MATHEMATICS AND COMPUTATIONAL SCIENCES

Vol 2(3), 2021, pp:14-21

DOI:10.30511/mcs.2021.532909.1027

ISSN:2717-2708

http://mcs.qut.ac.ir/

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Mittag-Leffler-Hyer-Ulam-Rassias stability of cubic functional equation

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Abstract : In this paper, we prove the Mittag-Leffler-Hyers-Ulam-Rassias stability for cubic functional equation by using the fixed point alternative theorem. As a consequence, we show that the cubic multipliers are superstable under some conditions.

Keywords : Quadratic functional; Superstable; Cubic multipliers.
2020 Mathematics Subject Classification : 45M10, 65P40, 65L20, 70H14.
Receive: 26 June 2021, **Accepted:** 6 August 2021

1 Introduction

The stability theory for functional equations started with a problem related to the stability of group homomorphism that was considered by Ulam in 1940(see [27, 28]). Ulam considered the following question: "Let $\mathcal{G}_1$ be a group and let $\mathcal{G}_2$ be a group endowed with a metric ρ. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $\vartheta : \mathcal{G}_1 \to \mathcal{G}_2$ satisfies the inequality

$$\rho\big(\vartheta(y \cdot z), \vartheta(y)\vartheta(z)\big) < \delta,$$

for each y, $z \in \mathcal{G}_1$, then we can find a homomorphism $\hat{h} : \mathcal{G}_1 \to \mathcal{G}_2$ such that

$$\rho\big(\vartheta(y), \hat{h}(y)\big) < \varepsilon,$$

for almost all $y \in \mathcal{G}_1$?". In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. An affirmative answer to this equation was given by Hyers in[12], for the case of Banach spaces. This answer, in this case, says that the Cauchy functional equation is stable in the sense of Hyers - Ulam. In 1950, T. Aoki was the second author to treat this problem for additive mapping (for more details, see[5, 9]). In 1978, Rassias generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences ([20]). He has introduced a new type of stability which is called the Hyers - Ulam - Rassias stability. Research in stability theory is now very extensive and many papers and books have been published (for instance, consider [13, 17, 21, 23–25]). Recently, the Ulam stability of first order, second order and third order differential equations were investigated in a series of papers [3, 18, 29]. Murali and Selvan investigated the Mittag - Leffler - Hyers - Ulam stability and Mittag - Leffeler - Hyers - Ulam - Rassias stability of a homogeneous and non-homogeneous linear differential equation of first order $u'(t) + \ell u(t) = 0$ and $u'(t) + \ell u(t) = r(t)$, for all $t \in I := [a, b]$, $u(t)$, $r(t) \in C(I)$, by using the Laplace Transforms Methods [18].

The functional equation

$$\mathfrak{u}(y + z) + \mathfrak{u}(y - z) = 2\mathfrak{u}(y) + 2\mathfrak{u}(z), \tag{1.1}$$

is called quadratic functional equation. Also, every solution (for example $\mathfrak{u}(y) = ay^2$) of functional Equation (1.1) is said to be a quadratic mapping. A Hyers - Ulam stability problem for the quadratic functional equation was proved by Skof for mappings $\mathfrak{u} : \mathcal{N} \to \mathcal{B}$, where $\mathcal{N}$ is a normed space and $\mathcal{B}$ is a Banach space ([26]). Jun *et al.* in [15] introduced the functional equation

$$\mathfrak{u}(2y + z) + \mathfrak{u}(2y - z) = 2\mathfrak{u}(y + z) + 2\mathfrak{u}(y - z) + 12\mathfrak{u}(y), \tag{1.2}$$

which is somewhat different from Eq. (1.1). It is easy to see that function $\mathfrak{u}(y) = ay^3$ is a solution of Eq. (1.2). Thus, it is natural that Equation (1.2) is called a cubic functional equation and every solution of this cubic functional equation

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is said to be a cubic function. One year after that in [16], they solved the generalized Hyers - Ulam - Rassias stability of a cubic functional equation

$$\mathfrak{u}(y+2z)+\mathfrak{u}(y-2z)+6\mathfrak{u}(y)=\mathfrak{u}(y+z)+4\mathfrak{u}(y-z). \tag{1.3}$$

