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An optimal family of methods for obtaining the zero of nonlinear equation

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Abstract: This manuscript presents a developed fourth-order iterative family of methods for determining the zero of nonlinear equations that is optimal in line with Kung-Traub conjecture. The family of methods was constructed by using weight function technique. One iteration cycle of any concrete member of the family of methods requires the evaluation of three functions. Consequently, the efficiency index of any concrete member of the family is 1.5873. The method convergence analysis was carried out via the Taylor series technique and numerical examples are provided to illustrate its performance as compared with its contemporary existing methods for obtaining the zero of nonlinear equation.

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1 Introduction

The development of new iterative methods (IMs) for obtaining zero of nonlinear equation (NE)

$$w(x) = 0 \tag{1.1}$$

is an area in numerical analysis that have attracted the attention of many scholars in the past three decades. In developing new IMs, the golden rule is to put forward methods that have advantage(s) over existing methods. Some of these advantages includes one or combination of IMs characteristics such as: simplicity, efficiency, fast convergence or optimality. The Newton IM given as

$$x_{k+1} = x_k - \frac{w(x_k)}{w'(x_k)}, \tag{1.2}$$

is a classical IM for obtaining the zero of NE. It is of convergence order (CO) two. Some advantages of the Newton method includes simplicity and optimality as conjectured by Kung and Traub [11], that an IM without memory requiring p number of distinct function evaluations per iteration will have a maximum CO that is bounded by 2^{p-1}. When this bound is attained by the iterative method, it is said to be optimal. Several techniques have been employed by authors in constructing new or modifying existing IM for obtaining the zero of NE. For instance, the decomposition and Adomian techniques were used to put forward several iterative structures in ([15], [7], [3],[12], [1]) and some references inside. In ([20], [22]), the divided

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difference techniques was employed to develop some optimal IMs. A good literature on the geometric, functional, sampling and rational function techniques for developing IMs can be found in [2]. The combination of quadrature and decomposition techniques was used in ([19], [15]) to put forward some good IMs. In recent years, the weight function (W-F) techniques have been utilized to improve the efficiency and CO of some existing methods ([17], [13], [14], [16]). In ([10], [4], [14], [8]), various forms of converging power series were used as W-F in constructing many families of IMs that are of CO four.
Consequent upon above research trends, this manuscript presents a family of optimal CO four IM for obtaining the zero of (1.1) is developed using the W-F technique. Apart from its simplicity and high precision of the developed IM, other advantages of the method includes non-requirement of higher derivatives and requirement of evaluation of one $w'(\cdot)$ and two $w(\cdot)$ functions as against evaluation of two $w'(\cdot)$ and one $w(\cdot)$ functions of most existing optimal methods with CO four.

2 The Method

Consider the iterative procedure in the form:

$$x_{k+1} = x_k + (\Omega_1 \tau_1 + \Omega_2 \tau_2) H(\mu), \tag{2.1}$$

where

$$\tau_1 = \frac{w(x_k)}{w'(x_k)}, \tag{2.2}$$

$$\tau_2 = \frac{w(x_k + \Omega_3 \tau_1)}{w'(x_k)}, \tag{2.3}$$

$$\mu = \frac{w(x_k + \Omega_3 \tau_1)}{w(x_k)}, \tag{2.4}$$

k an iteration counter, $H(\mu)$ is a real valued weight function and $\Omega_i; \{i = 1; 2; 3\}$ are suitable constants to be determined. It is important to note that for $\Omega_1 = 1; \Omega_i = 0$ and $H(\mu) = 1$ in (3), the NM is obtained. Furthermore, for $\Omega_1 = 0; \Omega_2 = -\frac{3+\sqrt{3}}{2}; \Omega_3 = \frac{1-\sqrt{5}}{2}$ and $H(\mu) = 1$, the CO three IM in Ghorbani and Gachpazan [9] is obtained.
Here, our interest is to determine the constants Ω_i and the weight function $H(\mu)$ so that the method (2.1) can obtain the zero of (1.1) with CO four. The consequence of this is that, method (2.1) requiring the evaluation of three functions will be optimal according to Kung-Traub conjecture. The next section presents the convergence analysis of the method under some assumptions.

