr)ds)$$

Theorem 3.1. *Let assumptions A_1, A_2, A_3 be satisfied. Then the problem (1.1) has at least one positive solution.*

Proof. *Let* $x_0(t) = 2t - t^2$,

$$\Im = \{(u,v) \in K^2 | (u,v) = T(u,v) + \eta(x_0, x_0)\}, \eta \geqslant 0\}, \quad (3.1)$$

If $u_0, v_0 \in \Im$ *then for* $t \in [0,1]$ *we have*

$$u_0(t) = \frac{1}{1-\sum_{i=1}^n a_i}\sum_{i=1}^n a_i \int_0^t \varphi_{p_1}^{-1}(\int_s^1 (f_1(u_0(r))g_1(v_0(r)))dr)ds + \eta_0 x_0(t)$$

and

$$v_0(t) = \frac{1}{1-\sum_{i=1}^n b_i}\sum_{i=1}^n b_i \int_0^t \varphi_{p_2}^{-1}(\int_s^1 (f_2(u_0(r))g_2(v_0(r)))dr)ds + \eta_0 x_0(t)$$

So we have,

$$u_0(t) \geqslant \frac{1}{1-\sum_{i=1}^n a_i}\sum_{i=1}^n a_i \int_0^t \varphi_{p_1}^{-1}(\int_s^1 (f_1(u_0(r))g_1(v_0(r)))dr)ds \quad (3.2)$$

$$v_0(t) \geqslant \frac{1}{1-\sum_{i=1}^n b_i}\sum_{i=1}^n b_i \int_0^t \varphi_{p_2}^{-1}(\int_s^1 (f_2(u_0(r))g_2(v_0(r)))dr)ds$$

It is easy to see that u_0, v_0 *are concave and increasing, so by lemma 2.3 we have,*

$$u_0^{p_1'-1}(t) \geqslant \frac{1}{1-\sum_{i=1}^n a_i}\sum_{i=1}^n a_i \int_0^t (\int_s^1 ((f_1(u_0(r))g_1(v_0(r))))^{p_1'} dr)ds =$$

$$\frac{1}{1-\sum_{i=1}^n a_i}\sum_{i=1}^n a_i \int_0^1 (min\{t,s\})((f_1(u_0(r))g_1(v_0(r)))^{p_1'})ds, \quad (3.3)$$

$$v_0^{p_2'-1}(t) \geqslant \frac{1}{1-\sum_{i=1}^n b_i}\sum_{i=1}^n b_i \int_0^t (\int_s^1 ((f_2(u_0(r))g_2(v_0(r))))^{p_2'} dr)ds =$$

$$\frac{1}{1-\sum_{i=1}^n b_i}\sum_{i=1}^n b_i \int_0^1 (min\{t,s\})((f_2(u_0(r))g_2(v_0(r)))^{p_2'})ds,$$

Applying lemma 2.3 and conditin A_2 *we find*

$$u_0^{p_1'-1}(t) \geqslant \int_0^1 (min\{t,s\}) m_1^{p_1 \cdot}(v_0^{p_2'-1}(s))ds - \gamma_1, \quad (3.4)$$

$$v_0^{p_2'-1}(t) \geqslant \int_0^1 (min\{t,s\}) n_1^{p_2 \cdot}(u_0^{p_1'-1}(s))ds - \gamma_2$$

then, we have

$$u_0^{p_1'-1}(t) \geqslant \int_0^1 (min\{t,s\}) m_1^{p_1 \cdot}(\int_0^1 (min\{r,s\}) n_1^{p_2 \cdot}(u_0^{p_1'-1}(r))drds - \gamma_3$$

$$\geqslant \iota \int_0^1 (min\{t,s\})(min\{r,s\}) u_0^{p_1'-1}(r)drds - \gamma_4$$

so,

$$\int_0^1 u_0^{p_1'-1}(t) sin\frac{\pi}{2}tdt \geqslant \frac{16\alpha}{\pi^4}\int_0^1 u_0^{p_1'-1}(t) sin\frac{\pi}{2}tdt - \frac{2\gamma_4}{\pi},$$

$$\int_0^1 u_0^{p_1'-1}(t)sin\frac{\pi}{2}tdt \leq \frac{2\gamma_4\pi^3}{16\iota-\pi^4}, \qquad (3.5)$$

