

# Solvability for a typical system of boundary value problems by fixed point theory

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**Abstract :** In this paper we study the existence of positive solutions for a class of boundary value problems. We introduce a cone and completely continuous operator. We provide sufficient conditions under which this system according to the fixed point index theorem, has positive solution.

**Keywords :** boundary value problem; positive solution; fixed point index; Jensen's inequality

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## 1 Introduction

This paper is concerned with a multi-point boundary value problem system. We study the existence of positive solutions for a class of boundary problems:

$$\begin{cases} \Delta_{p_1} u = -f_1(u)g_1(v) \\ \Delta_{p_2} v = -f_2(u)g_2(v) \end{cases} \quad (1.1)$$

$$\begin{cases} u(0) = \sum_{i=1}^n a_i u(\xi_i) & , & u(1) = \sum_{i=1}^n a_i u(\eta_i) \\ v(0) = \sum_{i=1}^n b_i v(\xi_i) & , & v(1) = \sum_{i=1}^n b_i v(\eta_i) \end{cases} \quad (1.2)$$

where

$$\Delta_{p_i} s = \phi_{p_i}(s'), \phi_{p_i}(s) = |s|^{p_i-2} s, p_i > 1, \phi_{q_i} = (\phi_{p_i})^{-1}, \frac{1}{p_i} + \frac{1}{q_i} = 1, a_i \geq 0, b_i \geq 0, 0 \leq \sum_{i=1}^n a_i < 1,$$

$$0 \leq \sum_{i=1}^n b_i < 1, 0 < \xi_1 < \xi_2 < \dots < \xi_n < \frac{1}{2}, \xi_i + \eta_i = 1, i = 1, 2, \dots, n.$$

and  $f_i, g_i \in C([0, +\infty), [0, +\infty))$ .

In recent years, the existence and multiplicity of positive solutions to boundary value problems have been studied by many authors [1, 3, 4, 6, 10].

In [5] author, studied the problem

$$\begin{cases} \varphi_{p_1}(u_1') + h_1(t)f_1(u_1, u_2) = 0 \\ \varphi_{p_2}(u_2') + h_2(t)f_2(u_1, u_2) = 0 \\ u_1(0) = u_1(1) = u_2(0) = u_2(1) = 0 \end{cases}$$

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By introducing a cone and completely continuous operator, the author proves the existence of positive solutions for this system.

In [8] the author consider the boundary value problem

$$\begin{aligned}(\varphi_p(x'(t)))' + q(t)f(t, x(t), x'(t)) &= 0, \quad t \in (0, 1) \\ x(0) &= 0 = x(1) \\ x'(0) &= 0 = x'(1).\end{aligned}$$

The author define a cone and completely continuous operator and then by using the fixed point theorems in cone, they prove the existence of solutions to this problem .

In [10] authors, investigated the existence of solution to the problem

$$\begin{aligned}x''(t) + a(t)x'(t) + b(t)x(t) + f(t, x(t), x'(t)) &= 0, \quad t \in (0, 1) \\ x(0) = 0, x(1) &= \sum_{i=1}^{m-2} \alpha_i x(\xi_i) \\ x'(0) = 0, x(1) &= \sum_{i=1}^{m-2} \alpha_i x(\xi_i).\end{aligned}$$

A completely continuous operator is defined and by using the fixed point theorem in cones, the existence of multiple solutions is proved.

## 2 The preliminary lemmas

**Definitions** [7]. Let  $E$  be a real Banach space. A nonempty convex closed set  $K \subset E$  is said to be a cone provided that

- i)  $au \in K$  for all  $u \in K$  and  $a \geq 0$  and
- ii)  $u, -u \in K$  implies that  $u = 0$ .

Let  $E := C([0, 1], \mathbf{R})$  and

$$K := \{u \in E : u(t) \geq 0, t \in [0, 1]\}, \|u\| := \max\{|u(t)| : t \in [0, 1]\}$$

and  $\|(u, v)\| := \max\{\|u\|, \|v\|\}, (u, v) \in E \times E$

$B_r = \{(u, v) \in E^2 : \|(u, v)\| < r\}$  for  $r > 0$ . Then  $(E, \|\cdot\|)$  is a real Banach space and  $K, K^2$  are cones.

