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Some results on schur multiplier of pairs of groups

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Abstract : In this paper, we study the concept of the c-nilpotent multiplier of a pair of groups and prove that the c-nilpotent multipliers of perfect pairs of groups are isomorphic. Also, we prove an inequality for the order of the Schur multiplier of a pair of groups.

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1 Introduction

The study of the Schur multipliers of groups dates back to 1904 [14]. In 1998, Ellis [6] extended the theory of the Schur multiplier for a pair of groups. Let (N, G) be a pair of groups, in which N is a normal subgroup of G. The Schur multiplier of the pair (N, G) is a abelian group $\mathcal{M}(N, G)$ whose principal feature is a natural exact sequence

$$\begin{gathered} H_3(G) \to H_3(G/N) \to \mathcal{M}(N, G) \to \mathcal{M}(G) \to \\ \mathcal{M}(G/N) \to N/[N, G] \to (G)^{ab} \to (G/N)^{ab} \to 1 \end{gathered}$$

in which $H_3(G)$ is the third homology group of G with integer coefficients. Ellis [6] proved that if N admits a complement in G, then

$$\mathcal{M}(N, G) \cong \ker\Big(\mu : \mathcal{M}(G) \to \mathcal{M}(G/N)\Big). \tag{1.1}$$

Let $1 \to R \to F \to G \to 1$ be a free presentation of G and S be a normal subgroup of F with $N \cong S/R$. If N admits a complement in G then (1.1) implies that

$$\mathcal{M}(N, G) \cong \frac{R \cap [S, F]}{[R, F]}.$$

We define the c-nilpotent multiplier ($c \geq 1$) of a pair (N, G) as

$$\mathcal{M}^{(c)}(N, G) = \frac{R \cap [S,_c F]}{[R,_c F]}.$$

The group $\mathcal{M}^{(c)}(N, G)$ is abelian and independent of the choice of the free presentation of G. If $N = G$, then $\mathcal{M}^{(c)}(G, G) = \mathcal{M}^{(c)}(G)$ is the c-nilpotent multiplier of G. See [2, 3, 7, 11, 13] for more information.

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2 Main results

Let G and M be two groups with an action of G on M. Then the G-commutator subgroup and G-center subgroup of M are defined, respectively, as follows:

$$[M,G] = \langle [m,g] = m^g m^{-1} \mid m \in M, g \in G\rangle,$$
$$Z(M,G) = \{m \in M \mid m^g = m, \quad \forall g \in G\}.$$

Also, the subgroups $[M,_c G]$ and $Z_c(M,G)$ for all $c \geq 1$, as follows:

$$[M,_c G] = \langle [m, g_1, \ldots, g_c] \mid m \in M, g_1, \ldots, g_c \in G\rangle,$$
$$Z_c(M,G) = \{m \in M \mid [m, g_1, \ldots, g_c] = 1, forall g_1, \ldots, g_c \in G\}.$$

Let (N,G) be a pair of groups. A relative central extension of the pair (N,G) is a homomorphism $\sigma : M \to G$ together with an action of G on M such that

(i) $\sigma(M) = N$

(ii) $\sigma(m^g) = g^{-1}\sigma(m)g$, for all $g \in G$, $m \in M$,

(iii) $m'^{\sigma(m)} = m^{-1}m'm$, for all $m, m' \in M$,

(iv) $\ker\sigma \subseteq Z(M,G)$.

In addition, the relative central extension $\sigma : M \to G$ is siad to be a cover of (N,G) if there exists a subgroup A of M such that

(i) $A \subseteq Z(M,G) \cap [M,G]$,

(ii) $A \cong \mathcal{M}(N,G)$.

(iii) $N \cong M/A$.

