Solvability for a typical system of boundary value problems by fixed point theory

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Abstract : In this paper we study the existence of positive solutions for a class of boundary value problems. We introduce a cone and completely continuous operator. We provide sufficient conditions under which this system according to the fixed point index theorem, has positive solution.

Keywords : boundary value problem; positive solution; fixed point index; Jensen’s inequality

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1 Introduction

This paper is concerned with a multi-point boundary value problem system. We study the existence of positive solutions for a class of boundary problems:

\[ \begin{cases} \Delta_{p_1} u = -f_1(u)g_1(v) \\ \Delta_{p_2} v = -f_2(u)g_2(v) \end{cases} \quad (1.1) \]

\[ \begin{cases} u(0) = \sum_{i=1}^{n} a_i u(\xi_i) \\ u(1) = \sum_{i=1}^{n} a_i u(\eta_i) \end{cases} \quad (1.2) \]

where

\[ \Delta_{p_i} s = \phi_{p_i}(s') \phi_{p_i}(s) = |s|^{p_i-2} s, p_i > 1, \phi_{p_i} = (\phi_{p_i})^{-1}, \frac{1}{p_i} + \frac{1}{q_i} = 1, a_i \geq 0, b_i \geq 0, 0 \leq \sum_{i=1}^{n} a_i < 1, \]

\[ 0 \leq \sum_{i=1}^{n} b_i < 1, 0 < \xi_1 < \xi_2 < ... < \xi_n < \frac{1}{2} \xi_i + \eta_i = 1, i = 1, 2, ..., n. \]

and \( f_i, g_i \in C([0, +\infty), [0, +\infty]). \)

In recent years, the existence and multiplicity of positive solutions to boundary value problems have been studied by many authors [1, 2, 3, 4, 5].

In [1] author, studied the problem

\[ \begin{cases} \varphi_{p_1}(u_1') + h_1(t)f_1(u_1, u_2) = 0 \\ \varphi_{p_2}(u_2') + h_2(t)f_2(u_1, u_2) = 0 \end{cases} \]

\[ u_1(0) = u_1(1) = u_2(0) = u_2(1) = 0 \]

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By introducing a cone and completely continuous operator, the author proves the existence of positive solutions for this system.

In [8] the author consider the boundary value problem

\[(\varphi_p(x'(t))' + q(t)f(t, x(t), x'(t)) = 0, \quad t \in (0, 1)\]

\[x(0) = 0 = x(1)\]

\[x'(0) = 0 = x'(1).\]

The author define a cone and completely continuous operator and then by using the fixed point theorems in cone, they prove the existence of solutions to this problem.

In [10] authors, investigated the existence of solution to the problem

\[x''(t) + a(t)x'(t) + b(t)x(t) + f(t, x(t), x'(t)) = 0, \quad t \in (0, 1)\]

\[x(0) = 0, x(1) = m_2 \sum_{i=1}^{n} x(i)\]

\[x'(0) = 0, x(1) = m_2 \sum_{i=1}^{n} x(i).\]

A completely continuous operator is defined and by using the fixed point theorem in cones, the existence of multiple solutions is proved.

2 The preliminary lemmas

Definitions [9]. Let \(E\) be a real Banach space. A nonempty convex closed set \(K \subseteq E\) is said to be a cone provided that

i) \(au \in K\) for all \(u \in K\) and \(a \geq 0\) and
ii) \(u, -u \in K\) implies that \(u = 0\).

Let \(E := C([0, 1], \mathbb{R})\) and

\[K := \{u \in E : u(t) \geq 0, t \in [0, 1]\}, \|u\| := \max\{|u(t)| : t \in [0, 1]\}\]

and \(\|(u, v)\| := \max\{\|u\|, \|v\|\}, (u, v) \in E \times E\)

\(B_r = \{(u, v) \in E^2 : \|(u, v)\| < r\}\) for \(r > 0\). Then \((E, \|\cdot\|)\) is a real Banach space and \(K, K^2\) are cones.