We define the following operators:

$$L(u, v) = (L_1(u, v)(t), L_2(u, v)(t))$$

such that

$$\begin{aligned}L_1(u, v)(t) &:= \frac{1}{1 - \sum_{i=1}^n a_i} \sum_{i=1}^n a_i \int_0^t \varphi_{p_1}^{-1} \left( \int_s^1 (f_1(u(r))g_1(v(r))) dr \right) ds \\ L_2(u, v)(t) &:= \frac{1}{1 - \sum_{i=1}^n b_i} \sum_{i=1}^n b_i \int_0^t \varphi_{p_2}^{-1} \left( \int_s^1 (f_2(u(r))g_2(v(r))) dr \right) ds\end{aligned}$$

Hence  $L : K^2 \rightarrow K^2$  is an completely continuous operator.

The main tool of this paper is the following theorem.

**Theorem 2.1.** [2]. Let  $E$  be a real Banach space and  $K \subset E$  a cone. Suppose that  $\Omega \subset E$  is a bounded open set and  $T : \overline{\Omega} \cap K \rightarrow K$  is a completely continuous operator. Let  $x_0 \in K \setminus \{0\}$

(I) If  $x - Tx \neq \eta x_0$  for  $\eta \geq 0, x \in \delta\Omega \cap K$ , then  $i(T, \Omega \cap k, k) = 0$ , where  $i$  indicates the fixed point index on  $K$ .

(II) Let  $E$  be a real Banach space and  $K$  a cone in  $E$ . Suppose that  $\Omega \subset E$  is a bounded open set with  $0 \in \Omega$  and  $T : \overline{\Omega} \cap K \rightarrow K$  is a completely continuous operator. If  $x - \eta Tx \neq \eta x_0$  for  $\eta \in [0, 1], x \in \delta\Omega \cap K$ , then  $i(T, \Omega \cap k, k) = 1$ .

**Remark 2.2.** Suppose that  $x \in K$  is concave on  $[0, 1]$ ,  $\|x\| = x(1)$ . then  $\|x\| \leq \frac{\pi^2}{4} \int_0^1 x(t) \sin \frac{\pi}{2} t dt$ .

**Lemma 2.3.** (Jensen's Integral Inequality for Nonnegative Concave Functions) Suppose that  $u \in C([a, b], \mathbf{R}), \phi \in C(\mathbf{R}^+, \mathbf{R}^+)$ . If  $\phi$  is concave, then

$$\phi\left(\frac{1}{b-a} \int_a^b u(t) dt\right) \geq \frac{1}{b-a} \int_a^b \phi(u(t)) dt.$$

In particular, if  $b - a \leq 1$ , then we have  $(\int_a^b u(t) dt)^\alpha \geq (b - a)^{\alpha-1} \int_a^b u^\alpha(t) dt \geq \int_a^b u^\alpha(t) dt$  for  $0 < \alpha \leq 1$ . Let  $C$  is a cone in a Banach space  $(K, \|\cdot\|)$ , the Bonsall cone spectral radius of  $T$  is defined by  $R_C(T) := \lim_{m \rightarrow \infty} \|T^m\|^{\frac{1}{m}} = \inf_{m \geq 1} \|T^m\|^{\frac{1}{m}}$

**Lemma 2.4.** [9], Let  $C$  is a cone in a Banach space  $(K, \|\cdot\|)$ , and  $T : C \rightarrow C$  is a countinuous homogeneous. If  $R_C(T) < 1, u, u_0 \in C$  satisfy  $u \leq Tu + u_0$ , then  $u \leq (I - T)_C^{-1} u_0$ , where the Bonsall cone spectral radius of  $T$  is  $R_C(T) := \lim_{m \rightarrow \infty} \|T^m\|^{\frac{1}{m}} = \inf_{m \geq 1} \|T^m\|^{\frac{1}{m}}$  and  $(I - T)_C^{-1}$  is the inverse operator of  $I - T$  on  $C$ .

