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# Multiplicity of solutions for nonlinear systems with two-point BVP

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**Abstract:** In this paper, we investigate the existence of solutions to a class of non-linear system. Using some theorems, we prove some existence results for this system.

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## 1 Introduction

In this paper, we study the following system:

$$\begin{cases} A_u + a^{n-1} f(u) m(v) = 0 \\ A_v + a^{n-1} g(u) n(v) = 0 \end{cases} \quad (1) \qquad \begin{cases} u(1) = 0, u'(0) = 0 \\ v(1) = 0, v'(0) = 0 \end{cases} \quad (2)$$

Where $A_u = a^{n-1} div(|\nabla u|^{p-2} \nabla u), p > 1,\ a = |x|, x \in \Omega \subset \mathbb{R}^n$,
$i) f, g, m, n \in C^1(\mathbb{R}), uf(u)m(v) > 0, vg(u)n(v) > 0,$ when $u, v \neq 0, f, g \geq \epsilon_1 > 0\ on\ (0, +\infty], a \geq 0$ then $f(0)m(0) = 0, g(0)n(0) = 0$.
In recent years, many authors studied the existence of solutions for nonlinear boundary value problems. See, [1-12] and the references therein. For example, in [10] the problem

$$-\left(a^{n-1} \varphi_p(u_1')\right)' = a^{n-1} \psi(a) f(u), on\ [0,1], \qquad (3)$$

$$u'(0) = 0, u(1) = 0 \qquad (4)$$

$\psi \in C^1(\mathbb{R}^+), \varphi_p(x) = |x|^{p-2} x, p > 1,$ was studied.
We introduce the following eigenvalue problem

$$-\left(a^{n-1} \varphi_p(u_1')\right)' = \lambda a^{n-1} \psi(a) |u|^{\ p-2} u\ , \qquad (5)$$

$$u'(0) = 0, u(1) = 0,$$

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Consider

$$u(0)=\alpha, u'(0)=0, \qquad (6)$$

$|(u,v)|=|u|+|v|$.

**Lemma** 1 [10]. Let $\{r_i\}_{i=1}^{k}$ be zeros of an eigen function $y_k$ for (5) corresponding to $\lambda_k$ satisfying $0=r_0<r_1 \ldots < r_k=1. i)$ .

Assume $\lambda>\lambda_k$ , $1\le i\le k$ , there exist a solution $z_i$ of

$$-(a^{n-1}|z'|^{p-2}z')'+\lambda a^{n-1}w(z)|z|^{p-2}z=0, \qquad (7)$$

it has at least two zeros in

$$(r_{i-1},r_i). ii) \text{ for } \lambda<\lambda_k, 1\le i\le k,$$

there exist a solution $\bar{z_i}$ of (1) satisfying $\bar{z_i}(a)$ on $[r_{i-1},r_i]$.

**Lemma 2** [10]. Let $M>0$ , $w^*=max\{\psi(a)|a\in[0,1]\}$ and $\alpha$ satisfy $k_2+k_1F(\alpha)>w^*F(M)$.

We define

$$\delta\equiv Mp^{-\frac{1}{p}}\left(k_2+k_1F(\alpha)-w^*F(M)\right)^{-\frac{1}{p}}>0. \qquad (8)$$

Then the solution $u(a,\alpha)$ of (3) , (6) has the following properties:

$i)$ if $u(r,\alpha)$ has no zero in $(r_1,r_2)$ and satisfies $|u(a,\alpha)|\le M$ on $[r_1,r_2]$ for some $r_1,r_2\in[0,1]$ , then we have $r_2-r_1\le\delta. ii)$ if $u(r,\alpha)$ has no zero in $(x,y)$ for some $x,y\in[0,1]$ satisfying $y-x>2\delta$, then $|u(a,\alpha)|>M$ for $r\in(x+\delta,y-\delta)$. For $M,\alpha>0$ define $I_{M,\alpha}=\{r\in[0,1]||u(r,\alpha)|<M\}$.