Since $u_0^{p_1'-1}$ *is concave and* u_0 *is increasing, remark 2.2 implies that*

$$\|u_0^{p_1'-1}\| = u_0^{p_1'-1}(1) \leq \frac{\pi^2}{4}\int_0^1 u_0^{p_1'-1}(t)sin\frac{\pi}{2}tdt \leq \frac{\gamma_4\pi^5}{32\iota-2\pi^4}$$

so

$$\|u_0\| \leq (\frac{\gamma_4\pi^5}{32\iota-2\pi^4})^{\frac{1}{p_1'-1}}, \qquad (3.6)$$

for all $(u_0, v_0) \in \Im$ *we have* v_0 *is concave. Then by remark 2.2 we have*

$$\int_0^1 v_0^{p_2'-1}(t)sin\frac{\pi}{2}tdt \geqslant \frac{4}{\pi^2} \parallel v_0^{p_2'-1}\|.$$

It follows from (3.4), (3.5),

$$\frac{2\gamma_4\pi^3}{16\iota-\pi^4} \geqslant \int_0^1 u_0^{p_1'-1}(t)sin\frac{\pi}{2}tdt$$

$$\geqslant \frac{4}{\pi^2}\int_0^1 m_1^{p_1\cdot}(v_0^{p_2'-1}(t))sin\frac{\pi}{2}tdt - \frac{2\gamma_1}{\pi}$$

$$= \frac{4}{\pi^2}\int_0^1 m_1^{p_1\cdot}(\frac{v_0^{p_2'-1}(t)}{\parallel v_0^{p_2'-1}\|} \parallel v_0^{p_2'-1}\|)sin\frac{\pi}{2}tdt - \frac{2\gamma_1}{\pi}$$

$$\geqslant \frac{16}{\pi^4}m_1^{p_1\cdot}(\parallel v_0^{p_2'-1}\|) - \frac{2\gamma_1}{\pi}$$

then,

$$m_1(\parallel v_0^{p_2'-1}\|) \leq (\frac{2\gamma_4\pi^7}{128\iota-8\pi^4} + \frac{\gamma_1\pi^3}{8})^{\frac{1}{p_1\cdot}}$$

By (3.6) and A_2, *we have* $lim_{x\to\infty}m_1(x) = \infty$. *So, for* $(u_0, v_0) \in \Im$ *there is* $e > o$ *such that* $\|v_0\| \leq e$, *it means that* $\Im$ *is bounded. Let* $sup\{\|(u,v)\| : (u,v) \in \Im\} < \Re$, *where* $\Re > 0$. *thus,* $L(u,v)+\eta(x_0,x_0) \neq (u,v)$, *where* $(u,v) \in \partial B_\Re \cap K^2, \eta \geqslant 0$.
Theorem 2.1 (I) implies that

$$i(L, B_\Re \cap K^2, K^2) = 0, \qquad (3.7)$$

Now, we define

$$\Im_1 = \{(u,v) \in \overline{B_r} \cap K^2 \mid (u,v) = \eta L(u,v), 0 \leq \eta \leq 1\}$$

Let $(u_0, v_0) \in \Im_1$, *so, there exists* $0 \leq \eta_0 \leq 1$ *such that* $(u_0, v_0) = \eta_0 L_1(u_0, v_0)$ *then for*

$$u_0(t) \leq \frac{1}{1-\sum_{i=1}^n a_i}\sum_{i=1}^n a_i \int_0^t \varphi_{p_1}^{-1}(\int_s^1 (f_1(u_0(r))g_1(v_0(r)))dr)ds$$

$$v_0(t) \leq \frac{1}{1-\sum_{i=1}^n b_i}\sum_{i=1}^n b_i \int_0^t \varphi_{p_2}^{-1}(\int_s^1 (f_2(u_0(r))g_2(v_0(r)))dr)ds$$

by A_3 *for* $0 \leq t \leq 1$ *we have*

$$u_0(t) \leq \frac{1}{1-\sum_{i=1}^n a_i}\sum_{i=1}^n a_i \int_0^t \varphi_{p_1}^{-1}(\int_s^1 (\alpha_1 u^{p_1-1}(r) + \beta_1 v^{p_1-1}(r)dr,$$

$$v_0(t) \leq \frac{1}{1-\sum_{i=1}^{n} b_i} \sum_{i=1}^{n} b_i \int_0^t \varphi_{p_2}^{-1}(\int_s^1 (\gamma_1 u^{p_2-1}(r) + \delta_1 v^{p_2-1}(r)dr$$

by lemma 2.4 we have $(u_0, v_0) = (0, 0)$, *thus* $\Im_1 = \{(0, 0)\}$, $\eta L(u, v) \neq (u, v)$ *where* $(u, v) \in \partial B_r \cap K^2, \eta \geqslant 0$. *theorem 2.1 (II) implies that*

$$i(L, B_r \cap K^2, K^2) = 1, \qquad (3.8)$$

from (3.7),(3.8) we obtain

$$i(L, (B_{\Re} \setminus \overline{B_r}) \cap K^2, K^2) = -1$$

This implise that L has a fixed point and problem (1.1) has at least one positive solution.

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