We define the following operators:

\[L(u, v) = (L_1(u, v)(t), L_2(u, v)(t))\]

such that

\[L_1(u, v)(t) := \frac{1}{1 - \sum_{i=1}^{n} a_i} \sum_{i=1}^{n} a_i \int_{0}^{1} \varphi_{p_{1}}^{-1} \left( \int_{s}^{1} (f_1(u(r))g_1(v(r)))dr \right)ds\]

\[L_2(u, v)(t) := \frac{1}{1 - \sum_{i=1}^{n} b_i} \sum_{i=1}^{n} b_i \int_{0}^{1} \varphi_{p_{2}}^{-1} \left( \int_{s}^{1} (f_2(u(r))g_2(v(r)))dr \right)ds\]

Hence \(L : K^2 \to K^2\) is an completely continuous operator.

The main tool of this paper is the following theorem.
Theorem 2.1. Let $E$ be a real Banach space and $K \subset E$ a cone. Suppose that $\Omega \subset E$ is a bounded open set and $T : \Omega \cap K \to K$ is a completely continuous operator. Let $x_0 \in K \setminus \{0\}$
(I) If $x - Tx \neq \eta x_0$ for $\eta \geq 0, x \in \delta \Omega \cap K,$ then $i(T, \Omega \cap k, k) = 0,$ where $i$ indicates the fixed point index on $K.$
(II) Let $E$ be a real Banach space and $K$ a cone in $E.$ Suppose that

$\Omega \cap K$ is a completely continuous operator. If $x - \eta Tx \neq \eta x_0$ for $\eta \in [0,1]$, $x \in \delta \Omega \cap K,$ then $i(T, \Omega \cap k, k) = 1.$

Remark 2.2. Suppose that $x \in K$ is concave on $[0,1]$, $\| x \| = x(1).$ then $\| x \| \leq \frac{1}{b-a} \int_a^b x(t) \sin \frac{\pi}{a-b} dt.$

Lemma 2.3. (Jensen’s Integral Inequality for Nonnegative Concave Functions) Suppose that $u \in C([a,b], R), \phi \in C(R^+, R^+).$ If $\phi$ is concave, then

$$\phi \left( \frac{1}{b-a} \int_a^b u(t) dt \right) \geq \frac{1}{b-a} \int_a^b \phi(u(t)) dt.$$ 

In particular, if $b - a \leq 1,$ then we have $(\int_a^b u(t) dt)^\alpha \geq (b - a)^{\alpha-1} \int_a^b u^\alpha(t) dt \geq \int_a^b u^\alpha(t) dt$ for $0 < \alpha \leq 1.$

Let $C$ is a cone in a Banach space $(K, \| . \|),$ the Bonsall cone spectral radius of $T$ is defined by $R_C(T) := \lim_{n \to \infty} \| T^n \|^{\frac{1}{n}} = \inf_{n \geq 1} \| T^n \|^\frac{1}{n}$

Lemma 2.4. Let $C$ is a cone in a Banach space $(K, \| . \|),$ and $T : C \to C$ is a continuous homogeneous. If $R_C(T) < 1, u, u_0 \in C$ satisfy $u \leq Tu + u_0,$ then $u \leq (I - T)_{C^{-1}} u_0,$ where the Bonsall cone spectral radius of $T$ is $R_C(T) := \lim_{n \to \infty} \| T^n \|^{\frac{1}{n}} = \inf_{n \geq 1} \| T^n \|^\frac{1}{n}$ and $(I - T)_{C^{-1}}$ is the inverse operator of $I - T$ on $C.$

Let $\Theta > 1,$ we define $\Theta' := \min\{2, \Theta\}, \Theta'' := \max\{2, \Theta\}, \Theta = \frac{\Theta'' - 1}{\Theta' - 1}, \Theta = \frac{\Theta'' - 1}{\Theta' - 1}$

