Vol 3(2), 2022, PP: 10-21
DOI:10.30511/mcs.2022.543230.1050
ISSN:2717-2708
http://mcs.qut.ac.ir/

Numerical Solution of System of Linear Fractional Integro-differential Equations by Least Squares Collocation Chebyshev Technique

O. Taiye[1], T. O. Adebayo[2], A. A. James[3], I. A. Adam[4], A. A. Muhammed[5]

Abstract: This study presents the approximate solutions of a system of Fractional Integro-Differential Equations (FIDEs) with least squares collocation Chebyshev technique. The technique reduces the problem to system of linear algebraic equations and then solved. The applicability of this method has been demonstrated by numerical examples. Numerical results show that the method is easy to implement and compares favorably with the exact results.

Keywords: System of linear fractional integro-differential equations; least squares collocation; Chebyshev polynomials.
2020 Mathematics Subject Classification: 34A08, 65PXX
Receive: 20 November 2021, **Accepted:** 26 April 2022

1 Introduction

The Fractional integro-differential equation has played a significant role in modelling of real world physical problems e.g. the modelling of earthquakes, reducing the spread of viruses, control the memory behaviour of electric socket and many others. Fractional calculus is a field dealing with integral and derivatives of arbitrary orders, and their applications in science, engineering and other fields. The idea is from the ordinary calculus. According to [1,6,7], It was discovered by Leibniz in the year 1695 few years after he discovered ordinary calculus but later forgotten due to the complexity of the formula. Since most FIDEs cannot be solved analytically,

[1] Department of Applied Science, Federal College of Dental Technology and Therapy, Enugu, Nigeria, oyedepotaiye@yahoo.com
[2] Department of Mathematics, University of Ilorin, Ilorin, Nigeria, tomotayoadebayo@yahoo.com
[3] Department of Mathematics, America University of Nigeria, Yola, Nigeria & adewale.james@aun.edu.com
[4] Department of Physical Sciences, Al-Hihmah University, Ilorin, Nigeria, kadam4best@gmail.com
[5] Department of Mathematics, Modibbo Adama University, Yola, Nigeria, abdullahim@mautech.edu.ng

therefore, much attention has been devoted to searching for approximate and numerical techniques to the solution of FIDEs. Recently, many methods have been developed by researchers for providing approximate solutions to FIDEs. In [17] FIDEs have been solved using Bernstein polynomials as basic functions while Laguerre polynomials were employed in [5]. Collocation techniques has been applied to find the solution of FIDEs in References [8,11,19]. Sumudu transform method and Hermite Spectral collocation method for the solution of FIDEs were introduced in [3]. Authors in [2] introduced approximate solutions of Volterra-Fredholm integro-differential equations of fractional order. Least Squares Method (LSM) for the solution of FIDEs has been applied in References [12, 13]. Taylor series expansion and Galerkin method based on the second kind of Chebyshev polynomials have been applied in [14, 15, 21] to solve fractional singular Integro-Differential Equations (IDEs). The authors in [19] applied a numerical solution of Fredholm-Volterra FIDEs with nonlocal boundary conditions. In [16] Bernstein modified homotopy perturbation method was employed to solve Volterra FIDEs. The objective of this work is to introduce a new technique for the first time in the literature called Least Squares Collocation Chebyshev Technique (LSCCT) with different weight functions that provides less rigorous works in terms of computational cost with improved accuracy. The general form of the class of problem considered in this work is given as:

$$D^{\alpha}u_i(x) = p_i(x)u_i(x) + f_i(x) + \int_0^x k_i(x,t)\Big(\sum_{r=1}^{n} u_r(t)\Big)\,dt, \qquad i = 1,2,\dots,n, \qquad o \le x,t \le 1, \tag{1.1}$$

With the following supplementary conditions:

$$u_i^{(j)}(x_0) = u_{ij} \quad i = 0,1,2,\dots,m-1, \quad m-1 < \alpha \le m,\ m \in, N \tag{1.2}$$

Where $D^{\alpha}u_i(x)$ indicates the $\propto th$ Caputo fractional derivative of $u_i(x)$; $p_i(x), f_i(x),\ k_i(x,t)$ are given smooth functions, x and t are real variables varying [0, 1] and $u_i(x)$ is the unknown function to be determined.

2 Some Relevant Basic Definitions

Definition 2.1: Riemann – Lowville fractional integral is defined as [9]

$$J^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(x)}{(x-t)^{1-\alpha}}dt,\ \alpha > 0, x > 0, \tag{2.1}$$

J^{α} denotes the fractional integral of order $\propto$

Definition 2.2: The Caputo fractional Derivative is defined [9]:

$$D^{\alpha}f(x) = \frac{1}{\Gamma(n-\propto)}\int_0^x (x-s)^{n-\propto-1} f^m(s)ds \tag{2.2}$$

Where m is a positive integer with the property that $n-1 <\propto< n$

For example if $0 <\propto< 1$ the caputo fractional derivative is

$$D^{\alpha}f(x) = \frac{1}{\Gamma(1-\propto)}\int_0^x (x-s)^{-\propto} f^1(s)ds \tag{2.3}$$

Hence, we have the following properties:

(1) $J^{\alpha} J^{v} f = j^{\alpha+v} f, \alpha, v > 0, f \in C_{\mu}, \mu > 0$

(2) $J^{\alpha}x^{\gamma} = \frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma,}\ \alpha > 0, \gamma > -1, x > 0$

(3) $J^{\alpha} D^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} f^k(0)\frac{x^k}{k!}, \qquad x > 0, n-1 < \alpha \le n$