Recently, Bodaghi *et al.* in [6, 7, 19] proved the superstability of quadratic double centralizers and of quadratic multiplies on Banach algebras by fixed point methods. They studied n-variable mappings that are quartic in each variable, showed that the conditions defining such mappings can be unified in a single functional equation and by using an alternative fixed point method to prove the Hyers Ulam stability for the multiquartic functional equations in the normed spaces. In fact a mapping $\mathfrak{c}:\mathcal{G}^n\to\mathcal{S}$ is called multicubic if it is cubic, i.e., it satisfies the cubic functional equation

$$\mathfrak{c}(2y+z)+\mathfrak{c}(2yz)=2\mathfrak{c}(y+z)+2\mathfrak{c}(yz)+12\mathfrak{c}(y), \tag{1.4}$$

in each variable. where $\mathcal{G}$ is a commutative group and $\mathcal{S}$ is a linear space ([8]). Also the stability and super stability of cubic double centralizers of Banach algebras which are strongly without order had been established in [11] by Eshaghi Gordji *et al.*.
In this paper, we investigate the Mittag - Leffler - Hyers - Ulam - Rassias stability (MLHUR-stability) by using the alternative fixed point for the cubic functional Equation (1.2) and their correspondent cubic multipliers.

2 Main results

Throughout this section, $\mathcal{N}$ is a normed vector space and $\mathcal{B}$ is a Banach space.

Now, for the given mapping $\mathfrak{u}:\mathcal{N}\to\mathcal{B}$, we consider the functional equation

$$\mathbb{D}[\mathfrak{u}](y,z):=\mathfrak{u}(2y+z)+\mathfrak{u}(2u-z)-2\mathfrak{u}(y+z)-2\mathfrak{u}(y-z)-12\mathfrak{u}(y). \tag{2.1}$$

We need the following known fixed point theorem, which is useful for our results.

Theorem 2.1. *[10] (The fixed point alternative) Suppose that (Υ,ρ) be a complete generalized metric space. and let $\mathfrak{J}:\to\Upsilon\to\Upsilon$ be a strictly contractive mapping with the Lipschitz constant $\ell<1$. Then for each element $y\in\Upsilon$, either*

$$\rho\left(\mathfrak{J}^n(y),\mathfrak{J}^{n+1}(y)\right)=\infty,$$

for all $n\geq 0$, or there exists a natural number N_0 such that:

(i) For all $n\geq N_0$, $\rho\left(\mathfrak{J}^n(y),\mathfrak{J}^{n+1}(y)\right)=\infty$.

(ii) The sequence $\{\mathfrak{J}^n(y)\}$ is convergent to a fixed point y^* of $\mathfrak{J}$.

(iii) y^* is the unique fixed point of $\mathfrak{J}$ in

$$\Omega=\Big\{\omega\in\Upsilon\,:\,\rho\big(\mathfrak{J}^{N_0}(y),\omega\big)<\infty\Big\}.$$

(iv) For all $\omega\in\Omega$,

$$\rho(\omega,y^*)\leq\frac{1}{1-\ell}\rho\big(\mathfrak{J}(\omega),\omega\big).$$

2.1 Stability of cubic function equations

Definition 2.2. *The Mittag - Leffler function of one parameter is denoted by $\mathcal{E}_\zeta(z)$ and defined as*

$$\mathcal{E}_\zeta(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\zeta+1)},$$

where $z,\zeta\in\mathcal{C}$ with $Re\ (\zeta)>0$.

Theorem 2.3. *Let $\mathfrak{u}:\mathcal{N}\to\mathcal{B}$ be a mapping with $\mathfrak{u}(0)=0$ and let $\psi:\mathcal{N}\times\mathcal{N}\to[0,\infty)$ be a function satisfying*

$$\lim_{n\to\infty}\frac{\psi\left(2^ny,2^nz\right)}{8^n}=0,$$

and

$$\|\mathbb{D}[\mathfrak{u}](y,z)\|\leq\psi(y,z)\mathcal{E}_\zeta(y), \tag{2.2}$$

for all $y, z \in \mathcal{N}$. If there exists $\ell \in (0,1)$ such that

$$\psi(2y,0)\mathcal{E}_\zeta(2y) \leq 8\ell\psi(y,0)\mathcal{E}_\zeta(y), \tag{2.3}$$

for each $y \in \mathcal{N}$, then there exists a uniqe cubic mapping $\mathfrak{c} : \mathcal{N} \to \mathcal{B}$ such that for $y \in \mathcal{N}$,

$$\|\mathfrak{u}(y) - \mathfrak{c}(y)\| \leq \frac{\ell}{2(1-\ell)}\psi(y,0)\mathcal{E}_\zeta(y). \tag{2.4}$$

