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A method on generation of finite level(s) in real and complex systems

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Abstract : The Rath's mannequin approximation is generalized in this work to generate finite strength level(s) quantum structures in Hermitian and non-Hermitian quantum systems. On one hand, in Hermitian systems, an extensive variety of bounded single properly,triple nicely, and pentic proper systems is utilized. On the other hand, isospectral Hermitian structures are considered. Further mannequin is extended to PT-symmetric structures involving an isospectral condition.

Keywords: Exponential potentials; Single well; Multiple well; Isospectral; PT-symmetry; limited distinct discrete states
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1 Introduction

Bounded quantum structures are associated with actual strength levels [2]. Similarly, unbroken PT-symmetry operators can also supply actual power levels [1]. In actual operators, bounded structures satisfy the following:

$$\Psi(\pm x \to \infty) \to 0 \tag{1.1}$$

However, in complex PT-symmetric systems, we obtain [1]

$$|\Psi|^2(\pm x \to \infty) \to 0 \tag{1.2}$$

Here, P stands for parity which has the property of space reflection as follows:

$$P^{-1}xP = -x, \tag{1.3}$$

$$P^{-1}pP = -p, \tag{1.4}$$

$$P^{-1}|x|P = |x|. \tag{1.5}$$

Similarly, T represents the time reversal operator which has the following property:

$$T^{-1}iT = -i, \tag{1.6}$$

$$T^{-1}xT = x, \tag{1.7}$$

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$$T^{-1}pT = -p. \tag{1.8}$$

Further, PT-symmetry implies

$$[H, PT] = 0. \tag{1.9}$$

In fact, these structures have been widely studied in other research works with limitless quantum levels [2, 1]. However, quantum systems with countable or limited energy levels have been rarely discussed or even not discussed at all. According to our investigation, a few number of studies about this particular topic can be seen in [3, 7, 8]. As a result, the mannequin proposed by Rath's potential function [3, 7, 8] is generalized in this work. Therefore, all our results are new and worthy.

2 Generation of finite discrete levels

All quantum operators satisfy the following condition:

$$V(x) \ll 1. \tag{2.1}$$

This can be mathematically expressed as [3, 8]:

$$\frac{V(x)}{V_0} = [1 - e^{-\lambda F(x)}] \ll 1. \tag{2.2}$$

This thought provokes the main idea that was suggested by Rath [3, 7, 8]. Instead, this can be named as a Rath's potential function. From the above, $F(x)$ is an even feature such that x^{2N}, and λ is a simple constant, that can additionally be accounted for dimensionless or practicable cases. The energy of the potential can be additionally exchanged. Let us reflect on the following shape of potentials as:

$$V_N(x) = V_0[1 - e^{-\lambda x^{2N}}]_{N=1,2,3..} \tag{2.3}$$

in the Hamiltonian, we have:

$$H_1 = p^2 + V_0[1 - e^{-\lambda x^{2N}}]_{N=1,2,3..} \tag{2.4}$$

In Table 1, the energy levels of the systems are tabulated. By selecting $V_0 = 1$, all the structures can solely have one level, and the value of the energy level is always $\frac{E}{V_0} \ll 1$. A regular single proper mannequin is represented in Fig. 1.

2.1 Double well model

A standard double nicely dolable is reflected in Fig. 2.

2.2 Triple well model

Here, we barely exchange the potential as follows:

$$H_2 = p^2 + 5[1 - e^{-2(x-Sgn(x))^2}]. \tag{2.5}$$

The nature of this triple well is represented in Fig. 3. Further, we extend this mannequin to the following:

$$H_3 = p^2 + 5[1 - e^{-2(x-Sgn(x))^4}], \tag{2.6}$$

$$H_4 = p^2 + 5[1 - e^{-2(x-Sgn(x))^4}]. \tag{2.7}$$

In Table 2, we construct the number of discrete energy levels.