3 Convergence Analysis of the method

The following definition will be useful in proving the convergence of the IM put forward in (2.1).

Definition 3.1. *Suppose $d_k = |x_k - x^*|$ is the error in kth iteration of an iterative method $w(x_k)$, then the equation*

$$d_{k+1} = M d_k^p + O\left(d_k^{p+1}\right); \tag{3.1}$$

is known as the error equation of $w(x_k)$, where M is constant and ρ is its CO [6] .

Theorem 3.2. *Let $w : D \subset R \to R$ be a real value function that is sufficiently differentiable such that $w'(\cdot) \neq 0$ in D an open interval and $x^* \subset D$. Furthermore, suppose an initial guess x_0 is close to x^* , then the generated sequence of approximations $\{x_k\}_{k\geq 0}; (x_k \in D)$ by the families of IM (2.1), converges to x^* with CO four provided $\Omega_i = 1; \{i = 1; 2; 3\}$, $H(0) = 1; H'(0) = 0; H''(0) = 4$ and $H'''(0) < \infty$.*

Proof. Set $x = x_k$ in the Taylor series expansion of $w(x)$ and $w'(x)$ around x_k, and the results below are obtained.

$$w(x_k) = w'(x^*)\left[d_k + \sum_{n=2}^{4} c_n d_k^n + O\left(d_k^5\right)\right], k = 0, 1, 2, \cdots \tag{3.2}$$

and

$$w'(x_k) = w'(x^*)\left[1 + \sum_{n=2}^{4} c_n d_k^{n-1} + O\left(d_k^5\right)\right], k = 0, 1, 2, \cdots \tag{3.3}$$

where $d_k = \frac{1}{k!}\frac{w^{(k)}(x^*)}{w'(x^*)}, k = 2, 3, \cdots$

Using (3.2) and (3.3), the equations (2.2), (2.3) and (2.4) can be expressed in series as follows:

$$\tau_1 = d_k - c_2 d_k^2 - \left(2c_2^2 - 2c_3\right)d_k^3 + \left(-4c_2^3 + 7c_2c_3\right)d_k^4 + O\left(d_k^5\right); \tag{3.4}$$

$$\begin{aligned}\tau_2 = {} & (1+\Omega_3)\,d_k + c_2\left(\Omega_3^2 - \Omega_3 - 1\right)d_k^2 + (\Omega_3 - 1)\left(-2c_2\left(1 + 2\Omega_3\right) + c_3\left(2 + 4\Omega_3 + \Omega_3^2\right)\right)d_k^3 \\ & + \left(c_2\left(-4 - 4\Omega_3 + 13\Omega_3^2\right) + c_3^2\left(7 + 7\Omega_3 - 19\Omega_3^2 - 5\Omega_3^3\right) + c_4\left(-3 - 3\Omega_3 + 6\Omega_3^2 + 4\Omega_3^3 + \Omega_3^4\right)\right) \\ & + O\left(d_k^5\right),\end{aligned} \tag{3.5}$$

$$\begin{aligned}\mu = {} & (1+\Omega_3) + c_2\Omega_2 d_k + \Omega_3^2\left(-3c_2^2 + c_3(3+\Omega_3)\right)d_k^2 + \Omega_3^2\left(8c_2^3 - 2c_2c_3(7+2\Omega_3) + c_4\left(6 + 4\Omega_3 + \Omega_3^2\right)\right)d_k^3 \\ & + O\left(d_k^5\right),\end{aligned} \tag{3.6}$$

