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Taylor collocation method for approximate solutions of q–difference equations

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Abstract: This manuscript suggests an efficient scheme to find an approach for a class of differential equations arsing in the quantum calculus. The present scheme considers the solution in the form of a truncated Taylor series near zero with unknown coefficients. Then, by placing this approach into the problem and collocating the relation which is obtained at some nodes, a system of algebraic equations is achieved. The solution of this algebraic system is the unknown coefficients of the series. The ability of present method is examined by some examples.

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1 Introduction

A q-analog is a mathematical concept which develops an expression and transforms it to the expression in the limit $q \to 1^-$. It is notable that there are q-analogs of the factorial, binomial coefficient, derivative, integral, Fibonacci numbers, and etc. The q-calculus or quantum calculus is a design in comparison with common study of the calculus but it is based on the q-analogous results without the utilization of the limits. The basis device of this branch is the q-derivative [21, 24].

The q-difference equations are considered as a type of the important problems arising in the q-calculus. They have applications in many fields. For instance, these equations have been used in the fractal sets [8], the deformation of the quantum mechanics [9], and the nonextensive statistical mechanics [22]. Other aspects may be found in [1, 12, 15–17].

Consider

$$D_q[y(t)] = f(t, y(t)), \tag{1.1}$$

under the initial condition

$$y(0) = \alpha, \;\; t \in [0, T], \tag{1.2}$$

so that D_q denotes to the q-derivative operator, $\alpha, T \in \mathbb{R}$ are two fixed constants, $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$ is a known function and y is an unknown function which must be identified. In this manuscript, we assume f is a polynomial type function. The existence and uniqueness of the solution for (1.1) under the initial

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condition (1.2) have been investigated in [18, 23].

In spite of many applications of these equations, there exist a few articles in the literature which focus on the solutions of them. For solving (1.1), Jafari et al. proposed a decomposition method [11], Semary et al. applied the homotopy analysis method [19], and Vijesh et al. suggested a Legendre wavelet quasi-linearization technique [23].

The collocation methods provide proper procedures for solving the differential equations [2, 3]. One of them that can be formulated easily for solving a large variety of differential equations is the Taylor collocation method. This method is based on the Taylor expansion and collocation points. It has been applied successfully to obtain the approximate solutions of the delay equations [4, 5], the Bagley-Torvik equation [6], the integro-differential equations [10, 14, 25], the complex differential equations [20].

Throughout this manuscript, we suppose the approximate solution of (1.1) be in the form of

$$y(t) = \sum_{n=0}^{N} a_n t^n, \tag{1.3}$$

where the Taylor coefficients to be specified are

$$a_n = \frac{y^{(n)}(0)}{n!}, \quad n = 0, 1, \ldots, N.$$

Here, N is any chosen positive integer number. By substituting (1.3) into (1.1) and collocating the identity determined at the points defined as

$$t_j = \frac{T}{N} j, \quad j = 0, 1, \ldots, N,$$

an algebraic system including $(N+1)$ equations and $(N+1)$ unknowns is produced. The solution of aforesaid system is the coefficients a_n. If these coefficients are placed into (1.3), the solution of (1.1) is uncovered.

The structure of this manuscript is as follows. In Section 2, the definitions of q-calculus are announced. Section 3 describes a procedure for obtaining the solution. In Section 4, the numerical experiments are expressed to illustrate the applicability of present method. Section 5 is related to a conclusion.

2 Basic definitions

At the beginning, let us review some relevant materials of q-calculus [7, 13].