Proof. We consider the set

$$\Upsilon := \Big\{\mu : \mathcal{N} \to \mathcal{B} \,|\, \mu(0) = 0\Big\},$$

and introduce the generalized metric on Υ as follows:

$$\rho(\mu_1,\mu_2) := \inf\Big\{C \in (0,\infty] \,:\, \|\mu_1(y) - \mu_2(y)\| \leq C\psi(y,0)\mathcal{E}_\zeta(y),\, \forall y \in \mathcal{N}\Big\},$$

where $\mathcal{E}_\zeta$ is Mittag - Leffler function. If there is not exists such constant C then $\rho(\mu_1,\mu_2) = \infty$, otherwise. One can prove that the metric space (Υ,ρ) is complete. Now, we define the mapping $\mathcal{J} : \Upsilon \to \Upsilon$ by $\mathcal{J}[\mu](y) = \frac{1}{8}\mu(2y)$, for $y \in \mathcal{X}$. If μ_1, $\mu_2 \in \Upsilon$ such that $\rho(\mu_1,\mu_2) < C$, by definition of ρ and $\mathcal{J}$, we have

$$\left\|\frac{1}{8}\mu_1(2y) - \frac{1}{8}\mu_2(2y)\right\| \leq \frac{1}{8}C\psi(2y,0)\mathcal{E}_\zeta(2y),$$

for each $y \in \mathcal{N}$. By using Eq. (2.3), we get

$$\left\|\frac{1}{8}\mu_1(2y) - \frac{1}{8}\mu_2(2y)\right\| \leq C\ell\psi(y,0)\mathcal{E}_\zeta(y),$$

for all $y \in \mathcal{N}$. The above inequality shows that

$$\rho\big(\mathcal{J}[\mu_1],\mathcal{J}[\mu_2]\big) \leq \ell\rho(\mu_1,\mu_2),$$

for almost all μ_1, $\mu_2 \in \Upsilon$. Hence, $\mathcal{J}$ is a strictly contractive mapping on Υ with Lipschitz constant ℓ. Putting $z = 0$ in inequality (2.2), using (2.3), and dividing both sides of the resulting inequality by 16, we have

$$\left\|\frac{1}{8}\mathfrak{u}(2y) - \mathfrak{u}(y)\right\| \leq \frac{1}{16}\psi(y,0)\mathcal{E}_\zeta(y) \leq \frac{\ell}{2}\psi\left(\frac{y}{2},0\right)\mathcal{E}_\zeta\left(\frac{y}{2}\right),$$

for all $y \in \mathcal{N}$. Thus, $\rho\left(\mathfrak{u},\mathcal{J}[\mathfrak{u}]\right) \leq \frac{1}{2}\ell < \infty$. By Theorem 2.1, the sequence $\{\mathcal{J}[\mathfrak{u}]\}$ converges to a fixed point $\mathfrak{c} : \mathcal{N} \to \mathcal{B}$ in the set

$$\Upsilon_1 = \{\mu \in \Upsilon \,:\, \rho\left(\mathfrak{u},\mu\right) < \infty\},$$

that is

$$\mathfrak{c}(y) = \lim_{n\to\infty}\frac{\mathfrak{u}(2^n y)}{8^n}. \tag{2.5}$$

By Theorem 2.1, we have

$$\rho\left(\mathfrak{u},\mathfrak{c}\right) \leq \frac{\rho(\mathfrak{u},\mathcal{J}[\mathfrak{u}])}{1-\ell} \leq \frac{\ell}{2(1-\ell)}. \tag{2.6}$$