Substitute (3.4), (3.5) and (3.5) in (2.1) yields;

$$\begin{aligned}x_{k+1} = {} & x^* + \left(1 + \frac{1}{6}\left(\Omega_1 + \Omega_2 + \Omega_2\Omega_3\right)\right)(6H(0) + (1+\Omega_3)\left(6H'(0) + (1+\Omega_3)(3H''(0) + H'''(0)\right) \\ & + \Omega_3 H'''(0)))d_k + \frac{1}{6}c_2\left(3\Omega_3\left(\Omega_1 + \Omega_2 + \Omega_2\Omega_3\right)\left(2H'(0) + (1+\Omega_3)\left(2H''(0) + H'''(0) + \Omega_3 H'''(0)\right)\right)\right) \\ & - \left(\Omega_1 + \Omega_2\left(1 + \Omega_3 - \Omega_3^2\right)\right)\left(6H(0) + (1+\Omega_3)\right)\left(6H(0) + (1+\Omega_3)\left(3H'''(0) + \Omega_3 H'''(0)\right)\right)d_k^2 \\ & + \frac{1}{6}(-3c_2^2\Omega_3^2(\Omega_1 + \Omega_2\left(1 + \Omega_3 - \Omega_3^2\right))(2H'(0) + (1+\Omega_3)(2H''(0) + \Omega_3 H'''(0))) \\ & + 3\Omega_3^2(\Omega_1 + \Omega_2 + \Omega_2\Omega_3)(-(c_2^2(6H'(0) + (6 + 6\Omega_3 - \Omega_3^2)H''(0) + (3 + 6\Omega_3 + 2\Omega_3^2 - \Omega_3^3)H'''(0))) \\ & + c_3(3+\Omega_3)(2H'(0) + (1+\Omega_3)(2H''(0) + H'''(0) + \Omega_3 H'''(0)))) + (2\Omega_1(c_2^2 - c_3) \\ & + \Omega_2(\Omega_3 - 1)(-2c_2^2(1 + 2\Omega_3) + c_3(2 + 4\Omega_3 + \Omega_3^2)))(6H(0) + (1+\Omega_3)(6H'(0) \\ & + (1+\Omega_3)(3H''(0) + H'''(0) + \Omega_3 H'''(0)))))d_k^3 \\ & + (3c_2\Omega_3^2(2\Omega_1(c_2^2 - c_3) + \Omega_2(\Omega_3 - 1)(-2c_2^2(1 + 2\Omega_3) + c_3(2 + 4\Omega_3 + \Omega_3^2)))) \cdots \\ & \cdots H(0) + \frac{1}{6}\left(1 + H(0)\right)\left(6H'(0) + (1+\Omega_3)\left(3H''(0) + \Omega_3 H''(0)\right)\right)d_k^4 + O\left(d_k^5\right),\end{aligned} \tag{3.7}$$

For the method (2.1) to approximate the zero of the NLE (1.1) with CO four, the coefficients of d_k, d_k^2 and d_k^3 in (3.7) must vanish. This is achievable when $\Omega_i = 1; \{i = 1, 2, 3\}; H(0) = 1; H'(0) = 0; H''(0) = 4$ and $H'''(0) < \infty$. When these conditions are substituted in (3.7), the error equation below is obtained.

$$d_{k+1} = x^* - \left(-c_2c_3 - c_2^2\left(3 - \frac{H'''(0)}{6}\right)\right) d_k^4 + O\left(d_k^5\right) \tag{3.8}$$

From Defintion 1, the error equation of the developed IM (2.1) obtained in (3.8) is of order 4. This completes the proof.

□

Remark 3.3. *Many functions can be constructed to satisfy the conditions of the weight function $H(\mu)$. This includes $H(\mu) = 1 + 2\mu^2 + \theta\mu^3$.*

Remark 3.4. *For a choice of $H(\mu) = 1 + 2\mu^2 + \theta\mu^3$ in (1.1), a family of optimal CO four method for obtaining the zero of NE (1.1) given as:*

$$x_{k+1} = x_k - \left[\frac{w(x_k)}{w'(x_k)} + \frac{w\left(x_k - \frac{w(x_k)}{w'(x_k)}\right)}{w(x_k)}\right]\left[1 + 2\left(\frac{w\left(x_k - \frac{w(x_k)}{w'(x_k)}\right)}{w(x_k)}\right)^2 + \theta\left(\frac{w\left(x_k - \frac{w(x_k)}{w'(x_k)}\right)}{w(x_k)}\right)^3\right] \tag{3.9}$$

is obtained.