Table 1: Sinle well model of Rath's potential

N	λ	No of level	Energy value
1	1	1	0.646 0
	2		0.760 9
	3		0.818 0
	4		0.852 8
	5		0.876 3
	6		0.893 2
	7		0.906 1
	8		0.916 2
	9		0.924 3
	10		0.931 0
2	1		0.606 0
	2		0.674 0
	3		0.712 0
	4		0.737 4
	5		0.756 3
	6		0.771 1
	7		0.783 2
	8		0.793 3
	9		0.802 0
	10		0.809 6
3	1		0.587 9
	5		0.696 0
	10		0.732 8

Table 2: Triple-well model of Rath's potential

N	λ	Levels $E \ll V$	No of wells
1	0	1	1
1	1	4	3
2	0	2	1
2	1	4	3
3	0	2	1
3	1	4	3

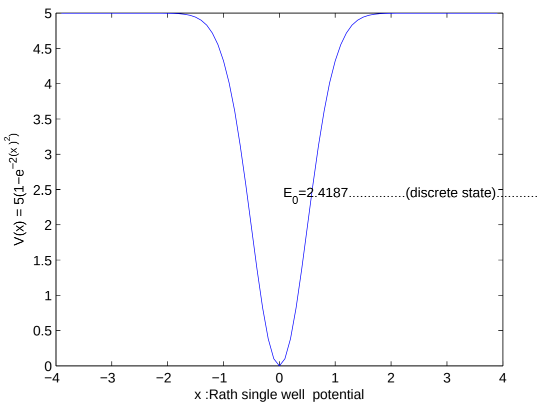

Figure 1: Rath's Single Well Potential

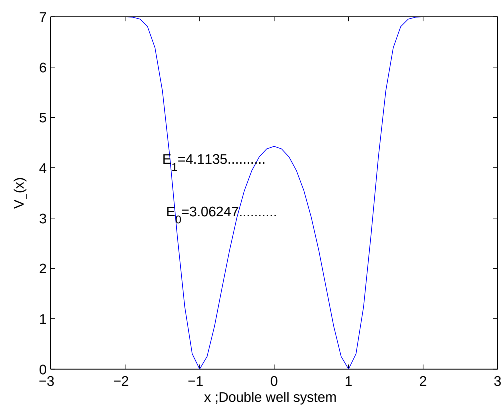

Figure 2: Rath's Double Well Potential

2.3 Pentic (Five) well model

Before the discussion of this part, let us say that a mannequin triple well model was previously investigated via Keung, Kovac and Sukhatme [4] which has inspired us to endorse a practicable mannequin as mentioned in Fig. 4.

2.4 Isospectral Hermitian systems

Let us here reflect on superpotential $W(x)$ as follows:

$$W(x) = 5[1 - e^{-2x^2}], \tag{2.8}$$

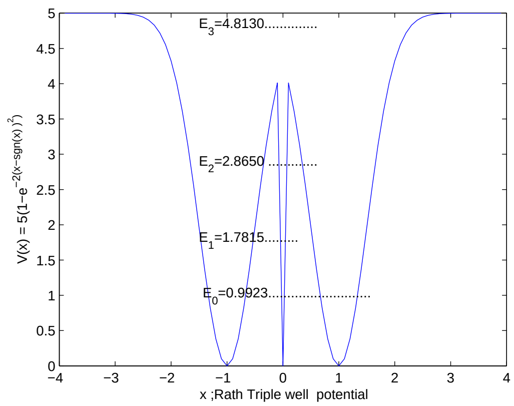

Figure 3: Rath's Triple Well Potential

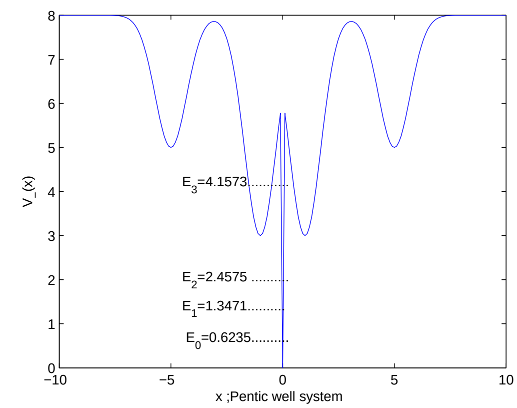

Figure 4: Pentic(five) Well System using Rath's Potential

and by using the corresponding SUSY Hamiltonians, we have:

