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On dimension of Lie Algebras and nilpotent Lie algebras

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Abstract: Schur proved that if the center of a group G has finite index, then the derived subgroup G' is also finite. Moneyhun proved that if L is a Lie algebra such that $\dim(L/Z(L)) = n$, then $\dim(L^2) \leq 1/2n(n-1)$. In this paper, we extend the converse of Moneyhun's Theorem. Also, we prove a well-known result of nilpotent Lie algebras by using a different technique.

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1 Introduction

Let G be a group. In 1904, I. Schur proved that if the order of $G/Z(G)$ is finite, then the derived subgroup G' is finite (see [16]). Wiegold in [17] proved that if $|G/Z(G)| = n$, then $|G'| \leq n^{\frac{1}{2}\log_2^n}$. P. Hall [15] showed that if G' is finite, then $G/Z_2(G)$ is finite. Also, I. D. Macdonald [9] gave an explicit bound for $|G/Z_2(G)|$. In 2005, Podoski and Szegedy improved the Macdonald's bound. They proved that

$$|G/Z_2(G)| \leq |G'|^{c\log_2|G'|}.$$

B. H. Neumann [12] proved that if G' is finite and G is finitely generated, then $G/Z(G)$ is finite. This result is extended by P. Niroomand [13]. He proved that $G/Z(G)$ is finite, if G' is finite and $G/Z(G)$ is finitely generated.

K. Moneyhun [10] proved that if L is a Lie algebra such that $\dim L/Z(L) = n$, then $\dim L^2 \leq 1/2n(n-1)$. The author and Saeedi [4] proved some results concerning the converse of Moneyhun's Theorem. In this paper, we generalize a result of [4] by using an interesting technique.

Nilpotent Lie algebras are important in the classification theory of Lie algebras. The first research about nilpotent Lie algebras is due to K. Umlauf in the 19th century (see [6] for more information). In this paper, we prove a result of [5] using a different technique.

Let $Z_n(L)$ and L^{n+1} denote the n-th terms of the upper and lower central series of a Lie algebra L, respectively. define inductively by $L^1 = L$ and $L^{n+1} = [L^n, L]$ for $n \geq 1$, where $[,]$ denotes the Lie bracket and $Z_1(L) = Z(L)$ and $Z_{n+1}/Z_n(L) = Z(L/Z_n(L))$ for $n \geq 1$. A Lie algebra L is said to be abelian, if $[x, y] = 0$, for all $x, y \in L$ and the n-dimensional abelian Lie algebra is denoted by $A(n)$. Also $\Phi(L)$ denotes the Frattini subalgebra of L. The Frattini subalgebra $\Phi(L)$ of a Lie algebra L is the intersection of all maximal subalgebras of L. Moreover, we recall that, a Lie algebra L is nilpotent if $L^s = 0$, for some non-negative integer s.

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2 Main Results

In this section, first we give an upper bound for the dimension of $L/Z_n(L)$, when L^{n+1} is finite dimensional and $L/Z(L)$ is finitely generated. We prove that

$$\dim L/Z_n(L) \leq d^n \dim(L^{n+1})$$

where, $d = d(L/Z(L))$ is the minimal number of generators of $L/Z(L)$. In [3], Theorem 4.4, we gave an upper bound for the dimension of $L/Z_n(L)$, when L^{n+1} is finite dimensional and $L/Z_n(L)$ is finitely generated. Note that if $L/Z(L)$ is finitely generated, then $L/Z_n(L)$ is finitely generated. So, Theorem 4.4 of [3] is a stronger version than Theorem 2.2 of the present paper. In [3], Theorem 4.4, we used the idea of n-isoclinism, which gives us a different method from the technique applied in Theorem 2.2. The proof of Theorem 2.2 is similar to the work of A. Faramarzi Sales (2011). In [1], Theorem 2.3, the author generalized [3], Theorem 4.4 to a pair of Lie algebras (see [2, 8, 13]).

The following lemma is useful in the proof of Theorem 2.2.

Lemma 2.1. *([4], Corollary 2.5). Let L be a Lie algebra such that L^2 is finite dimensional. Then*

$$\dim(L/Z(L)) \leq d.\dim(L^2),$$

where $d = d(L/Z(L))$ is the minimal number of generators of $L/Z(L)$.