Definition 2.1. *For $0 < q < 1$, the q-derivative of a real valued function f is defined as follows*

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad D_q f(0) = \lim_{t \to 0} D_q f(t).$$

Also, higher order q-derivatives are defined as

$$D_q^{(0)} f(t) = f(t), \;\; D_q^{(n)} f(t) = D_q D_q^{(n-1)} f(t), \;\; n \in \mathbb{N}.$$

Corollary 2.2. *From the Definition 2.1, it can be verified that*

$$D_q t^n = \begin{cases} 0, & n = 0, \\ \frac{1-q^n}{1-q} t^{n-1}, & n \in \mathbb{N}. \end{cases} \tag{2.1}$$

Definition 2.3. *The q-integral of f defined on the interval $[a, b]$ is expressed by*

$$\int_a^t f(x) d_q x = \sum_{n=0}^{\infty} (1-q) q^n \left(t f(t q^n) - a f(q^n a) \right), \ \ t \in [a, b],$$

and for $a = 0$, we have

$$I_q f(t) = \int_0^t f(x) d_q x = \sum_{n=0}^{\infty} (1-q) q^n t f(t q^n).$$

Moreover,

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

Corollary 2.4. *According to the Definition 2.3, one can write*

$$I_q t^j = \int_0^t x^j d_q x = \sum_{n=0}^{\infty} (1-q) q^n t (t q^n)^j = (1-q) t^{j+1} \sum_{n=0}^{\infty} \left(q^{j+1} \right)^n = \left(\frac{1-q}{1-q^{j+1}} \right) t^{j+1},$$

where $j \in \mathbb{N} \cup \{0\}$.

Corollary 2.5. *It can be also shown easily that*

$$D_q(f(t)g(t)) = f(qt) D_q(g(t)) + g(t) D_q(f(t)) = g(qt) D_q(f(t)) + f(t) D_q(g(t)).$$

In this way,

$$\int_0^t D_q f(x) d_q x = f(t) - f(0).$$

3 Method of solution

We first assume that the desired solution of (1.1) has been considered in the form of the truncated series (1.3). Therefore, $y(t)$ can be written in the matrix form

$$y(t) = \mathbf{X}(t)\mathbf{A}, \tag{3.1}$$

in which

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & a_2 & \ldots & a_N \end{bmatrix}^T, \quad \mathbf{X}(t) = \begin{bmatrix} 1 & t & t^2 & \ldots & t^N \end{bmatrix}.$$

We consider two cases for the function f in (1.1).

i) f is linear. In this case, (1.1) can be written as

$$D_q[y(t)] = k(t) y(t) + g(t), \tag{3.2}$$

where k and g are two real known functions. If we substitute (3.1) into (3.2) and use (2.1), it is concluded that

$$\mathbf{X}_q(t)\mathbf{A} = k(t)\mathbf{X}(t)\mathbf{A} + g(t), \tag{3.3}$$

where

$$\mathbf{X}_q(t) = D_q\left[\mathbf{X}(t)\right] = \begin{bmatrix} 0 & 1 & \frac{1-q^2}{1-q}t & \frac{1-q^3}{1-q}t^2 & \dots & \frac{1-q^N}{1-q}t^{N-1} \end{bmatrix}.$$

Now, collocating (3.3) at the points defined by

$$\{t_0, t_1, t_2, \dots, t_{N-1}, t_N\} = \left\{0, \frac{T}{N}, \frac{2T}{N}, \dots, \frac{T(N-1)}{N}, T\right\}, \tag{3.4}$$

the following matrix equation is produced

$$\left\{\overline{\mathbf{X}}_q - \mathbf{K}\overline{\mathbf{X}}\right\}\mathbf{A} = \mathbf{G}, \tag{3.5}$$

so that

$$\overline{\mathbf{X}} = \begin{bmatrix} \mathbf{X}(t_0) \\ \mathbf{X}(t_1) \\ \vdots \\ \mathbf{X}(t_N) \end{bmatrix}, \overline{\mathbf{X}}_q = \begin{bmatrix} \mathbf{X}_q(t_0) \\ \mathbf{X}_q(t_1) \\ \vdots \\ \mathbf{X}_q(t_N) \end{bmatrix}, \mathbf{K} = \begin{bmatrix} k(t_0) & 0 & \dots & 0 \\ 0 & k(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & k(t_N) \end{bmatrix}, \mathbf{G} = \begin{bmatrix} g(t_0) \\ g(t_1) \\ \vdots \\ g(t_N) \end{bmatrix}.$$

Note $\mathbf{A}$ and $\mathbf{G}$ are column vectors consisting of $(N+1)$ entries and the dimension of other matrices is $(N+1)\times(N+1)$.

