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Frame and G-frame in Hilbert spaces

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Abstract: In this paper, we investigate frames and g-frames and show that constructs the direct sum of frames for a finite number of frames. also, We show under what condition it becomes g-frames to T^*-g-frames. Finally, we generalize Sun 's theorem to Parseval frames.

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1 Introduction

Frames for Hilbert spaces were first introduced by Duffin and Shaeffer [10] in 1952 to study some problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies et al. and popularized from then in [1, 2, 9, 10]. Frames are generalizations of bases in Hilbert spaces, a frame as well as an orthonormal basis allows each element in the underlying Hilbert space to be written as an unconditionally convergent series in linear combinations of the frame elements, however, in contrast to the situation for a basis, the coefficients might not be unique. Frames are very useful in characterization of function spaces and other fields of applications such as filter bank theory, sigma-delta quantization, signal and image processing and wireless communication, see[3, 14, 7, 11, 13].
Nowadays, frame theory is a standard notion in applied mathematics, computer science and engineering, but technical advances and massive amounts of data which cannot be handled with a single processing system have increased the demand for the extensions of frame e.g, fusion frames, g-frames, weaving frames, etc. [2, 6, 13]. Fusion frames are generalized frames and were introduced in [13]. Fusion frames have important applications e.g. in distributed processing, sensor networks and packet encoding. Over the years, various extensions of the frame theory have been investigated, several of them were contained in the elegant theory of g-frames. Sun [15] introduced g-frames as another generalized frames. He showed that obliqe frames, pseudo-frames and fusion frames are especial cases of g-frames. Some authors call it the operator-valued frame. Kaftal et.al developed operator theoretic method for dealing with multiwavelets and multiframes, see[1, 3, 15]. Throughout this paper, H denotes a separable Hilbert space with inner product $\langle .,. \rangle$ and I is a finite or countable subset of $\mathbb{Z}$, and $\{H_i : i \in I\}$ is a sequence of separable Hilbert spaces. Also, for every $i \in I, B(H, H_i)$ is the set of all bounded linear operators from H to H_i, and $B(H, H)$ is denoted by $B(H)$.

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Definition 1.1. *A family of vectors* $\{f_i\}_{i\in I}$ *in a Hilbert space* H *is said to be a frame if there are constants* $0 < A \leq B < \infty$ *such that, for every* $f \in H$,

$$A \parallel f \parallel^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B \parallel f \parallel^2,$$

where A *and* B *are lower frame bound and upper frame bound, respectively.*
A frame is called a tight frame if $A = B$*, and is called a Parseval frame if* $A = B = 1$*. If a sequence* $\{f_i\}_{i\in I}$ *satisfies the upper bound condition, then* $\{f_i\}_{i\in I}$ *is called a Bessel sequence.*

Remark 1.2. *Let* $\{f_i\}_{i\in I}$ *be a Bessel sequence in a Hilbert space* H*. Then*

$$T : \ell^2(I) \longmapsto H, \qquad T(c_i)_{i\in I} := \sum_{i\in I} c_i f_i$$

defines a bounded linear operator. The adjoint operator is given by

$$T^* : H \longmapsto \ell^2(I), \qquad T^* f = \langle f, f_i\rangle_{i\in I}.$$

Furthermore,

$$\sum_{i\in I} |\langle f, f_i\rangle|^2 \leq \parallel T \parallel^2 \parallel f \parallel^2, (f \in H).$$

We call the adjoint of the synthesis operator, the analysis operator.

Composing T *with its adjoint* T^**, we obtain the frame operator*

$$S : H \longmapsto H, \qquad Sf = TT^* f = \sum_{i\in I} \langle f, f_i\rangle f_i.$$

Note that in terms of the frame operator,

$$\langle Sf, f\rangle = \sum_{i\in I} |\langle f, f_i\rangle|^2 \qquad (f \in H).$$

Theorem 1.3. *Let* $\{f_i\}_{i\in I}$ *be a frame for* H *with frame operator* S*. Then the following hold:*
$(i)$$S$ *is invertible and self- adjoint.*
(ii) *Every* $f \in H$ *can be represented as*

$$f = \sum_{i\in I} \langle f, S^{-1} f_i\rangle f_i = \sum_{i\in I} \langle f, f_i\rangle S^{-1} f_i.$$

Proof. Ref [7]. □

Note that because $S : H \longmapsto H$ is bijective, the sequence $\{S^{-1} f_i\}_{i\in I}$ is also a frame. It is called the canonical dual frame of $\{f_i\}_{i\in I}$.