## 2 Main results

In this section I introduce Prufer-type substitution for the solution $u(a,\alpha)$ of (1) and (2) by using the generalized sine function $S_p(a)$ has been well studied by [2]. The function $S_p$ ,$S_q$ satisfies

$$\begin{cases}\left|S_p(a)\right|^p+\dfrac{\left|S'_p(a)\right|^p}{p-1}=1\\ \left|S_q(a)\right|^q+\dfrac{\left|S'_q(a)\right|^q}{q-1}=1\end{cases}, \qquad (9)$$

$$\begin{cases}(\left|S'_p\right|^{p-2}S'_p)'+\left|S_p\right|^{p-2}S_p=0\\ (\left|S'_q\right|^{q-2}S'_q)'+\left|S_q\right|^{q-2}S_q=0\end{cases}, \qquad (10)$$

I define phase-plane coordinates $\rho_i>0 \; and \; \theta_i$ for solutions $u(a,\alpha)$, $v(a,\alpha)$ of (1) and (2) as following. With

$$\theta_1(0,\alpha)=\tfrac{\pi_p}{2}, \theta_2(0,\alpha)=\tfrac{\pi_q}{2}. \begin{cases}\rho_1^{\frac{p}{p-1}}(a,\alpha)=|u(a,\alpha)|^p+\dfrac{a^{\frac{p(n-1)}{p-1}}}{p-1}|u'(a,\alpha)|^p\\ \rho_2^{\frac{q}{q-1}}(a,\alpha)=|v(a,\alpha)|^q+\dfrac{a^{\frac{q(n-1)}{q-1}}}{q-1}|v'(a,\alpha)|^q\end{cases} \qquad (*)$$

After differentiating with respect to a, we have

$$\theta'_1(a,\alpha)=\frac{a^{n-1}f(u)m(v)}{(p-1)|u|^{p-2}u}\left|S_p(\theta_1(a,\alpha))\right|^p+a^{\frac{1-n}{p-1}}\left|S'_p(\theta_1(a,\alpha))\right|^p, (11)$$

$$,\theta'_2(a,\alpha)=\frac{a^{n-1}g(u)n(v)}{(q-1)|v|^{q-2}v}\left|S_q(\theta_2(a,\alpha))\right|^q$$

Also

$$\frac{\rho'_2(a,\alpha)}{\rho_2(a,\alpha)}=(a^{\frac{1-n}{q-1}}-\frac{a^{n-1}g(u)n(v)}{|v|^{q-2}v}\left|S_q(\theta_2(a,\alpha))\right|^{q-2}S_q(\theta_2(a,\alpha))S'_q(\theta_2(a,\alpha)), a^{\frac{1-n}{q-1}}\left|S'_q(\theta_2(a,\alpha))\right|^q , \qquad (12)$$

he phase function for (1), (2) with $\lambda=\lambda_k$ ,

$$\phi'_{1k}(a,\lambda_k) = F(a,\lambda_k,\phi_1), \qquad (13),$$

$$\phi'_{2k}(a,\lambda_k) \equiv G(a,\lambda_k,\phi_2), \phi_{1k}(0,\lambda_k) = \frac{\pi_p}{2}, \phi_{1k}(1,\lambda_k) = k\pi_p, \ \phi_{2k}(0,\lambda_k) = \frac{\pi_q}{2}, \phi_{2k}(1,\lambda_k) = k\pi_q,$$

**Lemma 3**.

$i$) Suppose

$$\lim sup_{|u|\to 0} \frac{f(u)m(v)}{|u|^{p-2}u} < \lambda_k, \ \lim sup_{|v|\to 0} \frac{g(u)n(v)}{|v|^{q-2}v} < \lambda_k, \text{ for } k \in \mathbb{N},$$

then there exists $\alpha_* > 0$ such that

$$\theta_1(1,\alpha) < k\pi_p, \ \theta_2(1,\alpha) < k\pi_q, \text{for all } \alpha \in (0,\alpha_*).$$

That is the solution $(u(a,\alpha), v(a,\alpha))$ of (1) and (2) has at most k-1 zeros in (0,1) for $\alpha \in (0,\alpha_*]$.

$ii$) Suppose

$$\lim inf_{|u|\to 0} \frac{f(u)m(v)}{|u|^{p-2}u} > \lambda_k, \ \lim inf_{|v|\to 0} \frac{g(u)n(v)}{|v|^{q-2}v} > \lambda_k, \text{for } k \in \mathbb{N},$$

then there exists $\alpha_* > 0$ such that

$$\theta_1(1,\alpha) > k\pi_p, \ \theta_2(1,\alpha) > k\pi_q, \text{when } \alpha \in (0,\alpha_*].$$

That is, the solution $(u(a,\alpha), v(a,\alpha))$ of (1) and (2) has at least k-1 zeros in (0,1) for $\alpha \in (0,\alpha_*]$.