Theorem 2.2. *Let L be a Lie algebra such that L^{n+1} is finite dimensional and $L/Z(L)$ be finitely generated, then*

$$\dim(L/Z_n(L)) \leq d^n.\dim(L^{n+1})$$

where $d = d(L/Z(L))$ is the minimal number of generators of $L/Z(L)$.

Proof. We proceed inductively. If $n = 1$, then the result follows by Lemma 2.1. Suppose that $n > 1$ and $L/Z(L)$ denotes the Lie algebra generated by $l_1 + Z(L), \ldots, l_d + Z(L)$. Define

$$\psi : L^n/C_{L^n}(l_i) \to \bigoplus_{i=1}^{d}[L^n, L]$$
$$\psi(l + C_{L^n}(l_i)) = ([l, l_1], \ldots, [l, l_d])$$

where $C_{L^n}(l_i)$ is the centralizer of l_i in L^n. One may easily check that ψ is a one-to-one linear transformation. On the other hand, we have $Z(L) = \bigcap_{i=1}^{d} C_L(l_i)$. Hence

$$\begin{aligned}\dim(L^n/(L^n \cap Z(L))) &\leq \dim(L^n/(Z(L) \cap C_{L^n}(l_i)) \\ &= \dim(L^n/C_{L^n}(l_i)) \\ &\leq d.\dim(L^{n+1}). \end{aligned} \tag{2.1}$$

So, $L^n/(L^n \cap Z(L)) = (L/Z(L))^n$ is finite dimensional. Since the Lie algebra $(L/Z(L))/Z(L/Z(L)) = L/Z_2(L)$ is finitely generated, hence by (2.1), we obtain

$$\begin{aligned}\dim(L/Z_n(L)) &= \dim((L/Z(L))/Z_{n-1}(L/Z(L))) \\ &\leq d^{n-1}.\dim((L/Z(L))^n) \\ &\leq d^{n-1}.(d.\dim(L^{n+1})) \\ &= d^n.\dim(L^{n+1}).\end{aligned}$$

□

Finally, we prove a well-known result in nilpotent Lie algebras using a different technique. The following theorem was proved by Chao in [5].

Theorem 2.3. *Let L be a finite dimensional Lie algebra and N be a nilpotent ideal of L. Then L is nilpotent if and only if L/N^2 is nilpotent.*

The result is a necessary and sufficient condition for the nilpotency of a finite dimensional Lie algebra. Here, the author gives another proof of the condition, which is simple. The proof of Theorem 2.5 is similar to the work of S. Montague and G. Thomas (1965). The next lemma is useful in the proof of Corollary 2.6.

Lemma 2.4. *([11], Corollary 2). The Frattini subalgebra of a finite dimensional nilpotent Lie algebra equals the derived subalgebra of L.*

Theorem 2.5. *If L is a Lie algebra of finite dimensional with a subalgebra H such that $\Phi(H)$ is ideal in L and $L/\Phi(H)$ is nilpotent, then L is nilpotent.*

Proof. Let L be a Lie algebra of finite dimensional, also H be a subalgebra of L and N be a subalgebra of H such that N is ideal in L and $N \leq \Phi(H)$, we prove that $N \leq \Phi(L)$. For this, let U be a maximal subalgebra of L such that $N \nleq U$. We have

$$\begin{aligned} H &= L \cap H \\ &= (N + U) \cap H \\ &= N + (U \cap H) \\ &= U \cap H. \end{aligned}$$

Thus, $H \leq U$. So, we obtain a contradiction. Now put $N = \Phi(H)$, so, $\Phi(H) \leq \Phi(L)$. Therefore $L/\Phi(L)$ is nilpotent and so L is nilpotent.

□

Corollary 2.6. *If L is a Lie algebra of finite dimension with an ideal H such that H is nilpotent and L/H^2 is nilpotent, then L is nilpotent.*

Proof. Since H is nilpotent, then $\Phi(H) = H^2$ by Lemma 2.4. Therefore $L/\Phi(H)$ is nilpotent by Theorem 2.5. □

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