$$H^{(\pm)} = p^2 + W^2 \pm \frac{dW}{dx}. \tag{2.9}$$

The outcomes are plotted in Fig. 5.
Similarly, we consider SUSY quartic model as:

$$W(x) = 5[1 - e^{-2x^4}], \tag{2.10}$$

The resulting energy levels are plotted in Fig. 6.
Now, we consider SUSY sextic model as:
Similarly, we consider SUSY sextic model as:

$$W(x) = 5[1 - e^{-2x^6}], \tag{2.11}$$

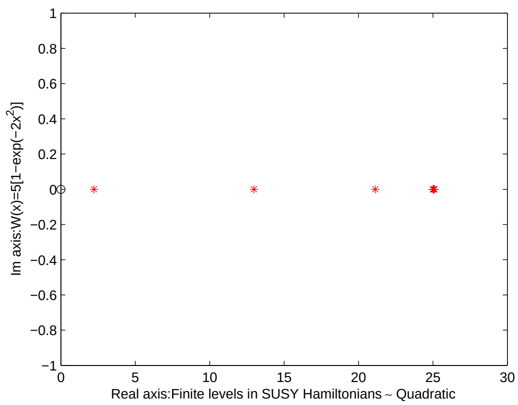

Figure 5: Iso-spectral Hermitian Hamiltonians

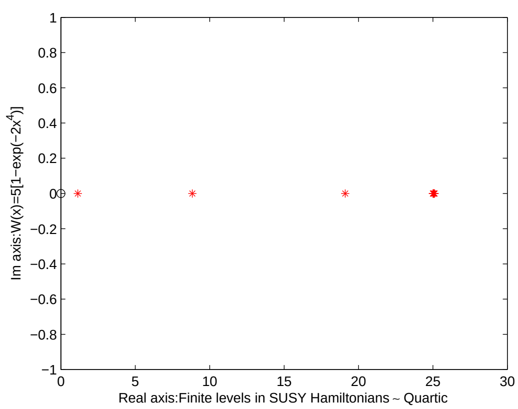

Figure 6: Iso-spectral Hermitian Hamiltonians Quartic Model

The corresponding energy levels are plotted in Fig. 7.

In fact all the three cases have been plotted on the same scale in order to show that finite discrete levels which are are well-marked.

2.5 Iso-spectral PT-symmetric Hamiltonians with finite energy levels

By simply using the above relations after making a appropriate trade, we get the following:

$$H^{(\pm)} = p^2 + W^2 \pm i\frac{dW}{dx}. \tag{2.12}$$

The corresponding energy levels are reflected in Fig. 8.