3 Main Results

Suppose that $f, g$ satisfy:

$A_1)$ There are two constants $\alpha > 0, d > 0$ and two nonnegative functions $m_1, n_1 \in C([0, +\infty), [0, +\infty))$ such that

i) $m_1^{\frac{1}{\alpha}}$ is concave on $[0, +\infty)$

ii) $f_1(u)g_1(v) \geq m_1(v^{\alpha - 1}) - c, f_2(u)g_2(v) \geq n_1(v^{\alpha - 1}) - c$ for all $u, v \in [0, +\infty),$ 

iii) $m_1^{\frac{1}{\alpha}} (n_1^{\frac{1}{\alpha - 1}} (w)) \geq uw - d$ for all $w \in [0, +\infty)$

$A_2)$ There are nonnegative constants $\alpha_1, \beta_1, \gamma_1, \delta_1, R$ such that $R_{K^2}(T_1) < 1$ and

$$f_1(u)g_1(v) \leq \alpha_1 u^{\alpha_1 - 1} + \beta_1 v^{\alpha_1 - 1}, f_2(u)g_2(v) \leq \gamma_1 u^{\gamma_1 - 1} + \delta_1 v^{\gamma_1 - 1}$$

for $u, v \in [0, R), t \in [0, 1]$ we define $T_1 : K^2 \to K^2$ by $T_1(u, v)(t) =

\left( \frac{1}{1 - \sum_{i=1}^{n} a_i} \right) \sum_{i=1}^{n} a_i \int_0^t \phi_1^{\frac{1}{\alpha_1}}(\int_s^t (\alpha_1 u^{\alpha_1 - 1} (r) + \beta_1 v^{\alpha_1 - 1} (r)) dr) ds,

\left( \frac{1}{1 - \sum_{i=1}^{n} b_i} \right) \sum_{i=1}^{n} b_i \int_0^t \phi_2^{\frac{1}{\gamma_1}}(\int_s^t (\gamma_1 u^{\gamma_1 - 1} (r) + \delta_1 v^{\gamma_1 - 1} (r)) dr) ds$
Theorem 3.1. Let assumptions $A_1, A_2, A_3$ be satisfied. Then the problem (1.1) has at least one positive solution.

Proof. Let $x_0(t) = 2t - t^2$,

$$\mathcal{Z} = \{(u, v) \in K^2 | (u, v) = T(u, v) + \eta(x_0, x_0), \eta \geq 0\}, \quad (3.1)$$

If $u_0, v_0 \in \mathcal{Z}$ then for $t \in [0, 1]$ we have

$$u_0(t) = \frac{1}{1 - \sum_{i=1}^{n} a_i} \sum_{i=1}^{n} a_i \int_{0}^{t} \varphi_{p_i}^{-1} \left( \int_{s}^{1} (f_1(u_0(r))g_1(v_0(r)))dr \right)ds + \eta_0 x_0(t)$$

and

$$v_0(t) = \frac{1}{1 - \sum_{i=1}^{n} b_i} \sum_{i=1}^{n} b_i \int_{0}^{t} \varphi_{q_i}^{-1} \left( \int_{s}^{1} (f_2(u_0(r))g_2(v_0(r)))dr \right)ds + \eta_0 x_0(t)$$

So we have,

$$u_0(t) \geq \frac{1}{1 - \sum_{i=1}^{n} a_i} \sum_{i=1}^{n} a_i \int_{0}^{t} \varphi_{p_i}^{-1} \left( \int_{s}^{1} (f_1(u_0(r))g_1(v_0(r)))dr \right)ds$$

$$v_0(t) \geq \frac{1}{1 - \sum_{i=1}^{n} b_i} \sum_{i=1}^{n} b_i \int_{0}^{t} \varphi_{q_i}^{-1} \left( \int_{s}^{1} (f_2(u_0(r))g_2(v_0(r)))dr \right)ds$$