Example 1.4. *Let* $\{e_i\}_{i=1}^2$ *be an orthonormal basis for a two-dimensional vector space* H *with the inner product. Let*

$$f_1 = 2e_2, \quad f_2 = 3e_1, \quad f_3 = 2e_1 + 3e_2.$$

For every $f \in \mathbb{R}^2$,

$$4 \parallel f \parallel^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq 22 \parallel f \parallel^2,$$

Then $\{f_i\}_{i=1}^3$ *is a frame for* H*. Using the definition of the frame operator,*

$$Sf = \sum_{i=1}^{3} \langle f, f_i \rangle f_i,$$

we obtain that

$$Se_1 = 13e_1 + 6e_2, \qquad Se_2 = 6e_1 + 13e_2.$$

Thus,

$$S^{-1}e_1 = \frac{26}{133}e_1 - \frac{12}{133}e_2, \qquad S^{-1}e_2 = \frac{6}{133}e_1 + \frac{13}{133}e_2.$$

By linearity, the canonical dual frame is

$$\{S^{-1}f_i\}_{i=1}^3 = \{\frac{12}{133}e_1 + \frac{26}{133}e_2, \frac{78}{133}e_1 - \frac{36}{133}e_2, \frac{70}{133}e_1 - \frac{15}{133}e_2\}.$$

Theorem 1.5. *Let* $\{f_i : i \in I\}$ *be a frame for* H *and* $\Lambda \in B(H)$ *be invertible. Then* $\{\Lambda f_i : i \in I\}$ *is a frame for* H*.*

Proof. Ref [7]. □

Corollary 1.6. *Let* $\{f_i : i \in I\}$ *be a Bessel sequence for* H *and* $\Lambda \in B(H)$ *be surjective. Then* $\{\Lambda f_i : i \in I\}$ *is a frame for* H*.*

Remark 1.7. *Let* H *and* K *be two Hilbert spaces. We recall that* $H \bigoplus K := \{(f, g) : f \in H, g \in K\}$ *is a Hilbert space with Pointwise operations and inner product*

$$\langle (f, g), (f^{'}, g^{'}) \rangle = \langle f, f^{'} \rangle_H + \langle g, g^{'} \rangle_K \qquad (f, f^{'} \epsilon H, g, g^{'} \epsilon K).$$

Also, if $\Lambda \in B(H, V), \Gamma \in B(K, Y)$ *then for each* $f \in H,\ g \in K$ *we define*

$$\Lambda \oplus \Gamma \in B(H \oplus K, V \oplus Y) \text{ by } (\Lambda \oplus \Gamma)(f, g) := (\Lambda f, \Gamma g).$$

Theorem 1.8. *Let* $\{f_i : i \in I\}$ *be a frame with bounds* A, B *let* $\{g_j : j \in J\}$ *be a frame with bounds* C, D *. Then* $\{f_i \oplus g_j \in H \oplus K : (i, j) \in I \times J\}$ *is a frame. Furthermore, if* S_f, S_g *and* $S_{f \oplus g}$ *are frame operators respectively, then we have* $S_{f \oplus g} = S_f \oplus S_g$*.*

Proof. Let $\{f_i : i \in I\}$ and $\{g_j : j \in J\}$ be frames for H and K, respectively. Since A, B and C, D are their bounds respectively. Then for every $f \oplus g \in H \oplus K$ we have

$$A \parallel f \parallel^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B \parallel f \parallel^2,$$

$$C \parallel g \parallel^2 \leq \sum_{j \in J} |\langle g, g_i \rangle|^2 \leq D \parallel g \parallel^2 .$$