In the other notation, (3.5) may be shown as

$$\mathbf{W}\mathbf{A} = \mathbf{G},$$

so that

$$\mathbf{W} = \overline{\mathbf{X}}_q - \mathbf{K}\overline{\mathbf{X}}.$$

To represent the matrix form of (1.2), we write based on (3.1)

$$y(0) = \overline{\mathbf{X}}_0\mathbf{A} = \alpha,$$

in which

$$\overline{\mathbf{X}}_0 = \mathbf{X}(0) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix}_{1\times(N+1)}. \tag{3.6}$$

Replacing the first row of $\mathbf{W}$ by $\overline{\mathbf{X}}_0$, we construct a matrix named $\overline{\mathbf{W}}$. Consequently, the following new matrix equation is considered

$$\overline{\mathbf{W}}\mathbf{A} = \overline{\mathbf{G}}, \tag{3.7}$$

where

$$\overline{\mathbf{G}} = \begin{bmatrix} \alpha & g(t_1) & g(t_2) & \dots & g(t_N) \end{bmatrix}^T.$$

Clearly, (3.7) is a linear algebraic system whose solution, $\mathbf{A} = \left(\overline{\mathbf{W}}\right)^{-1}\overline{\mathbf{G}}$, is the coefficients $a_0, a_1, \dots, a_N$. The solution of (3.2) with the condition (1.2) is obtained by applying (3.1).

ii) f is nonlinear. Let

$$D_q[y(t)] = r(t)y^m(t) + s(t), \tag{3.8}$$

where $m > 1$ is a natural number and r , s are two real known functions. By substituting the collocation points (3.4) into $y^m(t)$, one can write

$$\begin{bmatrix} y^m(t_0) \\ y^m(t_1) \\ \vdots \\ y^m(t_N) \end{bmatrix} = \begin{bmatrix} y^{m-1}(t_0) & 0 & \dots & 0 \\ 0 & y^{m-1}(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y^{m-1}(t_N) \end{bmatrix} \begin{bmatrix} y(t_0) \\ y(t_1) \\ \vdots \\ y(t_N) \end{bmatrix} = \left(\overline{\overline{\mathbf{Y}}}\right)^{m-1} \overline{\mathbf{Y}}, \tag{3.9}$$

so that

$$\overline{\mathbf{Y}} = \begin{bmatrix} y(t_0) \\ y(t_1) \\ \vdots \\ y(t_N) \end{bmatrix} = \overline{\mathbf{X}}\mathbf{A}, \quad \overline{\overline{\mathbf{Y}}} = \overline{\overline{\mathbf{X}}}\,\overline{\mathbf{A}}, \tag{3.10}$$

where $\overline{\overline{\mathbf{X}}}$ is an $(N+1) \times (N+1)^2$ matrix and $\overline{\mathbf{A}}$ is an $(N+1)^2 \times (N+1)$ one written in the form of

$$\overline{\overline{\mathbf{X}}} \begin{bmatrix} \mathbf{X}(t_0) & 0 & \dots & 0 \\ 0 & \mathbf{X}(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{X}(t_N) \end{bmatrix}, \quad \overline{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & 0 & \dots & 0 \\ 0 & \mathbf{A} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{A} \end{bmatrix}.$$

We are ready to construct the main matrix equation corresponding to (3.8). First, the collocation points (3.4) are substituted into (3.8), that is

$$D_q[y(t_j)] = r(t_j)y^m(t_j) + s(t_j), \quad j = 0, 1, \dots, N,$$

and then this identity is written in the matrix form below

$$\overline{\mathbf{X}}_q\mathbf{A} - \mathbf{R}\left(\overline{\overline{\mathbf{Y}}}\right)^{m-1} \overline{\mathbf{Y}} = \mathbf{S}, \tag{3.11}$$

where

$$\mathbf{R} = \begin{bmatrix} r(t_0) & 0 & \dots & 0 \\ 0 & r(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r(t_N) \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} s(t_0) & 0 & \dots & 0 \\ 0 & s(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s(t_N) \end{bmatrix}.$$