It follows from inequality (2.6), that inequality (2.4) holds for all $y \in \mathcal{X}$. Substituting y, z by $2^n y$, $2^n z$ in (2.2), we have

$$\begin{aligned}\|\mathbb{D}[\mathfrak{c}](y,z)\| &= \lim_{n\to\infty}\frac{1}{8^n}\left\|\mathbb{D}[\mathfrak{u}](2^n y, 2^n z)\right\| \\ &\leq \lim_{n\to\infty}\frac{1}{8^n}\psi\left(2^n y, 2^n z\right)\mathcal{E}_\zeta(2^n y) = 0,\end{aligned}$$

for all $y \in \mathcal{N}$. Therefore $\mathfrak{c}$ is a cubic mapping which is unique. □

Corollary 2.4. *Let η and λ be non-negative real numbers such that $\eta < 3$. Suppose that $\mathfrak{u} : \mathcal{N} \to \mathcal{B}$ is a mapping satisfying*

$$\|\mathbb{D}[\mathfrak{u}](y,z)\| \leq \lambda\big(\|y\|^\eta + \|z\|^\eta\big)\mathcal{E}_\zeta(y),$$

for all y, $z \in \mathcal{N}$. Then, there exists a uniqe cubic mapping $\mathfrak{c} : \mathcal{N} \to \mathcal{B}$ such that

$$\|\mathfrak{u}(y) - \mathfrak{c}(y)\| \leq \frac{2^\eta\lambda}{2(8-2^\eta)}\|y\|^\eta\mathcal{E}_\zeta(y),$$

for all $x \in \mathcal{N}$.

Proof. The result follows from Theorem 2.3 by using $\psi(y,z) = \lambda(\|y\|^{\eta} + \|z\|^{\eta})$. □

Corollary 2.5. *Let η_1, η_2, λ be non-negative real numbers such that η_1 and $\eta_2 \in (3,\infty)$. Suppose a mapping $\mathfrak{u} : \mathcal{N} \to \mathcal{B}$ satisfies*

$$\|\mathbb{D}[\mathfrak{u}](y,z)\| \leq \lambda \|y\|^{\eta_1}\|z\|^{\eta_2}\mathcal{E}_\zeta(y), \tag{2.7}$$

for all y, $z \in \mathcal{N}$. Then, $\mathfrak{u}$ is cubic mapping on $\mathcal{N}$.

Proof. Letting $y = z = 0$ in (2.7), we get $\mathfrak{u}(0) = 0$. Once more, if we put $z = 0$ in (2.7), we have $\mathfrak{u}(2y) = 8\mathfrak{u}(y)$ for all $y \in \mathcal{N}$. It is easy to see that by induction, we have $\mathfrak{u}(2^n y) = 8^n \mathfrak{u}(y)$, and so

$$\mathfrak{u}(y) = \frac{1}{8^n}\mathfrak{u}(2^n y),$$

for all $y \in \mathcal{N}$ and $n \in \mathbb{N}$. Now, it follows from Theorem 2.3 that $\mathfrak{u}$ is a cubic mapping. □

Note that in Corollary 2.5, if $\eta_1 + \eta_2 \in (0,3)$ and $\eta_1 > 0$ such that the inequality (2.7) holds, then by applying

$$\psi(y,z) = \lambda\|y\|^{\eta_1}\|z\|^{\eta_2},$$

in Theorem 2.3, $\mathfrak{u}$ is again a cubic be mapping.

Theorem 2.6. *Let $\mathfrak{u} : \mathcal{N} \to \mathcal{B}$ be a mapping with $\mathfrak{u}(0) = 0$, and let $\psi : \mathcal{N} \times \mathcal{N} \to [0,\infty)$ be a function satisfying*

$$\lim_{n\to\infty} 8^n \left(\frac{y}{2^n}, \frac{z}{2^n}\right) = 0, \tag{2.8}$$

and

$$\|\mathbb{D}[\mathfrak{u}](y,z)\| \leq \psi(y,z)\mathcal{E}_\zeta(y), \tag{2.9}$$

for each y, $z \in \mathcal{N}$. If there exists $\ell \in (0,1)$ such that

$$\psi(y,0)\mathcal{E}_\zeta(y) \leq \frac{1}{8}\ell\psi(2y,0)\mathcal{E}_\zeta(2y), \tag{2.10}$$

for almost all $y \in \mathcal{N}$, then there exists a unique cubic mapping $\mathfrak{c} : \mathcal{N} \to \mathcal{B}$ such that

$$\|\mathfrak{u}(y) - \mathfrak{c}(y)\| \leq \frac{\ell}{16(1-\ell)}\psi(y,0)\mathcal{E}_\zeta(y). \tag{2.11}$$