Hence

$$\min\{A, C\} \parallel f \oplus g \parallel^2 = \min\{A, C\}(\parallel f \parallel^2 + \parallel g \parallel^2) \leq$$
$$\sum_{(i,j) \in I \times J} |\langle (f, g), (f_i, g_j) \rangle|^2 = \sum_{(i,j) \in I \times J} |\langle f, f_i \rangle + \langle f_i, g_i \rangle|^2$$
$$\leq \max\{B, D\} \parallel f \oplus g \parallel^2 .$$

Therefore $\{f_i \oplus g_j \in H \oplus K : (i, j) \in I \times J\}$ is frame and moreover,

$$S_{f \oplus g}(f, g) = \sum_{(i,j) \in I \times J} \langle (f, g), (f_i, g_i) \rangle (f_i, g_i) = \sum_{(i,j) \in I \times J} (\langle f, f_i \rangle_H + \langle g, g_i \rangle_K)(f_i, g_i)$$
$$= S_f(f) \oplus S_g(g) = (S_f \oplus S_g)(f \oplus g).$$

Therefore $S_{f \oplus g} = S_f + S_g$. □

2 G-frame

We observe that various generalizations of frames have been proposed recently. For example, bounded quasi-projectors , frames of subspaces, pseudo-frames, oblique frames, and outer frames [2, 13, 14, 15]. All of these generalizations are proved to be useful in many applications. Here we point out that they can be regarded as special cases of g-frames and many basic properties can be derived within this more general context.

Definition 2.1. *We call a sequence $\Lambda = \{\Lambda_i \in B(H, H_i) : i \in I\}$ a generalized frame, or simply a g-frame, for H with respect to $\{H_i : i \in I\}$ if there are two positive constants A and B such that*

$$A \parallel f \parallel^2 \leq \sum_{i\in I} \parallel \Lambda_i f \parallel^2 \leq B \parallel f \parallel^2 \qquad (f \in H).$$

We call A and B the lower and upper frame bounds, respectively. We call $\{\Lambda_i : i \in I\}$ a tight g-frame if $A = B$ and a Parseval g-frame if $A = B = 1$.
If only the right-hand side inequality is required, Λ is a g-Bessel sequence.
If Λ is s g-Bessel sequence, then the synthesis operator for Λ is the linear operator,

$$T_\Lambda : (\sum_{i\in I} \oplus H_i)_{\ell^2} \longmapsto H \qquad T_\Lambda(f_i)_{i\in I} = \sum_{i\in I} \Lambda_i^* f_i.$$

We call the adjoint of the synthesis operator, the analysis operator. The analysis operator is the linear operator,

$$T_\Lambda^* : H \longmapsto (\sum_{i\in I} \oplus H_i)_{\ell^2} \qquad T_\Lambda^* f = (\Lambda_i f)_{i\in I}.$$

We call $S_\Lambda = T_\Lambda T_\Lambda^*$ the g-frame operator of Λ and $S_\Lambda f = \sum_{i\in I} \Lambda_i^* \Lambda_i f$, $(f \in H)$, for more details see [11, 15].

Theorem 2.2. *A frame is equivalent to a g-frame whenever $H_i = \mathbb{C}, i \in I$.*

Proof. Ref [15]. □

Definition 2.3. *Let $T \in B(H)$ and $\{\Lambda_i \in B(H, H_i) : i \in I\}$ be a g-frame. We say $\{\Lambda_i\}_{i\in I}$ is a T-g-frame if there exist $0 < A \leq B < \infty$ such that*

$$A \parallel T^* f \parallel^2 \leq \textstyle\sum_{i\in I} \parallel \Lambda_i f \parallel^2 \leq B \parallel f \parallel^2 \qquad (f \in H).$$

Theorem 2.4. *Let $\{\Lambda_i \in B(H, H_i) : i \in I\}$ be a g-frame for H w.r.t $\{H_i : i \in I\}$. Suppose that $T_i \in B(H_i, Z_i)$ and these exist $m, M > 0$, such that for each $i \in I$, $x_i \in H_i$*

$$m\|x_i\| \leq \|T_i x_i\| \leq M\|x_i\|,$$

where each Z_i is a Hilbert space and $T \in B(H)$. Then
(i) $\{T_i\Lambda_i T \in B(H, Z_i) : i \in I\}$, is a T^*-g-frame.
(ii) If T is invertible, then $\{T_i\Lambda_i T \in B(H, Z_i) : i \in I\}$, is a g-frame.

