MATHEMATICS AND COMPUTATIONAL SCIENCES

Vol 3(1), 2022, PP: 37-41
DOI:10.30511/mcs.2021.539538.1040
ISSN:2717-2708
http://mcs.qut.ac.ir/

Qom University of Technology

On the Pythagorean triangles with an irrational hypotenuse

T. M. Aliyev [1], S. M. Pur Riza [2]

Abstract: The paper investigates a number of incomplete exact roots of a series of natural numbers, in relation to the Pythagorean theorem that in a right-angled triangle the square of the hypotenuse is equal to the sum of the squares of the legs. It uses the fact that the equation $x^2 + y^2 = z^n, n = 2,3,4, ..$ always has an solution (x,y,z) in integer numbers $x, y, z \in Z = \{0, \pm 1, \pm 2, \dots\}$.

Keywords: incomplete square, Pythagorean theorem, solution in integer numbers.
2020 Mathematics Subject Classification: 11-XX
Receive: 23 September 2021, **Accepted:** 03 December 2021

1 Introduction

Even the ancient Greek scientist Pythagoras, when studying a number of natural numbers N={1,2,3,...}, discovered that there are an infinite number (x,y,z) of triples from a series of natural numbers N={1,2,3,...} for which there is $x^2 + y^2 = z^2$. And in the language of geometry, this means that the square of the hypotenuse of any right-angled triangle is equal to the sum of the squares of the sides of a triangle. This is the formulation of the Pythagorean theorem. And all the time it was used in solving agricultural and land problems. Naturally, Pythagoras knew that the diagonal of a square with a unit side is not a natural number, but is equal to an incomplete exact root $\sqrt{2}$=1.4142... Thus, in the case of an incomplete root $\sqrt{2}$, $[(\sqrt{2})^2] = 1^2 + 1^2$ and to a number $\sqrt{2}$ there corresponds a fixed point on the number axis.

From a series of natural numbers N, select a series of incomplete exact roots

$$\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}, \sqrt{11}, \sqrt{12}, \sqrt{13}, \dots \quad (1.1)$$

Based on $[(\sqrt{2})^2] = 1^2 + 1^2$ on the numerical axis, in principle, it is possible to construct all points corresponding to the numbers of the series (1.1), we have

$[(\sqrt{3})^2] = [(\sqrt{2})^2] + 1^2, [(\sqrt{5})^2] = 2^2 + 1^2, [(\sqrt{6})^2] = [(\sqrt{5})^2] + 1^2$ and so etc.

[1] ANAS, Control Systems Institute, Azerbaijan, Baku, B.Vagabzadeh str., 9, Email: tofig_eliev44@rambler.ru
[2] ANAS, Control Systems Institute, Azerbaijan, Baku, B.Vagabzadeh str., 9, Email: pqs@rambler.ru

These are the so-called "visible" numbers located on the number axis. Despite the fact that for an infinite number of incomplete roots $\sqrt[m]{N}, N \in N, N \neq n^{mk}, k = 1,2,3,.., \ m = 3,4,5,.. \ , n = 2,3,4,..,$ the Pythagorean theorem does not hold, all of them and the points corresponding to them are on the number axis. Examples of rows containing an infinite number of "invisible numbers" are [1]:

$\sqrt[3]{2}, \sqrt[3]{3}, \sqrt[3]{4}, \sqrt[3]{5}, \sqrt[3]{6}, \sqrt[3]{7}, \sqrt[3]{9},...$

$\sqrt[4]{2}, \sqrt[4]{3}, \sqrt[4]{4}, \sqrt[4]{5}, \sqrt[4]{6}, \sqrt[4]{8},...$

$\sqrt[5]{2}, \sqrt[5]{3}, \sqrt[5]{4}, \sqrt[5]{5}, \sqrt[5]{6}, \sqrt[5]{7}, \sqrt[5]{8}, ...$

....

2 Algorithm compilation

2.1. Definitions and Remarks. In the series (1.1), numbers $\sqrt{3}, \sqrt{6}, \sqrt{7}, \sqrt{11}, \sqrt{12}, \sqrt{14}$, ... cannot be represented as the sum of two squares of natural numbers. And the rest of the numbers in the series (2.1)

$$\sqrt{2}, \sqrt{5}, \sqrt{8}, \sqrt{10}, \sqrt{13}, \sqrt{17}, \sqrt{18}, \sqrt{20}, ... \qquad (2.1)$$

are such that their squares are represented as a sum of natural numbers.

For example

$(\sqrt{5})^2 = 2^2 + 1^2, (\sqrt{10})^2 = 3^2 + 1^2, (\sqrt{13})^2 = 3^2 + 2^2, (\sqrt{18})^2 = 3^2 + 3^2$ and etc.