Proof. We consider the set

$$\Upsilon := \left\{\mu : \mathcal{N} \to \mathcal{B} \,|\, \mu(0) = 0\right\}$$

and introduce the generalized metric on Υ

$$\rho(\mu_1,\mu_2) := \inf\left\{C \in (0,\infty) \,:\, \|\mu_1(y) - \mu_2(y)\| \leq C\psi(y,0)\mathcal{E}_\zeta(y), \forall y \in \mathcal{N}\right\},$$

if there exists such constant C and $\rho(\mu_1,\mu_2) = \infty$, otherwise. It is easy to show that (Υ,ρ) is complete. We will show that the mapping $\mathcal{J} : \Upsilon \to \Upsilon$ define by

$$\mathcal{J}[\mu](y) = 8\mu\left(\frac{y}{2}\right),$$

for $y \in \mathcal{N}$, is strictly contractive. For given μ_1, $\mu_2 \in \Upsilon$ such that $\rho(\mu_1,\mu_2) < C$, we have

$$\left\|8\mu_1\left(\frac{y}{2}\right) - 8\mu_2\left(\frac{y}{2}\right)\right\| \leq C\psi\left(\frac{y}{2},0\right)\mathcal{E}_\zeta\left(\frac{y}{2}\right),$$

for all $y \in \mathcal{N}$. By using inequality (2.10), we obtain

$$\left\|8\mu_1\left(\frac{y}{2}\right) - 8\mu_2\left(\frac{y}{2}\right)\right\| \leq C\ell\psi(y,0)\mathcal{E}_\zeta(y),$$

for all $y \in \mathcal{N}$. It follows from the last inequality that $\rho(\mathcal{J}[\mu_1],\mathcal{J}[\mu_2]) \leq \ell\rho(\mu_1,\mu_2)$ for all μ_1, $\mu_2 \in \Upsilon$. Hence, $\mathcal{J}$ is a strictly contractive mapping on Υ with Lipschitz constant ℓ. By putting $z = 0$ and replacing y by $\frac{y}{2}$ in inequality (2.9) and using inequality (2.10), then by dividing both sides of the resulting inequality by 2, we have

$$\left\|8\mathfrak{u}\left(\frac{y}{2}\right) - \mathfrak{u}(y)\right\| \leq \frac{1}{2}\psi\left(\frac{y}{2},0\right)\mathcal{E}_\zeta\left(\frac{y}{2}\right) \leq \frac{1}{16}\ell\psi(y,0)\mathcal{E}_\zeta(y),$$

for all $y \in \mathcal{N}$. Hence,

$$\rho(\mathfrak{u}, \mathcal{J}[\mathfrak{u}]) \leq \frac{L}{16} < \infty.$$

By applying the fixed point alternative, there exists a unique mapping $\mathfrak{v} : \mathcal{N} \to \mathcal{B}$ in the set $\Upsilon_1 = \{\mu \in \Upsilon \,:\, \rho(\mathfrak{u}, \mu) < \infty\}$, such that

$$\mathfrak{c}(y) = \lim_{n\to\infty} 8^n \mathfrak{u}\left(\frac{y}{2^n}\right), \tag{2.12}$$

for every $y \in \mathcal{N}$. Again, Theorem 2.1 shows that

$$\rho(\mathfrak{u}, \mathfrak{c}) \leq \frac{\rho(\mathfrak{u}, \mathcal{J}[\mathfrak{u}])}{1-\ell} \leq \frac{\ell}{16(1-\ell)}, \tag{2.13}$$

where the inequality (2.13) implies the relation (2.11). Replacing y, z by $2^n y$, $2^n z$ in (2.9), respectively, and using (2.8) and (2.12), we conclude