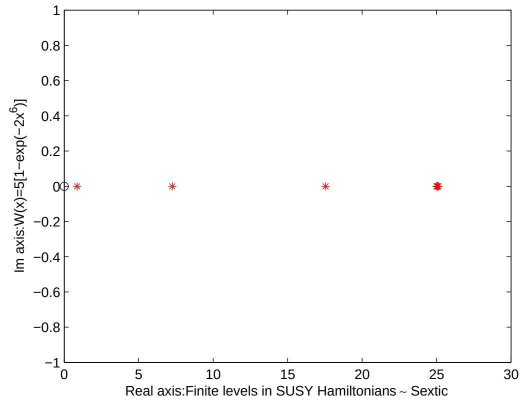

Figure 7: Iso-spectral Hermitian Hamiltonians Sextic Model

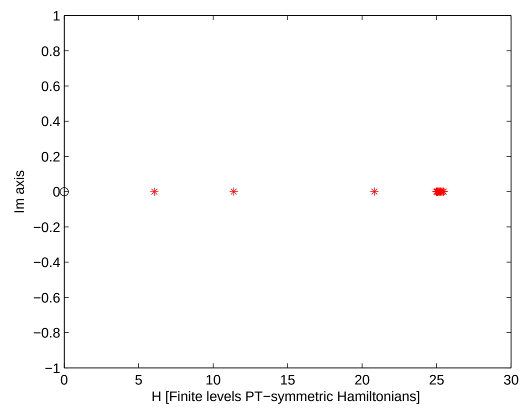

Figure 8: Spectra of Iso-Spectral PT-Symmetric Hamiltonians

2.6 Cubic, pentic, and sentic PT-symmetry operators

Let us now consider some PT-symmetric models with cubic, pentic, and sentic operators as follows:

$$H_1^{PT} = p^2 + 5[1 - e^{-(x^2 - ix)^2}]. \tag{2.13}$$

Energy level of this PT-operator is reflected in Fig. 9.

$$H_2^{PT} = p^2 + 5[1 - e^{-(x^4 - ix)^2}]. \tag{2.14}$$

Energy level of this PT-operator is reflected in Fig. 10.

$$H_1^{PT} = p^2 + 5[1 - e^{-(x^4 - ix^3)^2}]. \tag{2.15}$$

Energy level of this PT-operator is reflected in Fig. 11.

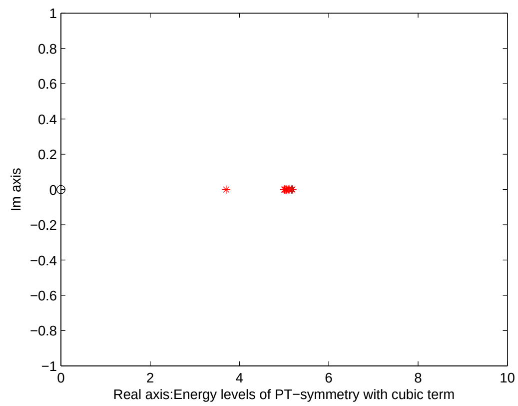

Figure 9: Spectra of Cubic PT-Symmetric Hamiltonians

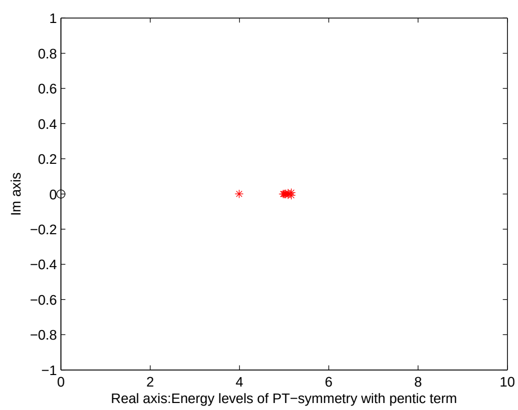

Figure 10: Spectra of Pentic PT-Symmetric Hamiltonians

3 Modified form of Rath's model

Now, the other choice of Eq. 1, is expressed as:

$$\frac{V(x)}{V_0} = 1 - \frac{1}{1+F(x)} = \frac{F(x)}{1+F(x)} \ll 1. \tag{3.1}$$

Let us now consider some cases as follows:

3.1 Quadratic potential

$$H = p^2 + \frac{x^2}{(1+x^2)}. \tag{3.2}$$

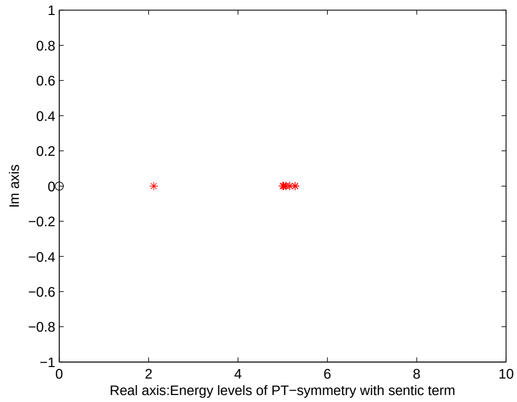

Figure 11: Spectra of Sentic PT-Symmetric Hamiltonians

Energy levels of this model Hamiltonian is presented in Fig. 12.