**Proof. *i***) The assumption implies that, there exists $\delta > 0$ and $\lambda > 0$ such that

$$\frac{f(u)m(v)}{|u|^{p-2}u} < \lambda < \lambda_k, \frac{g(u)n(v)}{|v|^{q-2}v} < \lambda < \lambda_k \text{ for } 0 < |u| + |v| < \delta.$$

Since $(u,v) \equiv 0$ is a solution of (1) , (2), there exists $\alpha_* > 0$ such that

$$\left|\big(u(a,\alpha), v(a,\alpha)\big)\right| < \delta \text{ for } 0 < \alpha < \alpha_* \text{ and } r \in [0,1].$$

From (11), (12) we have

$$\theta'_1(a,\alpha) < \frac{a^{n-1}\lambda_k}{(p-1)}\left|S_p\big(\theta_1(a,\alpha)\big)\right|^p + a^{\frac{1-n}{p-1}}\left|S_p\big(\theta_1(a,\alpha)\big)\right|^p = F(a,\lambda_k,\phi_1),$$

$$\theta'_2(a,\alpha) < \frac{a^{n-1}\lambda_k}{(q-1)}\left|S_q\big(\theta_2(a,\alpha)\big)\right|^q + a^{\frac{1-n}{q-1}}\left|S_q\big(\theta_2(a,\alpha)\big)\right|^q = G(a,\lambda_k,\phi_2).$$

Let $u_k, v_k$ be the solution of (5) with $\lambda = \lambda_k$ and $\phi_{1k}, \phi_{2k}$ be its Prüfer angle, then $u_k, v_k$ are eigenfunctions of (5). Thus

$$\phi_{1k}(1,\lambda_k) = k\pi_p, \ \phi_{2k}(1,\lambda_k) = k\pi_q,$$

The comparison theorem was studied by (Birkhoff & Rota 1989, p.30, in [3]), include that

$$\theta_1(1,\alpha) < \phi_{1k}(1,\lambda_k), \ \theta_2(1,\alpha) < \phi_{2k}(1,\lambda_k), 0 < \alpha < \alpha_*.$$

***ii***) By assumption, we have there exist exists $\delta > 0$ and $\lambda > 0$ such that

$$\frac{f(u)m(v)}{|u|^{p-2}u} > \lambda > \lambda_k, \ \frac{g(u)n(v)}{|v|^{q-2}v} > \lambda > \lambda_k \text{ when } 0 < |u| + |v| < \delta.$$

Similar to (i), there exists $\alpha_* > 0$ such that

$$0 < |(u(a,\alpha), v(a,\alpha))| < \delta \text{ for } 0 < \alpha < \alpha_*, a \in [0,1].$$

So,

$$\frac{f\big(u(a,\alpha)\big)m\big(v(a,\alpha)\big)}{|u(a,\alpha)|^{p-2}u(a,\alpha)} > \lambda_k, \ \frac{g\big(u(a,\alpha)\big)n\big(v(a,\alpha)\big)}{|v(a,\alpha)|^{q-2}v(a,\alpha)} > \lambda_k,$$

by (11), (12) we get

$$\theta'_1(a,\alpha) > \frac{a^{n-1}\lambda_k}{(p-1)}\left|S_p\big(\theta_1(a,\alpha)\big)\right|^p + a^{\frac{1-n}{1-p}}\left|S_p\big(\theta_1(a,\alpha)\big)\right|^p = F(a,\lambda_k,\phi_1),$$

$$\theta'_2(a,\alpha) > \frac{a^{n-1}\lambda_k}{(q-1)}\left|S_q\big(\theta_2(a,\alpha)\big)\right|^q + a^{\frac{1-n}{1-q}}\left|S_q\big(\theta_2(a,\alpha)\big)\right|^q = G(a,\lambda_k,\phi_2).$$

Similar as in (i), we have

$$\theta_1(1,\alpha) > k\pi_p\,,\ \theta_2(1,\alpha) > k\pi_q\,.$$

**Lemma 4.**

$i$) Assume that

$$\lim inf_{|u|\to\infty}\frac{f(u)m(v)}{|u|^{p-2}u} > \lambda_k, \lim inf_{|v|\to\infty}\frac{g(u)n(v)}{|v|^{q-2}v} > \lambda_k, \text{for } k\in\mathbb{N}\,,$$

then there exists $\alpha^* > 0$ such that the solution $(u(a,\alpha), v(a,\alpha))$ has at least k zeros in $(0,1)\times(0,1)$, $\alpha\in[\alpha^*,\infty)$.