Definition 2.1. By a Pythagorean triple with an "irrational" hypotenuse, we mean a triplet $(\sqrt{z}, |x|, |y|)$with values $z \in N = (\sqrt{z})^2 = |x|^2 + |y|^2$ or $z = |x|^2 + |y|^2$

In order to discover new classes of "invisible" numbers on the numerical axis, in addition to (2.1), let us stop under the smallest incomplete square root $\sqrt{2}$. It is known that this equation $x^2 = 2$ does not have a solution in rational numbers, that is $x \neq \frac{m}{n}, m, n \in Z = \{0, \pm 1, \pm 2, \pm 3, ...\}, n \neq 0, x \in R_1 = \sqrt[k]{\frac{m}{n}}$. In other $x = \sqrt{2}, k = 2, m = 2, n = 1$. And if we consider the equation $x^x = 2$.., then its solution, if it exists, does not belong to $R_1, x \notin R_1$. [2]. We have

$N \subset Z \subset Q \subset R_1 \subset R_2 \subset ...$

where Q - is the set of rational numbers, and $R_2 = \sqrt[s]{\sqrt[k]{\frac{m}{n}}}$ the extraction of the root of the so-called side. Extending this process to infinity, that is

$N \subset Z \subset Q \subset R_1 \subset R_2 \subset \cdots \subset R_k \subset R_{k+1} \subset ...$

We get the set of all real numbers, like $R_\infty = \lim_{k \to \infty} R_k$... and the classes R_k for $k = 2,3,4,5,$ ......, that is R_2, R_3, R_4... are the classes of "invisible" numbers on the numerical axis.

Note that the sum of the squares of any two integers $x, y \in Z$, in other $x^2 + y^2$ can always be represented as $\sqrt{z}$. So in $x = -2, y = 3, (-2)^2 + 3^2 = 13, (\sqrt{13})^2 = (-2)^2 + 3^2$,

in $x = -5, y = -2, (-5)^2 + (-2)^2 = 29, (\sqrt{29})^2 = (-5)^2 + (-2)^2$ and etc.

Therefore, if $x, y \in Z$, then always follows $(\sqrt{z})^2 = x^2 + y^2$ from $x^2 + y^2 = z$. And the opposite is not true. Not every incomplete root is represented as the sum of two squares. For example, these are the roots

$$\sqrt{3}, \sqrt{6}, \sqrt{7}, \sqrt{11}, \sqrt{12}, \sqrt{14}, \ldots$$

Let us give an algorithm for finding the entire class of Pythagorean triples $(\sqrt{z}, |x|, |y|)$ with an "irrational" hypotenuse.

Let's start with examples.

Consider the triplet (t^2, x, y) , $x, y \in Z$, moreover $t^2 = x^2 + y^2$. Along with this, consider $Z = x^2 + y^2, z \in N$. Then $Z = t^2$.

If $a, b \in Z$, then we have $(a + bi)^2 = (a^2 - b^2) + 2ab * i$

For x, y *and* t we take $(a^2 - b^2), 2ab$ *and* $a^2 + b^2$ accordingly, that is $x = a^2 - b^2, y = 2ab,\ t = a^2 + b^2$.

1º. $a = -2, b = 3, t = (-2)^2 + 3^2 = 13$

$x = a^2 - b^2 = (-2)^2 - 3^2 = 4 - 9 = -5$

$y = 2ab = 2 * (-2) * 3 = -12$

$z = t^2 = 13^2 = 169$

The required triple $(\sqrt{z}, x, y) = (\sqrt{169}, -5, -12) \equiv (\sqrt{169}, |-5|, |-12|)$.

2º. $a = 3, b = -2, t = 3^2 + (-2)^2 = 13$

$x = a^2 - b^2 = 3^2 - (-2)^2 = 9 - 4 = 5$

$y = 2ab = 2 * 3 * (-2) = -12$

$$z = t^2 = 13^2 = 169$$

and the required triple

$(\sqrt{z}, x, y) \equiv (\sqrt{169}, 5, -12) \equiv (\sqrt{169}, 5, |-12|)$.

3º. $a = -4, b = -5$

$t = (-4)^2 + (-5)^2 = 16 + 25 = 41$

$x = a^2 - b^2 = (-4)^2 - (-5)^2 = 16 - 25 = -9\ ;\ x = -9$

$y = 2ab = 2 * (-4) * (-5) = 40$

$z = t^2 = 1681$

The required triple

$(\sqrt{z}, x, y) \equiv (\sqrt{1681}, -9, 40) \equiv (\sqrt{1681}, |-9|, 40)$

Let's go to the general case.