Moreover, a pair (N,G) of groups is said to be perfect, if $[N,G] = N$. Also, let $\theta_i : M_i \to G, \quad (i = 1,2)$ be relative central extention of a pair (N,G). Then we say that θ_1 covers θ_2 if there exists a homomorphism $\varphi_1 : M_1 \to M_2$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \ker\theta_1 & \longrightarrow & M_1 & \xrightarrow{\theta_1} & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle 1_L} & & \\
1 & \longrightarrow & \ker\theta_2 & \longrightarrow & M_2 & \xrightarrow{\theta_2} & G & \longrightarrow & 1
\end{array}$$

Where, the homomorphism $Ker\theta_1 \to Ker\theta_2$ is the restriction of φ_1 to $Ker\theta_1$. The relative central extension θ_1 is called universal if it covers uniquely any relative central extension of (N,G). (See [12]). Let X, Y be two groups. Then $X \wedge Y$ is the exterior product of X and Y. (See [5] for more information).

Lemma 2.1. *([9, Theorem 2.4]). Let (N,G) be a pair of groups, such that N be a perfect group, $1 \to R \to F \to G \to 1$ be a free presentation of G and $N \cong S/R$ for a normal subgroup S in F. Then $\varphi : [S,F]/[R,F] \to G$ is a covering pair and universal of (N,G).*

Theorem 2.2. *The homomorphisms*

$$\gamma^*_{c+1}(N,G) \to \gamma^*_2(N,G) \quad and \quad \mathcal{M}^c(N,G) \to \mathcal{M}(N,G)$$

are isomorphisms for $c \geq 1$, where $\gamma^*_{c+1}(N,G) = [S,_c F]/[R,_c F]$.

Proof. One may check that if the relative central extensions $\theta_1 : M_1 \to G$ and $\theta_2 : M_2 \to G$ are universal, then there is an isomorphism $\psi : M_1 \to M_2$ such that $\psi(\ker\theta_1) = \ker\theta_2$. Hence, we prove that $\mu_{c+1} : \gamma^*_{c+1}(N, G) \to G$ is the universal relative central extension of (N, G) for all $c \geq 1$. Using Lemma 2.1 the case $c = 1$ is true. In ductively, assume that the result holds for $c \geq 1$. We can see that $\gamma^*_{c+1}(N, G)$ is perfect, thus

$$\eta_{c+1} : \gamma^*_{c+1}(N, G) \wedge \gamma^*_{c+1}(N, G) \to \gamma^*_{c+1}(N, G)$$

is the universal central extension of $\gamma^*_{c+1}(N, G)$. Put $\delta = \mu_{c+1}\eta_{c+1}$. So,

$$\delta : \gamma^*_{c+1}(N, G) \wedge \gamma^*_{c+1}(N, G) \to G$$

is the universal relative central extension of (N, G). Hence, there exists an isomorphism

$$\varphi : \gamma^*_{c+1}(N, G) \wedge \gamma^*_{c+1}(N, G) \to \gamma^*_{c+1}(N, G),$$

such that $\mu_{c+1}\varphi = \delta$. One can check that the following diagram is commutate:

$$\begin{array}{ccccc} \gamma^*_{c+1}(N, G) \wedge \gamma^*_{c+1}(N, G) & \xrightarrow{f} & \gamma^*_{c+1}(N, G) \wedge N & \xrightarrow{g} & \gamma^*_{c+2}(N, G) \\ & \searrow^{\varphi} & & \swarrow_{\beta} & \\ & & \gamma^*_{c+1}(N, G) & & \end{array}$$

Where β is the canonical homomorphism. On the other hand, N is perfect, so, f and g are isomorphisms. Therefore,

$$\gamma^*_{c+2}(N, G) \cong \gamma^*_{c+1}(N, G).$$

This completes the proof. □

We close this section by a result on the Schur multiplier of a pair of groups. Moghaddam, Salemkar and Chiti [8] proved the following theorems.

Theorem 2.3. *([8, Theorem 3.2(i)]). Let K and N be complements of finite group G such that $K \subseteq N$. Also, let $1 \to R \to F \to G \to 1$ be a free presentation of the group G, S and T are normal subgroups of the free group F such that $T \subseteq S$, $S/R \cong N$ and $T/R \cong K$. Then*

$$\Big|\mathcal{M}\Big(\frac{N}{K}, \frac{G}{K}\Big)\Big|\Big|\frac{[T, F]}{[R, F]}\Big| = |K \cap [N, G]| \cdot |\mathcal{M}(N, G)|.$$