$\boldsymbol{ii}$) suppose

$$\lim sup_{|u|\to\infty}\frac{f(u)m(v)}{|u|^{p-2}u} < \lambda_k, \lim sup_{|v|\to\infty}\frac{g(u)n(v)}{|v|^{q-2}v} < \lambda_k, \text{for } k\in\mathbb{N}\,,$$

then there exists $\alpha^* > 0$ such that the solution $(u(a,\alpha), v(a,\alpha))$ has at most (k -1) zeros in $(0,1)\times(0,1)$ for $\alpha^* < \alpha$.

**Proof.** $\boldsymbol{i}$) by assumption, there exist $\lambda > \lambda_k$ and $M > 0$ such that

$$\frac{f(u)m(v)}{|u|^{p-2}u} > \lambda > \lambda_k\,,\quad \frac{g(u)n(v)}{|v|^{q-2}v} > \lambda > \lambda_k \text{ when } |u|+|v|\geq M, \quad (14)$$

Let $u_k, v_k$ be the k-th eigenfunction of (5) corresponding to $\lambda_k$ and $\{r_i\}_{i=1}^k$ be zeros of $u_k, v_k$ with $r_0 = 0$ and $r_k = 1$. Lemma (1) implies that, there exists a solution $z_{1i}, z_{2i}$ of (7) having at least two zeros in $(r_{i-1}, r_i)$. Now, fix $i\in\{1,2,\dots,k\}$, let $t_1, t_2$ be zeros of $z_{1i}, z_{2i}$ satisfying

$$r_{i-1} < t_1 < t_2 < r_i\,.$$

By (8) and remark that $\delta$ tends to zero as $\alpha$ tends to infinity. For this $i$, we can choose an $\alpha_i > 0$ such that

$$r_i - r_{i-1} > 2\delta_i \text{ and } [t_1,t_2]\subset(r_{i-1}+\delta_i,\ r_i-\delta_i),$$

where $\alpha_i$ *and* $\delta_i$ are consistent with (8).

Let $\alpha\geq\alpha_i$, we prove $u(a,\alpha), v(a,\alpha)$ have at least one zero in $(r_{i-1}, r_i)$. Suppose $u(a,\alpha), v(a,\alpha)$ have no zero in $(r_{i-1}, r_i)$. Lemma (2) ($\boldsymbol{ii}$) implies that

$$|u(a,\alpha)| > M, |v(a,\alpha)| > M, \text{when } a\in(r_{i-1}+\delta_i,\ r_i-\delta_i).$$

From (17), we have

$$\lambda < \frac{f\big(u(a,\alpha)\big)m\big(v(a,\alpha)\big)}{(u(a,\alpha))^{p-1}},\ \lambda < \frac{g\big(u(a,\alpha)\big)n\big(v(a,\alpha)\big)}{(v(a,\alpha))^{q-1}}, \text{for } a\in[t_1,t_2]\subset(r_{i-1}+\delta_i,\ r_i-\delta_i).$$

So (in [8], p. 182) implies that $u(a,\alpha), v(a,\alpha)$ have at least one zero in $(t_1,t_2)$. This leads to a contradiction. Hence $u(a,\alpha), v(a,\alpha)$ with $\alpha\geq\alpha_i$ have at least one zero in $(r_{i-1}, r_i)$.