Proof. Let $\{\Lambda_i \in B(H, H_i) : i \in I\}$, be a g-frame with bounds A, B. Then for each $f \in H, T \in B(H)$, we have

$$\begin{aligned}\sum_{i\in I} \parallel T_i\Lambda_i T f \parallel^2 &\leq \sum_{i\in I} M^2 \parallel \Lambda_i T f \parallel^2 \leq M^2 B \parallel T f \parallel^2 \\ &\leq M^2 B \parallel T \parallel^2 \parallel f \parallel^2 .\end{aligned}$$

For lower bound, we have

$$\sum_{i\in I} \parallel T_i\Lambda_i T f \parallel^2 \geq \sum_{i\in I} m^2 \parallel \Lambda_i T f \parallel^2 \geq m^2 A \parallel Tf \parallel^2 =$$

$$m^2 A \parallel (T^*)^* f \parallel^2 .$$

(ii) If T is invertible operator for each $f \in H$, we have

$$\parallel f \parallel^2 = \parallel T^{-1}Tf \parallel^2 \leq \parallel T^{-1} \parallel^2 \parallel Tf \parallel^2 \leq \frac{1}{A} \parallel T^{-1} \parallel^2 \sum_{i\in I} \parallel T_i\Lambda_i f \parallel^2$$

$$\leq \frac{1}{Am^2} \parallel T^{-1} \parallel^2 \sum_{i\in I} \parallel T_i\Lambda_i T f \parallel^2 .$$

Therefore

$$Am^2 \parallel T^{-1} \parallel^{-2} \parallel f \parallel^2 \leq \parallel T^{-1} \parallel^2 \sum_{i\in I} \parallel T_i\Lambda_i T f \parallel^2 .$$

And hence (ii) is proved. □

Corollary 2.5. *Let $\{\Lambda_i \in B(H, H_i) : i \in I\}$ be a g-frame and S be the g-frame operator. Then $\{\Lambda_i S^{-\frac{1}{2}} : i \in I\}$. is a g-frame.*

Definition 2.6. *A sequence $\{\Lambda_i \in B(H, H_i) : i \in I\}$ is called*
(1) *g-complete, if $\{f : \Lambda_i f = 0, i \in I\} = \{0\}$.*
(2) *A g-Riesz basis for H with respect to $\{H_i\}_{i\in I}$, if $\{\Lambda_i \in B(H, H_i) : i \in I\}$ is g-complete and there exist two positive constants A and B such that for any finite subset $J \subseteq I$ and $g_j \in K_i$*

$$A \sum_{j\in J} \parallel g_j \parallel^2 \leq \parallel \sum_{j\in J} \Lambda_j^* g_j \parallel^2 \leq B \sum_{j\in J} \parallel g_j \parallel^2 .$$

(3) *A near g-Riesz basis, if there exists a finite subset σ of I for which $\{\Lambda_i\}_{i\in I\setminus\sigma}$ is a g-Riesz basis for H with respect to $\{H_i\}_{i\in I\setminus\sigma}$.*

Now we generalize Sun 's theorem [15, Theorem3.1] to Parseval frames.