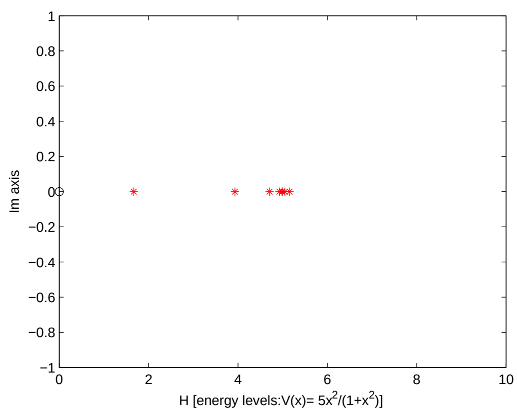

Figure 12: Spectra of Quadratic Hamiltonian

3.2 Quartic potential

$$H = p^2 + \frac{x^4}{(1 + x^4)}. \tag{3.3}$$

Energy levels of this model Hamiltonian is presented in Fig. 13.

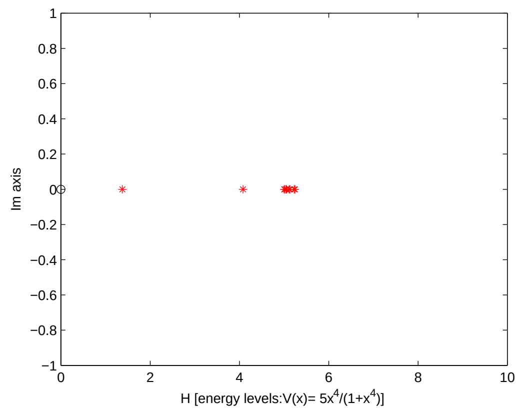

Figure 13: Spectra of Quartic Hamiltonian

3.3 Sextic potential

$$H = p^2 + \frac{x^6}{(1+x^6)}. \tag{3.4}$$

Energy levels of this model Hamiltonian is presented in Fig. 14.

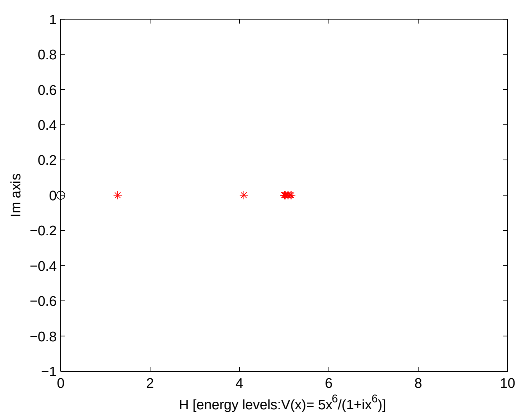

Figure 14: Spectra of Sextic Hamiltonian

3.4 Complex cubic potential

$$H = p^2 + \frac{ix^3}{(1+ix^3)}. \tag{3.5}$$

Energy levels of this model Hamiltonian is presented in Fig. 15.

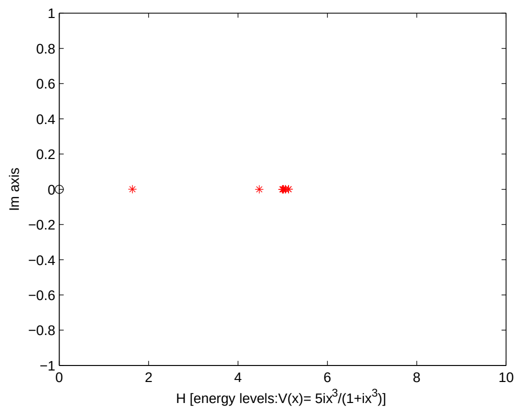

Figure 15: Spectra of Complex Cubic Hamiltonians

3.5 Complex pentic potential

$$H = p^2 + \frac{ix^5}{(1 + ix^5)}. \tag{3.6}$$

Energy levels of this model Hamiltonian is presented in Fig. 16.