Utilizing (3.11), (3.10) and (3.9), we have

$$\left\{\overline{\mathbf{X}}_q - \mathbf{R}\left(\overline{\overline{\mathbf{X}}}\,\overline{\mathbf{A}}\right)^{m-1} \overline{\mathbf{X}}\right\} \mathbf{A} = \mathbf{S},$$

or briefly

$$\overline{\overline{\overline{\mathbf{W}}}}\mathbf{A} = \mathbf{S},$$

where

$$\overline{\overline{\overline{\mathbf{W}}}} = \overline{\mathbf{X}}_q - \mathbf{R}\left(\overline{\overline{\mathbf{X}}}\,\overline{\mathbf{A}}\right)^{m-1} \overline{\mathbf{X}}.$$

Similar to previous case, replacing the first row of $\overline{\overline{\mathbf{W}}}$ by $\overline{\mathbf{X}}_0$ defined in (3.6), a matrix named $\widehat{\mathbf{W}}$ is constructed. Thus, the new matrix equation is considered as

$$\widehat{\mathbf{W}}\mathbf{A} = \widehat{\mathbf{S}}, \tag{3.12}$$

in which

$$\widehat{\mathbf{S}} = \begin{bmatrix} \alpha & s(t_1) & s(t_2) & \dots & s(t_N) \end{bmatrix}^T.$$

It is obvious that (3.12) is a nonlinear system of algebraic equations. We solve it by the *'fsolve'* command of MATLAB software with the initial guess

$$\mathbf{A}_0^T = \underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix}}_{N+1}.$$

The solution of (3.8) with the condition (1.2) is gained by utilizing (3.1).

4 Numerical examples

In this section, two examples are devoted to show the validity of the method. The computations of them have been done by using the MATLAB R2015a software.

Example 4.1. *The linear q-difference equation is given as*

$$D_{\frac{2}{3}}[y(t)] + 2y(t) = 2t^2 + \frac{5}{3}t + 2, \quad 0 < t \leq 1, \tag{4.1}$$

with the initial condition

$$y(0) = 1. \tag{4.2}$$

The exact solution of (4.1) with the condition (4.2) is $y(t) = t^2 + 1$.

Suppose $N = 2$. For this choice, the collocation points are

$$\{t_0, t_1, t_2\} = \left\{0, \frac{1}{2}, 1\right\}.$$

After some calculations, $\overline{\mathbf{W}}$ and $\overline{\mathbf{G}}$ applied in (3.7) are obtained in the following

$$\overline{\mathbf{W}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & \frac{4}{3} \\ 2 & 3 & \frac{11}{3} \end{bmatrix}, \quad \overline{\mathbf{G}} = \begin{bmatrix} 1 \\ \frac{10}{3} \\ \frac{17}{3} \end{bmatrix}.$$

Subsequently,

$$\mathbf{A} = \left(\overline{\mathbf{W}}\right)^{-1}\overline{\mathbf{G}} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{7}{5} & \frac{11}{10} & -\frac{2}{5} \\ \frac{3}{5} & -\frac{9}{10} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{10}{3} \\ \frac{17}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Finally, we have

$$y(t) = \mathbf{X}(t)\mathbf{A} = \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 + t^2,$$

which is the exact solution.

Example 4.2. *[23] Consider the Riccati type nonlinear q-difference equation as*

$$D_q[y(t)] - y^2(t) = \frac{2t^2}{15} - \frac{t^4}{9} - \frac{1}{25} - \frac{t}{2}, \quad 0 < t \leq 1, \tag{4.3}$$

with the initial condition

$$y(0) = \frac{1}{5}. \tag{4.4}$$

The exact solution of (4.3) under the condition (4.4) is $y(t) = \frac{1}{5} - \frac{t^2}{3}$ *when* $q = 0.5$.