$$\|\mathbb{D}[\mathfrak{c}](y,z)\| = \lim_{n\to\infty} 8^n \left\| \mathbb{D}[\mathfrak{u}]\left(\frac{y}{2^n}, \frac{z}{2^n}\right) \right\| \leq \lim_{n\to\infty} 8^n \psi\left(\frac{y}{2^n}, \frac{z}{2^n}\right) = 0,$$

for all $y \in \mathcal{N}$. Therefore $\mathfrak{c}$ is a cubic mapping. □

2.2 Stability of cubic multipliers

In this section, we investigate the Mittag - Leffler - Hyers - Ulam stability and the superstability of cubic multipliers.

Definition 2.7. *A cubic multiplier on an algebra $\mathcal{G}$ is a cubic mapping $\mathfrak{T} : \mathcal{G} \to \mathcal{G}$ such that*

$$y\mathfrak{T}(z) = \mathfrak{T}(y)z,$$

for each y, $z \in \mathcal{G}$.

The following theorem introduces a cubic multipliers on Banach algebras.

Theorem 2.8. *Let $\mathfrak{u} : \mathcal{G} \to \mathcal{G}$ be a mapping with $\mathfrak{u}(0) = 0$ and let $\psi : \mathcal{G}^4 \to [0, \infty)$ be afunction such that*

$$\|\mathbb{D}[\mathfrak{u}](y,z) + \mathfrak{u}(v)w - v\mathfrak{u}(w)\| \leq \psi(y,z,v,w)\mathcal{E}_\zeta(y), \tag{2.14}$$

for all y, z, v, $w \in \mathcal{G}$. If there exists a constant $\ell \in (0,1)$ such that

$$\psi(2y, 2z, 2v, 2w) \leq 8\ell\psi(y,z,v,w), \tag{2.15}$$

for all y, z, v, $w \in \mathcal{G}$, then there exists a unique cubic multiplier $\mathfrak{T}$ on $\mathcal{G}$ satisfying

$$\|\mathfrak{u}(y) - \mathfrak{T}(y)\| \leq \frac{\ell}{2(1-\ell)}\psi(y,y,0,0)\mathcal{E}_\zeta(y), \tag{2.16}$$

for each $y \in \mathcal{X}$.

Proof. It follows from the relation (2.15)that

$$\lim_{n\to\infty} \frac{\psi(2^n y, 2^n z, 2^n v, 2^n w)}{8^n} = 0, \tag{2.17}$$

for all y, z, v, $w \in \mathcal{G}$. Putting $z = v = w = 0$ in (2.14), we obtain

$$\|2\mathfrak{u}(2y) - 16\mathfrak{u}(y)\| \leq \psi(y,0,0,0)\mathcal{E}_\zeta(y),$$

for every $y \in \mathcal{G}$. Thus,

$$\left\| \frac{1}{8}\mathfrak{u}(2y) - \mathfrak{u}(y) \right\| \leq \frac{1}{16}\psi(y,0,0,0)\mathcal{E}_\zeta(y), \tag{2.18}$$

for all $y \in \mathcal{G}$. Now similar to the proof of theorem in previous section, we consider the set

$$\Upsilon := \{\mu : \mathcal{G} \to \mathcal{G} \,|\, \mu(0) = 0\},$$

and introduce the generalized metric on Υ as:

$$\rho(\mu_1, \mu_2) := \inf\left\{C \in \mathbb{R}^+ \,:\, \|\mu_1(y) - \mu_2(y)\| \leq C\psi(y,0,0,0)\mathcal{E}_\zeta(y), \forall y \in \mathcal{G}\right\},$$

if there exists such constant C, and $\rho(\mu_1, \mu_2) = \infty$, otherwise. The metric space (Υ, ρ) is complete, and by the same reasoning as in the proof of Theorem 2.3, the mapping $\Phi : \Upsilon \to \Upsilon$ defined by

$$\Phi[\mu](y) = \frac{1}{8}\mu(2y),$$

for $y \in \mathcal{G}$ is strictly contractive on $\mathcal{X}$ and has a unique fixed point $\mathfrak{T}$ such that