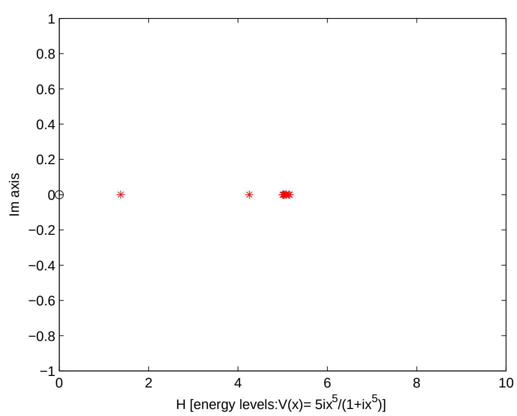

Figure 16: Spectra of Complex Pentic Hamiltonians

3.6 Complex sentic potential

$$H = p^2 + \frac{ix^7}{(1 + ix^7)}. \tag{3.7}$$

Energy levels of this model Hamiltonian is presented in Fig. 17.

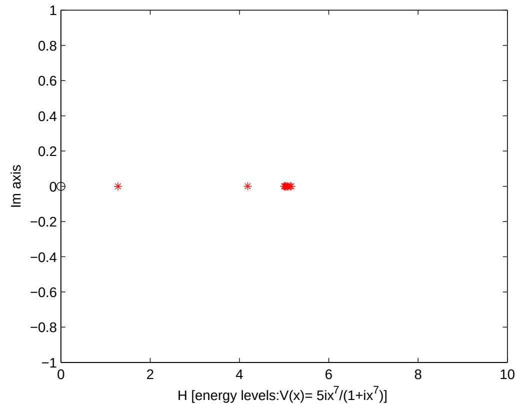

Figure 17: Spectra of Complex Sentic Hamiltonians

4 Method of calculation

In the previous sections, the matrix diagonalization method [5, 9] is used to mirror convergent energy levels, by solving the following eigenvalue relation:

$$H|\Psi >= E|\Psi >, \tag{4.1}$$

with

$$|\Psi >= \sum_m A_m|m >, \tag{4.2}$$

where $|m >$ satisfies the relation

$$[H_0 = p^2 + x^2]|m >= (2m + 1)|m > . \tag{4.3}$$

5 Energy conservation

In the above method of calculation, one can easily verify the energy conservation principle i.e

$$< T.E >_n =< K.E >_n + < P.E >_n . \tag{5.1}$$

Explicitly, we can write it as:

Total Energy = Kinetic Energy + Potential Energy

Here, the subscript n stands for state under consideration. This is different from the classical concept, where individual states are not considered. For example, we consider the following:

$$H = p^2 + 5\frac{x^2}{(1 + x^2)}, \tag{5.2}$$

and we reflect the individual contribution in Table 3.

Table 3: Energy conservation $H = p^2 + \frac{5x^2}{(1+x^2)}$

level	$< p^2 >$	$\frac{5x^2}{(1+x^2)}$	Total	Accurate
0	0.618 9	1.050 2	1.669 1	1.669 1
1	0.963 6	2.968 5	3.932 1	3.932 1
2	0.500 7	4.207 1	4.707 8	4.707 7

6 Conclusion

In conclusion, a general method on generating finite (distinct) discrete levels has been investigated. The method has been applied to both real and complex systems. Using a suitable value of V_0, one can generate only two level operators in both real and complex space and calculate tunneling exchange [4, 6]. In all of the mentioned cases, we notice distinct energy levels are confined to condition $E \ll V_0$. However, energy levels above V_0 even though are discrete but continuous in nature. In this approach, one can present kinetic energy/potential energy contribution of each state.

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