It is easy to see that $u_0, v_0$ are concave and increasing, so by lemma 2.3 we have,

$$u_0^{p_i - 1}(t) \geq \frac{1}{1 - \sum_{i=1}^{n} a_i} \sum_{i=1}^{n} a_i \int_{0}^{t} \left( \int_{s}^{1} ((f_1(u_0(r))g_1(v_0(r)))^{p_i}) dr \right) ds =$$

$$\frac{1}{1 - \sum_{i=1}^{n} a_i} \sum_{i=1}^{n} a_i \int_{0}^{1} (\min(t, s))(f_1(u_0(r))g_1(v_0(r)))^{p_i} ds, \quad (3.3)$$

$$v_0^{p_i - 1}(t) \geq \frac{1}{1 - \sum_{i=1}^{n} b_i} \sum_{i=1}^{n} b_i \int_{0}^{t} \left( \int_{s}^{1} ((f_2(u_0(r))g_2(v_0(r)))^{q_i}) dr \right) ds =$$

$$\frac{1}{1 - \sum_{i=1}^{n} b_i} \sum_{i=1}^{n} b_i \int_{0}^{1} (\min(t, s))(f_2(u_0(r))g_2(v_0(r)))^{q_i} ds,$$

Applying lemma 2.3 and condition $A_2$ we find

$$u_0^{p_i - 1}(t) \geq \int_{0}^{1} (\min(t, s))m_{p_i}^{p_i} (u_0^{p_i - 1}(s)) ds - \gamma_1, \quad (3.4)$$

$$v_0^{p_i - 1}(t) \geq \int_{0}^{1} (\min(t, s))n_{q_i}^{q_i} (u_0^{q_i - 1}(s)) ds - \gamma_2$$

then, we have

$$u_0^{p_i - 1}(t) \geq \int_{0}^{1} (\min(t, s))m_{p_i}^{p_i} \left( \int_{0}^{1} (\min(r, s))m_{p_i}^{p_i} (u_0^{p_i - 1}(r)) dr drds \right) - \gamma_3$$

$$\geq \int_{0}^{1} (\min(t, s))(\min(r, s))u_0^{p_i - 1}(r) dr drds - \gamma_4$$

so,

$$\int_{0}^{1} u_0^{p_i - 1}(t)\sin \frac{\pi}{2} dt \geq \frac{16\alpha}{\pi^4} \int_{0}^{1} u_0^{p_i - 1}(t)\sin \frac{\pi}{2} dt - \frac{2\gamma_4}{\pi}.$$
\[
\int_0^1 v_0^{p_1^{-1}}(t)\sin\frac{\pi}{2}tdt \leq \frac{2\gamma_4\pi^3}{16t - \pi^2},
\]
(3.5)

Since \(v_0^{p_1^{-1}}\) is concave and \(u_0\) is increasing, remark 2.2 implies that
\[
\|u_0^{p_1^{-1}}\| = u_0^{p_1^{-1}}(1) \leq \frac{\pi^2}{4} \int_0^1 u_0^{p_1^{-1}}(t)\sin\frac{\pi}{2}tdt \leq \frac{2\gamma_4\pi}{32t - 2\pi^2}
\]
so
\[
\|u_0\| \leq \left(\frac{2\gamma_4\pi}{32t - 2\pi^2}\right)^{\frac{1}{p_1^{-1}}},
\]
(3.6)
for all \((u_0, v_0) \in \mathcal{S}\) we have \(v_0\) is concave. Then by remark 2.2 we have
\[
\int_0^1 v_0^{p_1^{-1}}(t)\sin\frac{\pi}{2}tdt \geq \frac{4}{\pi^2} \|v_0^{p_1^{-1}}\|.
\]