We proceed the procedure described in the Section 3 for the nonlinear case. If $y_q(t)$ denotes to the Taylor collocation method (TCM) solution corresponding to the amount of q, then for $N = 5$

$$\begin{cases} y_{0.3}(t) = -0.028897t^5 + 0.10707t^4 - 0.087562t^3 - 0.36399t^2 - 0.0024804t + 0.2, \\ y_{0.5}(t) = (3.6413e-7)t^5 - (1.0904e-6)t^4 + (1.1285e-6)t^3 - 0.33333t^2 + (1.6129e-7)t + 0.2, \\ y_{0.7}(t) = -0.0013562t^5 - 0.017804t^4 + 0.022091t^3 - 0.29942t^2 + 0.00086435t + 0.2, \\ y_{0.9}(t) = -0.006016t^5 - 0.011922t^4 + 0.022083t^3 - 0.26784t^2 + 0.00086353t + 0.2. \end{cases}$$

The Table 1 gives the numerical results of applying the present method for $N = 5$ and the Legendre wavelets method (LWM) for $N_1 = 6$ published in [23]. The Figure 1 illustrates approximate solutions and the exact solution at $q = 0.5$.

Table 1: The numerical results of Example 4.2

t	$q = 0.3$		$q = 0.5$			$q = 0.7$		$q = 0.9$	
	LWM [23]	TCM	Exact	LWM [23]	TCM	LWM [23]	TCM	LWM [23]	TCM
0.1	0.1961	0.1960	0.1967	0.1967	0.1967	0.1971	0.1971	0.1974	0.1974
0.3	0.1650	0.1649	0.1700	0.1700	0.1700	0.1737	0.1738	0.1766	0.1766
0.5	0.1026	0.1026	0.1167	0.1167	0.1167	0.1272	0.1272	0.1353	0.1353
0.7	0.0108	0.0107	0.0367	0.0367	0.0367	0.0570	0.0570	0.0730	0.0731
0.9	-0.1076	-0.1077	-0.0700	-0.0700	-0.0700	-0.0382	-0.0381	-0.0115	-0.0115

5 Conclusion

We offered a convenient method to solve an important class of the q-difference equations. It generated the approximate solutions in the form of real polynomials. The numerical experiments indicated that a solution produced by the method of this manuscript is in a good agreement with the exact solution and an other method.

Since the computations were based on the matrix relations, the present method is computer-oriented and can be implemented by MATLAB or MAPLE without any difficulty. The examples demonstrated the process of present method is easy to perform and has reasonable results.

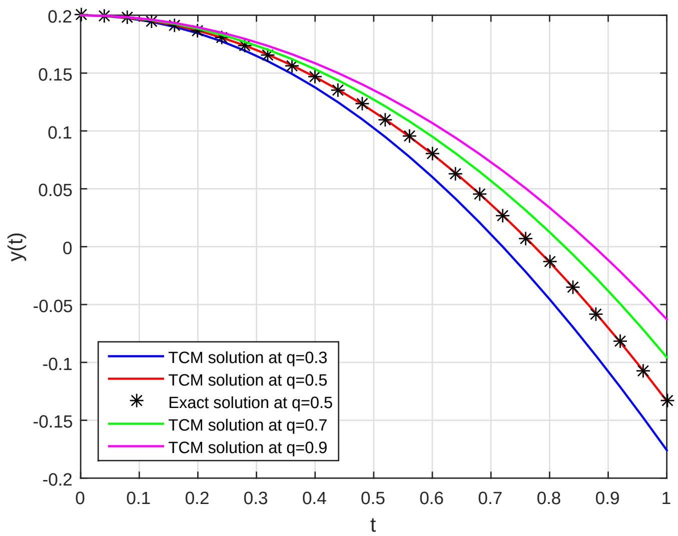

Figure 1: The solutions of Example 4.2 for $N = 5$

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