Let  $\Theta > 1$ , we define :  $\Theta' := \min\{2, \Theta\}, \Theta'' := \max\{2, \Theta\}, \Theta = \frac{\Theta' - 1}{\Theta - 1}, \Theta = \frac{\Theta'' - 1}{\Theta - 1}$

### 3 Main Results

Suppose that  $f, g$  satisfy:

A<sub>1</sub>)  $p_1, p_2 > 1, f_i, g_i \in C([0, +\infty), [0, +\infty))$

A<sub>2</sub>) There are two constants  $\alpha > \frac{\pi^4}{16}, d > 0$  and two nonnegative functions  $m_1, n_1 \in C([0, +\infty), [0, +\infty))$  such that

i)  $m_1^{p_1}$  is concave on  $[0, +\infty)$

ii)  $f_1(u)g_1(v) \geq m_1(v^{p_2-1}) - c, f_2(u)g_2(v) \geq n_1(u^{p_1-1}) - c$  for all  $u, v \in [0, +\infty)$ ,

iii)  $m_1^{p_1}(n_1^{p_2}(w)) \geq \omega w - d$  for all  $w \in [0, +\infty)$

A<sub>3</sub>) There are nonnegative constants  $\alpha_1, \beta_1, \gamma_1, \delta_1, R$  such that  $R_{K^2}(T_1) < 1$  and

$$f_1(u)g_1(v) \leq \alpha_1 u^{p_1-1} + \beta_1 v^{p_1-1}, f_2(u)g_2(v) \leq \gamma_1 u^{p_2-1} + \delta_1 v^{p_2-1}$$

for  $u, v \in [0, R], t \in [0, 1]$  we define  $T_1 : K^2 \rightarrow K^2$  by  $T_1(u, v)(t) =$

$$\left(\frac{1}{1 - \sum_{i=1}^n a_i} \sum_{i=1}^n a_i \int_0^t \varphi_{p_1}^{-1} \left(\int_s^1 (\alpha_1 u^{p_1-1}(r) + \beta_1 v^{p_1-1}(r)) dr\right) ds, \right.$$

$$\left. \frac{1}{1 - \sum_{i=1}^n b_i} \sum_{i=1}^n b_i \int_0^t \varphi_{p_2}^{-1} \left(\int_s^1 (\gamma_1 u^{p_2-1}(r) + \delta_1 v^{p_2-1}(r)) dr\right) ds\right)$$

**Theorem 3.1.** *Let assumptions  $A_1, A_2, A_3$  be satisfied. Then the problem (1.1) has at least one positive solution.*

**Proof.** Let  $x_0(t) = 2t - t^2$ ,

$$\mathfrak{S} = \{(u, v) \in K^2 | (u, v) = T(u, v) + \eta(x_0, x_0)\}, \eta \geq 0\}, \quad (3.1)$$

If  $u_0, v_0 \in \mathfrak{S}$  then for  $t \in [0, 1]$  we have

$$u_0(t) = \frac{1}{1 - \sum_{i=1}^n a_i} \sum_{i=1}^n a_i \int_0^t \varphi_{p_1}^{-1} \left( \int_s^1 (f_1(u_0(r))g_1(v_0(r))) dr \right) ds + \eta_0 x_0(t)$$

and

$$v_0(t) = \frac{1}{1 - \sum_{i=1}^n b_i} \sum_{i=1}^n b_i \int_0^t \varphi_{p_2}^{-1} \left( \int_s^1 (f_2(u_0(r))g_2(v_0(r))) dr \right) ds + \eta_0 x_0(t)$$

So we have,

$$\begin{aligned} u_0(t) &\geq \frac{1}{1 - \sum_{i=1}^n a_i} \sum_{i=1}^n a_i \int_0^t \varphi_{p_1}^{-1} \left( \int_s^1 (f_1(u_0(r))g_1(v_0(r))) dr \right) ds \quad (3.2) \\ v_0(t) &\geq \frac{1}{1 - \sum_{i=1}^n b_i} \sum_{i=1}^n b_i \int_0^t \varphi_{p_2}^{-1} \left( \int_s^1 (f_2(u_0(r))g_2(v_0(r))) dr \right) ds \end{aligned}$$