Theorem 2.7. *Let $\{\Lambda_i \in B(H, H_i) : i \in I\}$ and for each $i \in I$, $\{f_{ij} : j \in J_i\}$ be a Parseval frame for K_i. Then :*
(i) $\{\Lambda_i : i \in I\}$ is a g-frame in H w.r.t $\{H_i : i \in I\}$ (g-Bessel sequence) if and only if $\{(\Lambda_i)^*(f_{ij}) : i \in I, j \in J_i\}$ is a frame in H (Bessel sequence).
(ii) If $\{(\Lambda_i)^*(f_{ij}) : i \in I, j \in J_i\}$ is a Riesz basis, then $\{\Lambda_i : i \in I\}$ is a g-Riesz basis. Conversely if $\{\Lambda_i : i \in I\}$ is a g-Riesz basis and there exists $m > 0$ such that for each $i \in I_1$ and $(c_{ij})_{j\in I_1}$ for each finite $I_1 \subseteq J_i$,

$$m(\sum_{j\in I_1} |c_{ij}|^2)^{\frac{1}{2}} \leq \parallel \sum_{j\in I_1} c_{ij} f_{ij} \parallel,$$

then $\{(\Lambda_i)^*(f_{ij}) : i \in I, j \in J_i\}$ is a Riesz basis.

Proof. (i) For every $f \in H$, $i \in I$, $\Lambda_i f \in K_i$ and we have

$$\parallel \Lambda_i(f) \parallel^2 = \sum_{j\in J_i} |\langle \Lambda_i(f), f_{ij}\rangle|^2 = \sum_{j\in J_i} |\langle f, \Lambda_i^*(f_{ij})\rangle|^2.$$

Therefore $\{\Lambda_i : i \in I\}$ is a g-frame if and only if $\{(\Lambda_i)^*(f_{ij}) : i \in I, j \in J_i\}$ is a frame.
(ii) Let $\{(\Lambda_i)^*(f_{ij}) : i \in I, j \in J_i\}$ is a Riesz basis with bounds A and B. For each $g_i \in K_i$ we have $g_i = \sum_{j \in J_i} \langle g_i, f_{ij} \rangle f_{ij}$ and $\| g_i \|^2 = \sum_{j \in J_i} |\langle g_i, f_{ij} \rangle|^2$.
Moreover $\Lambda_i^*(g_i) = \sum_{j \in J_i} \langle g_i, f_{ij} \rangle \Lambda_i^*(f_{ij})$ and consequently

$$A \sum_{i \in I} \| g_i \|^2 = A \sum_{i \in I} \sum_{j \in J_i} | < g_i, f_{ij} > |^2 \leq \| \sum_{i \in I} \sum_{j \in J_i} < g_i, f_{ij} > \Lambda_i^*(f_{ij}) \|^2$$

$$= \| \sum_{i \in I} \Lambda_i^* g_i \|^2 \leq B \sum_{i \in I} \sum_{j \in J_i} | < g_i, f_{ij} > |^2 = B \sum_{i \in I} \| g_i \|^2 .$$

For the converse, we assume that $\{\Lambda_i : i \in I\}$ is a g-Riesz basis for H it follows that

$$A \sum_{i \in J_1} \| g_i \|^2 \leq \| \sum_{i \in J_1} \Lambda_i^* g_i \|^2 \leq B \sum_{i \in J_1} \| g_i \|^2,$$

where $J_1 \subset J$ is a finite set.
If $g_i = \sum_{j \in J_i} c_{ij} f_{ij}$ and

$$\| g_i \|^2 = \langle g_i, \sum_{j \in g_i} c_{ij} f_{ij} \rangle = \sum_{j \in J_1} \overline{c_{ij}} \langle g_i, f_{ij} \rangle$$

$$\leq (\sum_{j \in J_i} |c_{ij}|^2)^{\frac{1}{2}} (\sum_{j \epsilon J_1} |\langle g_i, f_{ij} \rangle|^2)^{\frac{1}{2}} = (\sum_{j \epsilon J_i} |c_{ij}|^2)^{\frac{1}{2}} \| g_i \|,$$

then for each $i \in I$

$$\| g_i \| \leq (\sum_{j \in J_i} |c_{ij}|^2)^{\frac{1}{2}}.$$

So, $\| \sum_{i \in J_1} \sum_{i \in J_i} c_{ij} \Lambda_i^*(f_{ij}) \|^2 \leq B(\sum_{j \in J_i} |c_{ij}|^2)^{\frac{1}{2}}$, and by the assumption we have the result.

□

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