Set

$$\alpha^* = max\{\alpha_i | i = 1,2,\dots,k\}.$$

If $\alpha\geq\alpha^*$, then $u(a,\alpha), v(a,\alpha)$ have at least one zero in $(r_{i-1}, r_i)$ for each $i = 1,2,\dots,k$. It means that $u(a,\alpha), v(a,\alpha)$ have at least k zeros in $(0,1)$ for $\alpha\in[\alpha^*,\infty)$. $\boldsymbol{ii}$) by assumption, there exist $\lambda < \lambda_k$ and $M > 0$ such that

$$\frac{f(u)m(v)}{|u|^{p-2}u} < \lambda < \lambda_k\,,\ \frac{g(u)n(v)}{|v|^{q-2}v} < \lambda < \lambda_k \text{ when } |u|+|v|\geq M, \quad (15)$$

For every $\alpha > 0$, let $\phi_i(a,\alpha)$ , $\phi_{ik}(a,\alpha)$ be the Prüfer angle of the solutions of (5) and (6) with $\lambda$ *and* $\lambda_k$ . So,

$$\phi_{1k}(1,\alpha) = k\pi_p\,,\ \phi_{2k}(1,\alpha) = k\pi_q\,,$$

hence by the comparison theorem,

$\phi_1(1,\alpha) = k\pi_p - \varepsilon$, $\phi_2(1,\alpha) = k\pi_q - \varepsilon, \varepsilon > 0$ and from (11) , (12) $\phi_i(r,\alpha)$ satisfying

$$\phi'_1(a,\alpha) = \frac{a^{n-1}\lambda}{(p-1)}\big|S_p\big(\phi_1(a,\alpha)\big)\big|^p + r^{\frac{1-n}{p-1}}\big|S_p\big(\phi_1(a,\alpha)\big)\big|^p \equiv F(a,\alpha,\phi_1),$$

$$\phi'_2(a,\alpha) = \frac{a^{n-1}\lambda}{(q-1)}\big|S_q\big(\phi_2(a,\alpha)\big)\big|^q + r^{\frac{1-n}{q-1}}\big|S_q\big(\phi_2(a,\alpha)\big)\big|^q \equiv G(a,\lambda_k,\phi_2), \quad (16)$$

Define:

$$R(a,\alpha)=\begin{cases}\frac{f(u(a,\alpha))m(v(a,\alpha))}{|u(a,\alpha)|^{p-2}u(a,\alpha)} & , \ |u(a,\alpha)|<M\\ \lambda & , \ |u(a,\alpha)|\geq M\end{cases},$$

$$T(a,\alpha)=\begin{cases}\frac{g(u(a,\alpha))n(v(a,\alpha))}{|v(a,\alpha)|^{q-2}v(a,\alpha)} & ,|v(a,\alpha)|<M\\ \lambda & ,|v(a,\alpha)|\geq M\end{cases}$$

By (11) and (12) and comparing with (16) there exists a sufficiently large $\alpha^*$, $\left|\frac{f(u(a,\alpha))(m(v(a,\alpha))}{\rho_1(a,\alpha)}\right|$,

$$\left|\frac{g(u(a,\alpha))n(v(a,\alpha))}{\rho_2(a,\alpha)}\right|$$

can be small for

$$|u(a,\alpha)|<M\ ,|v(a,\alpha)|<M \ \text{ and } \ \alpha\geq\alpha^*.$$

So $\theta_1(a,\alpha),\theta_2(a,\alpha)$ are bounded for $\alpha\geq\alpha^*$ --0and -$a\in[0,1]$.

The number of zeros of $u(a,\alpha),v(a,\alpha)$ of (3) and (6) is uniformly bounded for $\alpha\geq\alpha^*$. By ([10]), we have

$$\lim_{\alpha\to\infty}\|I_{M,\alpha}\|=0 \qquad (17)$$

let $\psi_i(a,\alpha)$ be the solution of the equation $\psi'_i(a,\alpha)=H_i((a,\alpha,\psi_i)$, satisfying

$$\psi_1(0,\alpha)=\frac{\pi_p}{2}\ ,\psi_2(0,\alpha)=\frac{\pi_q}{2} \ \text{ and from (13) with } \lambda=\lambda_k \qquad (18)$$

we obtain ($\alpha\geq\alpha^*$ and $a\in[0,1]$ ),

$$\psi_1(a,\alpha)-\phi_1(a,\alpha)=\int_0^a(H(s,\alpha,\psi_1)-F(s,\alpha,\psi_1)+F(s,\alpha,\psi_1)-F(s,\alpha,\phi_1))ds=\int_0^a\frac{s^{n-1}}{p-1}(R(s,\alpha)-\lambda)\left|S_p(\psi_1(s,\alpha))\right|^p ds+\int_0^a\frac{\partial}{\partial\phi_1}F(s,\alpha,\xi_1)\left(\psi_1(s,\alpha)-\phi_1(s,\alpha)\right)ds,$$

$$\psi_2(a,\alpha)-\phi_2(a,\alpha)=\int_0^a\frac{\partial}{\partial\phi_2}G(s,\alpha,\xi_2)\left(\psi_2(s,\alpha)-\phi_2(s,\alpha)\right)ds,$$

Where $\xi_i(s,\alpha)$ is between $\psi_i(s,\alpha)$ and $\phi_i(s,\alpha)$.