It is easy to see that  $u_0, v_0$  are concave and increasing, so by lemma 2.3 we have,

$$\begin{aligned} u_0^{p_1'-1}(t) &\geq \frac{1}{1 - \sum_{i=1}^n a_i} \sum_{i=1}^n a_i \int_0^t \left( \int_s^1 ((f_1(u_0(r))g_1(v_0(r))))^{p_1'} dr \right) ds = \\ &\frac{1}{1 - \sum_{i=1}^n a_i} \sum_{i=1}^n a_i \int_0^1 (\min\{t, s\}) ((f_1(u_0(r))g_1(v_0(r))))^{p_1'} ds, \quad (3.3) \\ v_0^{p_2'-1}(t) &\geq \frac{1}{1 - \sum_{i=1}^n b_i} \sum_{i=1}^n b_i \int_0^t \left( \int_s^1 ((f_2(u_0(r))g_2(v_0(r))))^{p_2'} dr \right) ds = \\ &\frac{1}{1 - \sum_{i=1}^n b_i} \sum_{i=1}^n b_i \int_0^1 (\min\{t, s\}) ((f_2(u_0(r))g_2(v_0(r))))^{p_2'} ds, \end{aligned}$$

Applying lemma 2.3 and condition  $A_2$  we find

$$\begin{aligned} u_0^{p_1'-1}(t) &\geq \int_0^1 (\min\{t, s\}) m_1^{p_1'} (v_0^{p_2'-1}(s)) ds - \gamma_1, \quad (3.4) \\ v_0^{p_2'-1}(t) &\geq \int_0^1 (\min\{t, s\}) n_1^{p_2'} (u_0^{p_1'-1}(s)) ds - \gamma_2 \end{aligned}$$

then, we have

$$\begin{aligned} u_0^{p_1'-1}(t) &\geq \int_0^1 (\min\{t, s\}) m_1^{p_1'} \left( \int_0^1 (\min\{r, s\}) n_1^{p_2'} (u_0^{p_1'-1}(r)) dr \right) ds - \gamma_3 \\ &\geq \iota \int_0^1 (\min\{t, s\}) (\min\{r, s\}) u_0^{p_1'-1}(r) dr ds - \gamma_4 \end{aligned}$$

so,

$$\int_0^1 u_0^{p_1'-1}(t) \sin \frac{\pi}{2} t dt \geq \frac{16\alpha}{\pi^4} \int_0^1 u_0^{p_1'-1}(t) \sin \frac{\pi}{2} t dt - \frac{2\gamma_4}{\pi},$$

$$\int_0^1 u_0^{p_1'-1}(t) \sin \frac{\pi}{2} t dt \leq \frac{2\gamma_4 \pi^3}{16\iota - \pi^4}, \quad (3.5)$$

Since  $u_0^{p_1'-1}$  is concave and  $u_0$  is increasing, remark 2.2 implies that

$$\|u_0^{p_1'-1}\| = u_0^{p_1'-1}(1) \leq \frac{\pi^2}{4} \int_0^1 u_0^{p_1'-1}(t) \sin \frac{\pi}{2} t dt \leq \frac{\gamma_4 \pi^5}{32\iota - 2\pi^4}$$

so

$$\|u_0\| \leq \left( \frac{\gamma_4 \pi^5}{32\iota - 2\pi^4} \right)^{\frac{1}{p_1'-1}}, \quad (3.6)$$

for all  $(u_0, v_0) \in \mathfrak{S}$  we have  $v_0$  is concave. Then by remark 2.2 we have

$$\int_0^1 v_0^{p_2'-1}(t) \sin \frac{\pi}{2} t dt \geq \frac{4}{\pi^2} \|v_0^{p_2'-1}\|.$$