If $x^2 + y^2 = t^n$, $x, y \in Z$, $t \in N$, then this equation for any values of $n = 2,3,4, \ldots$ has an infinite number of solutions (x, y, t). To be convinced of this, for any $a, b \in Z = \{0, \pm 1, \pm 2, \ldots\}$… consider decomposition [3]

$$(a + bi)^n = \sum_{k=0}^{n} C_n^k \, a^{n-k} (bi)^k, \qquad C_n^k = \frac{n!}{k!(n-k)!} \tag{2.2}$$

Given $a, b \in Z$ and $n = 2,3,4, ..$ for t we take $t = a^2 + b^2$, for x we take the real part of decomposition (2.2), and for y - the coefficient at i.

So, each triple $(x, y, t^n), n = 2,3,4, \ldots$ will match the triple $(\sqrt{z}, |x|, |y|)$, moreover $Z = x^2 + y^2, \quad Z = t^n$,

but x, y, t for any $a, b \in Z$ are found from decomposition (2.2).

Consider the case $n = 4$.

We have the triple (x, y, t^n) and $(a + bi)^4 = (a^4 - 6a^2b^2) + (4a^3b - 4ab^3) * i$ for any $a, b \in Z$.

We have $x = a^4 - 6a^2b^2 + b^4$, $y = 4a^3b - 4ab^3$, $t = a^2 + b^2$.

Example.

$a = -2, b = 3$

$t = x^2 + b^2 = 2^2 + 3^2 = 13, x = a^4 - 6a^2b^2 + b^4 = 2^4 - 6 * 2^2 * 3^2 + 3^4 = -119.$

$y = 4a^3b - 4ab^3 = 4 * 2^3 * 3 - 4 * 2 * 3^3 = -120$

$x^2 + y^2 = t^4, 13^4 = (-119)^2 + (-120)^2$

$Z = t^n = 13^4 = 28561, x^2 = (-119)^2 = 14161,$

$y^2 = (-120)^2 = 14400, z = x^2 + y^2$or $28561 = 14161 + 14400$

As a result, we will get a triplet

$(\sqrt{z}, x, y) \equiv (\sqrt{28561}, -119, -120) \equiv (\sqrt{28561}, |-119|, |-120|)$

Thus, each triple $(x, y, t^n), n = 2,3,4, \ldots..$ corresponds to a triple $(\sqrt{z}, |x|, |y|)$, but $z = x^2 + y^2$ and $z = t^n$, where the values x, y, t for any $a, b \in Z$ are found from decomposition (2.2). As a result, we have a class of all Pythagorean triangles $(\sqrt{z}, |x|, |y|)$ with an "irrational" hypotenuse. Note that from row (1.1) incomplete exact squares [4]

$$\sqrt{3}, \sqrt{6}, \sqrt{7}, \sqrt{11}, \sqrt{12}, \sqrt{13}, \sqrt{14}, \ldots \tag{2.3}$$

is outside of our study, since for them a necessary condition $(\sqrt{z})^2 = x^2 + y^2$ for Pythagorean triangles $(\sqrt{z}, |x|, |y|)$ with an "irrational" hypotenuse is not fulfilled.

3 Conclusion

In a row of square roots

$$\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \ldots$$

such an interesting series stands out

$$\sqrt{3}, \sqrt{6}, \sqrt{7}, \sqrt{11}, \sqrt{12}, \sqrt{13}, \sqrt{14}, \ldots,$$

the members of which are not the hypotenuse of any Pythagorean triangle. Proceeding from this, contrary to Fermat's Last Theorem, it is proved that the equation

$$x^2 + y^2 = z^n, n = 2,3,4, \ldots$$

Always has a solution (x, y, z) in integers $(x, y, z) \in Z = \{0, \pm 1, \pm 2, \ldots\}$.
Based on this fact, it is established that the series $\sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}$,... contains numbers, each of which is an “irrational” hypotenuse of some Pythagorean triangle. We introduced some fundamental linear quantum systems.

Reference

[1] T . M . Aliyev, N.A. Aliyev, K.M. Jabiyev, Mystery of finite and infinite quantities, ASPU Publishing House, Baku-2020.

[2] N.A. Aliyev, From the history of the creation of numbers, Chashyoglu Publishing House, Baku, 2020.

[3] I.L. Kantor, A.S. Solodovnikov, Hypercomplex Numbers, Russia, Moscow: Nauka, 1973.

[4] Y.V. Nesterenko, Number theory (in rus), Academy, Russia, Moscow, 272, 2008.