By (17),

$$\left|\int_0^a\frac{s^{n-1}}{p-1}(R(s,\alpha)-\lambda)\left|S_p(\psi_1(s,\alpha))\right|^p ds\right|\leq\int_{I_{M,\alpha}}^a\frac{s^{n-1}}{p-1}(R(s,\alpha)-\lambda)ds<\delta\ ,$$

$$\left|\int_0^a\frac{s^{n-1}}{q-1}(T(s,\alpha)-\lambda)\left|S_q(\psi_2(s,\alpha))\right|^q ds\right|\leq\int_{I_{M,\alpha}}^a\frac{s^{n-1}}{q-1}B(s)(T(s,\alpha)-\lambda)ds<\delta \ \text{ when } \ \alpha\geq\alpha^*,\ \delta>0\,.$$

Note that

$$\left|\frac{\partial}{\partial\phi_1}F(s,\alpha,\xi_1)\right| \ \text{ and } \ \left|\frac{\partial}{\partial\phi_2}G(s,\alpha,\xi_2)\right| \text{ are bounded by } \ k_1,k_2>0\,.$$

So, we have

$$|\psi_1(a,\alpha)-\phi_1(a,\alpha)|<\delta+\int_0^a k_1|\psi_1(s,\alpha)-\phi_1(s,\alpha)|ds,$$

$$|\psi_2(a,\alpha)-\phi_2(a,\alpha)|<\delta+\int_0^a k_2|\psi_2(s,\alpha)-\phi_2(s,\alpha)|ds\ ,$$

If $\delta<\varepsilon e^{-k_1}$, $\delta<\varepsilon e^{-k_2}$, then $|\psi_1(a,\alpha)-\phi_1(a,\alpha)|<\delta e^{k_1 a}<\varepsilon$,

$$|\psi_2(a,\alpha)-\phi_2(a,\alpha)|<\delta e^{k_2 a}<\varepsilon.$$

So

$$\psi_1(a,\alpha)<\phi_1(a,\alpha)+\varepsilon,\ \psi_2(a,\alpha)<\phi_2(a,\alpha)+\varepsilon,$$

$$\theta_1(a,\alpha)\leq\psi_1(a,\alpha)<\phi_1(a,\alpha)+\varepsilon=k_1\pi_p,\ \theta_2(a,\alpha)\leq\psi_2(a,\alpha)<\phi_2(a,\alpha)+\varepsilon=k_2\pi_q,$$

thus the proof completed.

**Theorem 5.** Suppose that there exists an integer $k\in\mathbb{N}$ such that

$$\lim sup_{|u|\to 0}\frac{f(u)m(v)}{|u|^{p-2}u}<\lambda_k<\lim inf_{|u|\to\infty}\frac{f(u)m(v)}{|u|^{p-2}u}, \qquad (19)$$

$$\lim sup_{|v|\to 0}\frac{g(u)n(v)}{|v|^{q-2}v}<\lambda_k<\lim inf_{|v|\to\infty}\frac{g(u)n(v)}{|v|^{q-2}v}\,. \qquad (20)$$

Then (1) and (2) have a solution with at most k-1 zeros in (0,1).

**Proof.** By (19) and lemma (3) (i), there exits $\alpha_* > 0$ such that

$$\theta_1(1,\alpha) < k\pi_p\,, \theta_2(1,\alpha) < k\pi_q \quad \text{for } \alpha \le \alpha_*\,.$$

Lemma (4) (i) implies that there exits $\alpha^* > 0$ such that

$$\theta_1(1,\alpha) > k\pi_p\,, \theta_2(1,\alpha) > k\pi_q \quad \text{for } \alpha \ge \alpha^*.$$

Since

$$\theta_1(1,\alpha) = k\pi_p\,, \theta_2(1,\alpha) = k\pi_q\,.$$

Similarly, (20) can be proved. Now the proof is completed.

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