Some specific forms of (3.10) obtained by assigning values to θ and their corresponding error equations are given in Table 1.

Table 1: Specific forms of (3.9) and their error equations

Methods	θ	d_{k+1}
M_1	0	$\left(3c_2^3 - c_2c_3\right) d_k^4$
M_2	3	$-\left(c_2c_3\right) d_k^4$

4 Numerical Experience

The numerical experience with some specific forms (M1 and M2) of the developed family of methods on some standard NEs in literature are presented in this section. The results obtained with M1 and M2 are compared with those from convergence order four methods such as Ghorbani and Gachpazan (GGM) [9] given as:

$$\begin{aligned} x_{k+1} &= x_k - [\tau_1 + \tau_2]; \\ \tau_1 &= \frac{w^2(x_k)}{w'(x_k)\left(w(x_k) - w\left(x_k - \frac{w(x_k)}{w'(x_k)}\right)\right)} \\ \tau_2 &= \frac{\tau_1 w\left(x_k - \tau_1\right)}{w(x_k)} \end{aligned} \tag{4.1}$$

the optimal CO four method in Sharma (SM) [21]:

$$x_{k+1} = x_k - \frac{4w(x_k)}{w'(x_k) + 3w'(y_k)}\left(1 + \tau_1^3\right) - \frac{9}{16}\left(\frac{\phi}{w'(x_k)}\right)^2 \tau_1^3; \tag{4.2}$$

where $\phi = \frac{w'(x_k) - w'(y_k)}{\tau_1}$, $y_k = x_k - \frac{2}{3}\tau_1$, and optimal CO four method developed in Chun (CM) [5] given as:

$$w_{k+1} = w_k - \frac{w'(w_k) + 3w'(y_k)}{2w'(w_k) - w'(y_k)} \frac{w(x_k)}{w'(x_k)}; \tag{4.3}$$

The computation programs for all methods were written and implemented using MAPLE 2017 version environment and all computations are to 1000 digits of mantissa. The condition $|w(x_{k+1})| \leq 10^{-100}$ was used as stopping criteria. The Number of Iterations (Iter), Absolute value of function of last iteration value $|w(x_{k+1})|$ and Petkovic Computational local convergence order (PCLCO) due to Petkovic [18] given as:

$$\rho_{coc} = \frac{log\left|w\left(x_{k+1}\right)\right|}{log\left|w\left(x_k\right)\right|} \tag{4.4}$$

were used as metrics for comparison.

Example 4.1. *Obtain the zero x^* of the NE:*

$$x - cos(x) = 0; \qquad x^* = 0.7390851332151606416\ldots$$

Example 4.2. *Obtain the zero x^* of the NE:*

$$x^2 - (1-x)^5 = 0; \qquad x^* = 0.3459548158482420179\ldots$$

Example 4.3. *Obtain the zero x^* of the NE:*

$$sin(2cosx) - 1 - x^2 + e^{sin(x^3)} = 0; \qquad x^* = -0.7848959876612125352\ldots$$

Example 4.4. *Obtain the zero x^* of the NE:*

$$\sqrt{x^2+2x+5} - 2sin(x) - x^2 + 3 = 0; \qquad x^* = 2.3319676558839640103\ldots$$

Example 4.5. *Obtain the zero x^* of the NE:*

$$x - sin\left(\tfrac{x}{5}\right) - \tfrac{1}{4} = 0; \qquad x^* = 0.4099920179891371316\ldots$$