It follows from (3.4), (3.5),
\[
\frac{2\gamma_4\pi^3}{16t - \pi^4} \geq \int_0^1 u_0^{p_1^{-1}}(t)\sin\frac{\pi}{2}tdt
\]
\[
\geq \frac{4}{\pi^2} \int_0^1 m_1^{p_1} (v_0^{p_1^{-1}}(t))\sin\frac{\pi}{2}tdt - \frac{2\gamma_1}{\pi}
\]
\[
= \frac{4}{\pi^2} \int_0^1 m_1^{p_1} (v_0^{p_1^{-1}}(t)) \|v_0^{p_1^{-1}}\|\sin\frac{\pi}{2}tdt - \frac{2\gamma_1}{\pi}
\]
\[
\geq \frac{16}{\pi^2} m_1^{p_1} (\|v_0^{p_1^{-1}}\|) - \frac{2\gamma_1}{\pi}
\]
then,
\[
m_1 (\|v_0^{p_1^{-1}}\|) \leq \left(\frac{2\gamma_4\pi^7}{128t - 8\pi^4} + \frac{\gamma_1\pi^3}{8}\right)^{\frac{1}{p_1^{-1}}}
\]
By (3.6) and \(A_2\), we have \(\lim_{x \to \infty} m_1(x) = \infty\). So, for \((u_0, v_0) \in \mathcal{S}\) there is \(e > 0\) such that \(\|v_0\| \leq e\), it means that \(\mathcal{S}\) is bounded. Let \(\sup\{\|u, v\| : (u, v) \in \mathcal{S}\} < R\), where \(R > 0\). Thus, \(L(u, v) + \eta(x_0, x_0) \neq (u, v)\), where \((u, v) \in \partial B_R \cap K^2, \eta \geq 0\).

Theorem 2.1 (I) implies that
\[
i(L, B_R \cap K^2, K^2) = 0,
\]
(3.7)

Now, we define
\[
\mathcal{S}_1 = \{(u, v) \in \overline{B_R} \cap K^2 | (u, v) = \eta L(u, v), 0 \leq \eta \leq 1\}
\]
Let \((u_0, v_0) \in \mathcal{S}_1\), so, there exists \(0 \leq \eta_0 \leq 1\) such that \((u_0, v_0) = \eta_0 L_1(u_0, v_0)\) then for
\[
u_0(t) \leq \frac{1}{1 - \sum_{i=1}^n a_i} \sum_{i=1}^n a_i \int_0^t \varphi_{p_1}^{-1} (\int_s^t f_1(u_0(r))g_1(v_0(r))dr)ds
\]
\[
u_0(t) \leq \frac{1}{1 - \sum_{i=1}^n b_i} \sum_{i=1}^n b_i \int_0^t \varphi_{p_1}^{-1} (\int_s^t f_2(u_0(r))g_2(v_0(r))dr)ds
\]
by \(A_3\) for \(0 \leq t \leq 1\) we have
\[
u_0(t) \leq \frac{1}{1 - \sum_{i=1}^n a_i} \sum_{i=1}^n a_i \int_0^t \varphi_{p_1}^{-1} (\int_s^t (\alpha_1 u^{p_1-1}(r) + \beta_1 v^{p_1-1}(r))dr,
\]

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\[ v_0(t) \leq \frac{1}{1 - \sum_{i=1}^{n} b_i \sum_{i=1}^{n} b_i} \int_0^t \varphi_{p_2}^{-1} \left( \int_s^t \gamma_1 u^{p_2-1}(r) + \delta_1 v^{p_2-1}(r)dr \right) \]

by lemma 2.4 we have \((u_0, v_0) = (0, 0)\), thus \(31 = \{(0, 0)\}\), \(\eta L(u, v) \neq (u, v)\) where \((u, v) \in \partial B_r \cap K^2, \eta \geq 0\).

Theorem 2.1 (II) implies that

\[ i(L, B_r \cap K^2, K^2) = 1, \quad (3.8) \]

From (3.7), (3.8) we obtain

\[ i(L, (B_R \setminus B_r) \cap K^2, K^2) = -1 \]

This implies that \(L\) has a fixed point and problem (1.1) has at least one positive solution.

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References