Theorem 2.4. *([8, Corollary 3.4(i)]). Let N be a complement of a finite group G and K be a normal subgroup of G such that $K \subseteq Z(G) \cap N$. Then*

$$|\mathcal{M}(N, G)||[N, G] \cap K| \quad \textit{divides} \quad \Big|\mathcal{M}\Big(\frac{N}{K}, \frac{G}{K}\Big)\Big||\mathcal{M}(K)|\Big|\frac{G}{K} \otimes K\Big|.$$

In [10, Corollary 1.2(iii)], the authors generalized Theorem 2.3 in the case of Lie algebras. Also, in [1, Theorem 2.3(v)], the author, generalized [10, Corollary 1.2(iii)] to the c-nilpotent multiplier of a pair of Lie algebras. Here, we prove Theorem 2.3 to a stronger version.

Let G be a group and N be a normal subgroup of G and K be a normal subgroup of G contained in N. Similar to [4] we have the following diagram:

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathcal{M}(K,G) & & \mathcal{M}(N,G) & \longrightarrow & \mathcal{M}(N/K,G/K) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & K \wedge G & \longrightarrow & N \wedge G & \longrightarrow & N/K \wedge G/K & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle [\,,\,]} & & \downarrow{\scriptstyle [\,,\,]} & & \downarrow{\scriptstyle [\,,\,]} & & \\ 1 & \longrightarrow & ([N,G] \cap K) & \longrightarrow & [N,G] & \longrightarrow & [N/K,G/K] & \longrightarrow & 1 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 1 & & 1 & & \end{array} \tag{2.1}$$

In this diagram, rows and columns are exact. If the left hand side square is commutate then the following sequence is exact:

$$1 \to \mathcal{M}(K,G) \to \mathcal{M}(N,G) \to \mathcal{M}(\frac{N}{K},\frac{G}{K}) \to \frac{[N,G] \cap K}{[K,G]} \to 1 \tag{2.2}$$

Now, we prove Theorem 2.3, to a stronger version.

Theorem 2.5. *Let (N,G) be a pair of finite groups, $1 \to R \to F \to G \to 1$ be a free presentation of G, K be a normal subgroup of G contained in N, $K \cong T/R$ for some normal subgroup T of F such that commutes the diagram* (2.1)*. Then*

$$\left|\mathcal{M}(N/K,G/K)\right|\left|\frac{[T,F]}{[R,F]}\right| \le |K \cap [N,G]||\mathcal{M}(N,G)|.$$

Proof. One may check that there is an epimorphism

$$\varphi : K \wedge G \to \frac{[T,F]}{[R,F]}$$

such that the following diagram is commutative

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \ker\psi & \longrightarrow & K \wedge G & \longrightarrow & [K,G] & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \frac{R\cap[T,F]}{[R,F]} & \longrightarrow & \frac{[T,F]}{[R,F]} & \longrightarrow & \frac{[T,F]}{R\cap[T,F]} & \longrightarrow & 1 \end{array}$$

where $\psi : K \wedge G \to [K,G]$ is an epimorphism give by

$$\psi(k \wedge g) = [k,g]$$

and $\varphi\big|$ is the restriction of φ on $\ker\psi$. Clearly, $\varphi\big|$ is onto. Thus,
$(R \cap [T,F])/[R,F]$ is a homomorphic image of $\ker\psi$. Thus, we obtain

$$|\mathcal{M}(K,G)| \ge \left|\frac{R \cap [T,F]}{[R,F]}\right|.$$

On the other hand, by sequence (2.2) we have

$$\big|\mathcal{M}(N/K, G/K)\big||\mathcal{M}(K,G)||[K,G]| = |\mathcal{M}(N,G)||K \cap [N,G]|.$$

Therefore,

$$|\mathcal{M}(K,G)||[K,G]| \geq \Big|\frac{R \cap [T,F]}{[R,F]}\Big|\Big|\frac{[T,F]}{R \cap [T,F]}\Big| = \Big|\frac{[T,F]}{[R,F]}\Big|,$$

which completes the proof of theorem. □

Remark 2.6. *Similar to Theorem 2.5 and by [6, Proposition 4.2] we can prove Theorem 2.4 to a stronger version.*

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