$$\lim_{n\to\infty} \rho(\Phi^n \mathfrak{u}, \mathfrak{T}) = 0,$$

i.e,

$$\mathfrak{T}(y) = \lim_{n\to\infty} \frac{\mathfrak{u}(2^n y)}{8^n}, \tag{2.19}$$

for all $y \in \mathcal{G}$. By Theorem 2.3 shows that $\mathfrak{T}$ is a cubic mapping. If we substitute v and w by $2^n v$ and $2^n w$ in (2.14), respectively, and put $y = z = 0$ and we divide the both sides of the obtained inequality by 2^{4n}, we get

$$\left\| v \frac{1}{8^n} \mathfrak{u}\left(2^n w\right) - \frac{1}{8^n} \mathfrak{u}\left(2^n v\right) w \right\| = 0.$$

Passing to the limit as $n \to \infty$ and from (2.17), we conclude that $v\mathfrak{T}(w) = \mathfrak{T}(v)w$ for all $v,\, w \in \mathcal{G}$. □

Corollary 2.9. *Let η, λ be non-negative real numbers with $\eta < 3$ and let $\mathfrak{u} : \mathcal{G} \to \mathcal{G}$ be a mapping with $\mathfrak{u}(0) = 0$ such that*

$$\|\mathbb{D}[\mathfrak{u}](y, z) + \mathfrak{u}(v)w - v\mathfrak{u}(w)\| \leq \lambda\big(\|y\|^\eta + \|z\|^\eta + \|v\|^\eta + \|w\|^\eta\big)\mathcal{E}_\zeta(y),$$

for all $y,\, z,\, v,\, w \in \mathcal{G}$. Then, there exists a unique cubic multiplier $\mathfrak{T}$ on $\mathcal{G}$ satisfying

$$\|\mathfrak{u}(y) - \mathfrak{T}(y)\| \leq \frac{2^{\eta-1}\lambda}{8 - 2^\eta}\|y\|^\eta,$$

for all $y \in \mathcal{G}$.

Proof. The proof follows from Theorem 2.8 by taking

$$\psi(y, z, v, w) = \lambda\Big(\|y\|^\eta + \|z\|^\eta + \|v\|^\eta + \|w\|^\eta\Big),$$

for all $y,\, z,\, v,\, w \in \mathcal{G}$. □

Corollary 2.10. *Let η_j for $j = 1, 2, 3, 4$, λ be non-negative real numbers with*

$$\sum_{j=1}^{4} \eta_j < 3,$$

and let $\mathfrak{u}$ maps $\mathcal{G}$ to $\mathcal{G}$ be a mapping with $\mathfrak{u}(0) = 0$ such that

$$\|\mathbb{D}[\mathfrak{u}](y, z) + \mathfrak{u}(v)w - v\mathfrak{u}(w)\| \leq \lambda\big(\|y\|^{\eta_1} + \|z\|^{\eta_2} + \|v\|^{\eta_3} + \|w\|^{\eta_4}\big)\mathcal{E}_\zeta(y),$$

for all $y,\, z,\, v,\, w \in \mathcal{G}$. Then, $\mathfrak{u}$ is a cubic multiplier on $\mathcal{G}$.

Proof. It is enough to let

$$\psi(y, z, v, w) = \lambda\big(\|y\|^{\eta_1} + \|z\|^{\eta_2} + \|v\|^{\eta_3} + \|w\|^{\eta_4}\big),$$

in Theorem 2.8. □

3 Conclusion

Quadratic functional equation was used to characterize inner product spaces [1, 4, 14]. Several other functional equations were also to characterize inner product spaces. A square norm on an inner product space satisfies the important parallelogram equality

$$\|y + z\|^2 + \|y + z\|^2 = 2\left(\|y\|^2 + \|z\|^2\right).$$

This research has made an attempt to analyse the MLHU stability and the MLHUR stability of linear differential equation with constant coefficients. Also we have showed that the Mittag - Leffler function plays an immodest role to prove the stability of differential equation. This new method of stability unifies different classes of differential equations, which may inspire further research in this domain

Declarations

Availability of Data and Materials
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Competing interests
The authors declare that they have no competing interests.

Funding
Not applicable.

Authors' contributions
The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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