Example 4.6. *Obtain the zero x^* of the NE:*

$$xe^{x^2} - sin^2x + 3cosx + 5 = 0; \qquad x^* = -1.2076478271309189270\ldots$$

Example 4.7. *Obtain the zero x^* of the NE:*

$$sinx\left(e^x\right) + ln\left(x^2+1\right) = 0; \qquad x^* = 0$$

Example 4.8. *Obtain the zero x^* of the NE:*

$$\left(1+x^3\right) cos\left(x^2\right) + \sqrt{1-x^2} - 2\left(9\sqrt{2} + 7\sqrt{3}\right) x^4 = 0; \qquad x^* = 0.4457514094614178461\ldots$$

Table 2: The computation results obtained from solving Example 4.1-4.8 using $M1$ and $M2$, and methods in ([9], [21], [5])

Examples	$Methods$	x_0	$iter$	$\|w(x_{k+1})\|$	ρ_{pcloc}
4.1	GGM	2	5	$2.9346e-397$	4.0101
	SM		4	$2.3633e-214$	4.1154
	CM		4	$2.3633e-214$	4.1154
	$M1$		4	$5.3409e-235$	4.0526
	$M2$		5	$2.5942e-245$	3.9722
4.2	GGM	1	4	$3.3225e-237$	3.9500
	SM		5	$1.3110e-198$	3.9600
	CM		5	$1.2902e-318$	3.9750
	$M2$		4	$1.9230e-116$	4.0000
	$M1$		5	$1.8881e-343$	3.9884
4.3	GGM	-1	4	$7.1526e-238$	4.0339
	SM		4	$7.4281e-157$	4.0256
	CM		4	$3.9144e-210$	4.0385
	$M1$		4	$1.9230e-192$	4.0000
	$M2$		4	$7.9599e-217$	4.0185
4.4	GGM	1.5	3	$2.6923e-103$	4.1200
	SM		6	$6.3068e-329$	4.0121
	CM		4	$6.3932e-222$	4.0364
	$M1$		4	$9.0269e-299$	4.0405
	$M2$		4	$2.0218e-295$	4.0411
4.5	GGM	0.5	4	$1.3019e-165$	3.9286
	SM		4	$1.4128e-161$	3.9268
	CM		4	$2.0199e-168$	4.0000
	$M1$		4	$4.4104e-137$	3.9143
	$M2$		4	$3.0835e-205$	4.0196
4.6	GGM	-2	6	$2.7173e-287$	4.0139
	SM		6	$1.6805e-193$	4.02083
	CM		5	$9.5698e-153$	4.1351
	$M1$		6	$4.0591e-205$	4.0196
	$M2$		6	$2.4440e-345$	4.0116
4.7	GGM	0.5	5	$2.1555e-298$	3.9864
	SM		5	$1.4057e-338$	3.9765
	CM		5	$2.5435e-306$	3.9740
	$M1$		5	$6.6273e-223$	3.9298
	$M2$		5	$8.0029e-328$	4.0000
4.8	GGM	0.7	5	$2.4931e-256$	4.0000
	SM		5	$1.0421e-209$	4.0192
	CM		5	$4.9490e-320$	4.0000
	$M1$		5	$3.1644e-198$	4.0408
	$M2$		5	$2.2485e-305$	4.0132

4.1 Results discussion

In Table 2, the results obtained when the proposed methods (M1, M2) and the compared methods were tested on Examples 41-4.8 are presented. Observe that the proposed methods did not only obtained the zero of NE in Examples 4.1-4.8 with competitive convergence speed as the compared methods but also their CLCO agreed with the theoretical value obtained in Section 3. Moreover, unlike the GGMs that requires four functions evaluation per iteration circle, the methods M1 and M2 requires just three functions evaluation.

5 Conclusion

In this manuscript, a family of iterative scheme with CO four that requires the evaluation of three distinct function per iteration cycle in obtaining zero of NE is developed. The computation experience from the application of some specific members of the family, shows that, the developed methods are efficient in obtaining zeros of NEs.

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