It follows from (3.4), (3.5),

$$\begin{aligned} & \frac{2\gamma_4 \pi^3}{16\iota - \pi^4} \geq \int_0^1 u_0^{p_1'-1}(t) \sin \frac{\pi}{2} t dt \\ & \geq \frac{4}{\pi^2} \int_0^1 m_1^{p_1} \cdot (v_0^{p_2'-1}(t)) \sin \frac{\pi}{2} t dt - \frac{2\gamma_1}{\pi} \\ & = \frac{4}{\pi^2} \int_0^1 m_1^{p_1} \cdot \left( \frac{v_0^{p_2'-1}(t)}{\|v_0^{p_2'-1}\|} \|v_0^{p_2'-1}\| \right) \sin \frac{\pi}{2} t dt - \frac{2\gamma_1}{\pi} \\ & \geq \frac{16}{\pi^4} m_1^{p_1} \cdot (\|v_0^{p_2'-1}\|) - \frac{2\gamma_1}{\pi} \end{aligned}$$

then,

$$m_1(\|v_0^{p_2'-1}\|) \leq \left( \frac{2\gamma_4 \pi^7}{128\iota - 8\pi^4} + \frac{\gamma_1 \pi^3}{8} \right)^{\frac{1}{p_1}}.$$

By (3.6) and  $A_2$ , we have  $\lim_{x \rightarrow \infty} m_1(x) = \infty$ . So, for  $(u_0, v_0) \in \mathfrak{S}$  there is  $e > 0$  such that  $\|v_0\| \leq e$ , it means that  $\mathfrak{S}$  is bounded. Let  $\sup\{\|(u, v)\| : (u, v) \in \mathfrak{S}\} < \mathfrak{R}$ , where  $\mathfrak{R} > 0$ . thus,  $L(u, v) + \eta(x_0, x_0) \neq (u, v)$ , where  $(u, v) \in \partial B_{\mathfrak{R}} \cap K^2, \eta \geq 0$ .

Theorem 2.1 (I) implies that

$$i(L, B_{\mathfrak{R}} \cap K^2, K^2) = 0, \quad (3.7)$$

Now, we define

$$\mathfrak{S}_1 = \{(u, v) \in \overline{B_r} \cap K^2 \mid (u, v) = \eta L(u, v), 0 \leq \eta \leq 1\}$$

Let  $(u_0, v_0) \in \mathfrak{S}_1$ , so, there exists  $0 \leq \eta_0 \leq 1$  such that  $(u_0, v_0) = \eta_0 L_1(u_0, v_0)$  then for

$$u_0(t) \leq \frac{1}{1 - \sum_{i=1}^n a_i} \sum_{i=1}^n a_i \int_0^t \varphi_{p_1}^{-1} \left( \int_s^1 (f_1(u_0(r)) g_1(v_0(r))) dr \right) ds$$

$$v_0(t) \leq \frac{1}{1 - \sum_{i=1}^n b_i} \sum_{i=1}^n b_i \int_0^t \varphi_{p_2}^{-1} \left( \int_s^1 (f_2(u_0(r)) g_2(v_0(r))) dr \right) ds$$

by  $A_3$  for  $0 \leq t \leq 1$  we have

$$u_0(t) \leq \frac{1}{1 - \sum_{i=1}^n a_i} \sum_{i=1}^n a_i \int_0^t \varphi_{p_1}^{-1} \left( \int_s^1 (\alpha_1 u^{p_1-1}(r) + \beta_1 v^{p_1-1}(r)) dr \right),$$

$$v_0(t) \leq \frac{1}{1 - \sum_{i=1}^n b_i} \sum_{i=1}^n b_i \int_0^t \varphi_{p_2}^{-1} \left( \int_s^1 (\gamma_1 u^{p_2-1}(r) + \delta_1 v^{p_2-1}(r)) dr \right)$$

by lemma 2.4 we have  $(u_0, v_0) = (0, 0)$ , thus  $\mathfrak{S}_1 = \{(0, 0)\}$ ,  $\eta L(u, v) \neq (u, v)$  where  $(u, v) \in \partial B_r \cap K^2$ ,  $\eta \geq 0$ . theorem 2.1 (II) implies that

$$i(L, B_r \cap K^2, K^2) = 1, \quad (3.8)$$

from (3.7),(3.8) we obtain

$$i(L, (B_{\mathbb{R}} \setminus \overline{B_r}) \cap K^2, K^2) = -1$$

This implies that  $L$  has a fixed point and problem (1.1) has at least one positive solution.

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