(4) $D^{\alpha} J^{\alpha} f(x) = f(x), \quad x > 0, n-1 < \alpha \le n,$

(5) $D^{\alpha} C = 0, C$ is the constant,

(6) $\begin{cases} 0, & \beta \in N_0, \beta < [\alpha], \\ D^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, & \beta \in N_0, \beta \ge [\alpha], \end{cases}$

Where $[\alpha]$ denoted the smallest integer greater than or equal to α and $N_0 = \{0,1.2, \dots\}$

Definition 2.3: The Chebyshev polynomials [13] of degree n over [0,1] is defined by the relation

$$Q_m(x) = cos[\sin^{-1}(2x-1)] \tag{2.4}$$

and the recurrence relation is given by

$$Q^*_{m+1}(x) = 2(2x-1)Q^*_m(x) - Q^*_{m-1}(x), \quad m \ge 1 \tag{2.5}$$

Where $Q^*_0(x) = 1$, $Q^*_1(x) = 2x - 1$

Definition 2.4: The Chebyshev polynomials [13]: A linear combination of Chebyshev basis polynomials:

$$u_i(x) = \sum_{j=0}^{m} a_j{}^i Q^*_j(x) \tag{2.6}$$

is the Chebyshev polynomials of degree n where $a_{j,} \quad j = 0,1,2, \dots \dots$ are constants.

3 Demonstration of Least-Squares Collocation Chebyshev Technique

The proposed method is based on approximating the unknown function $u_i(x)$ in Eq. (1.1) by assuming an approximate solution of the form in Eq. (2.6). Consider Eq. (1.1) operating with $J^{\propto}$ on both sides as follows:

$$J^{\alpha}[u_i(x)] = J^{\alpha}\, [p_i(x)u_i(x) + f_i(x) + \int_0^x k_i(x,t)(\textstyle\sum_{v=1}^{n} u_v(t))\, dt] \tag{3.1}$$

$$u_i(x) = \sum_{k=0}^{n-1} u_i{}^k(0) \frac{x^k}{k!} + J^{\alpha}\, [p_i(x)u_i(x) + f_i(x) + \int_0^x k_i(x,t)(\textstyle\sum_{r=1}^{n} u_r(t))\, dt] \tag{3.2}$$

Substituting Eq. (2.6) into Eq. (3.2).

$$\sum_{j=0}^{m} a_j{}^i Q^*_j(x) = \sum_{k=0}^{m-1} u_i{}^k(0) \frac{x^k}{k!} + J^{\alpha}[p_i(x) \textstyle\sum_{j}^{m} a_j{}^i Q^*_j(x) + f_i(x) +$$

$$\int_0^x k_i(x,t)(\textstyle\sum_{r=1}^{n}[\sum_{j=0}^{m} a_j{}^i Q^*_j(t)])\, dt] \tag{3.2}$$

Hence, the residual equation is obtained as

$$R(a_0{}^i, a_1{}^i, \dots, a_m{}^i) = \sum_{j=0}^{m} a_j{}^i Q^*_j(x) - \{\textstyle\sum_{k=0}^{m-1} u_i{}^k(0)\, \frac{x^k}{k!} +$$

$$[p_i(x) \textstyle\sum_{j=0}^{m} a_j{}^i Q^*_j(x) + f_i(x) + \int_0^x k_i(x,t)(\sum_{r=1}^{n}[\sum_{j=0}^{m} a_j{}^i Q^*_j(t)])\, dt]\} \tag{3.3}$$

Let

$$S(a_{0,}a_1 \dots, a_m) = [R(a_{0,}a_1, \dots \dots, a_m)]^2 w(x) \tag{3.4}$$

Where $w(x)$ is the positive weight function defined in the interval, [a, b]. In this work,

we take $w(x) = \sqrt{c + dx^i}$, $i = 2$, $c = 1$ and $d = -1$. Thus,

$$S(a_0{}^i, a_1{}^i, \dots, a_m{}^i) = \Big\{\textstyle\sum_{j=0}^{m} a_j{}^i Q^*_j(x) - \{\sum_{k=0}^{m-1} u_i{}^k(0)\, \frac{x^k}{k!} + J^{\propto}[p(x) \sum_{j=0}^{m} a_j{}^i Q^*_j(x) + f_i(x) +$$

$$\int_0^x k_i(x,t)(\textstyle\sum_{r=1}^{n}[\sum_{j=0}^{m} a_j{}^i Q^*_j(t)])\, dt]\}\Big\}^2 \tag{3.5}$$

In order to minimize Eq. (3.5), the values of a_j $(j \geq 0)$ is obtained by finding the minimum value of S as:

$$\frac{\partial S_i}{\partial a_j{}^i} = 0, j = 0,1,2 \dots, m \tag{3.6}$$

Applying Eq. (3.6) on Eq. (3.5) , we have

$$\left\{\sum_{j=0}^{m} a_j{}^i Q^*{}_j(x) - \{\sum_{k=0}^{m-1} u_i{}^k(0)\frac{x^k}{k!} + J^{\propto}[p(x)\sum_{j=0}^{m} a_j{}^i Q^*{}_j(x) + f_i(x) \ + J^{\propto}[\int_0^x k_i(x,t)\left(\sum_{r=1}^{n}[\sum_{j=0}^{m} a_j{}^i Q^*{}_j(t)]\right) dt\right]\right\} \times$$

$$\left\{u_j(x) - J^{\propto}[p(x)Q^*{}_j(x)] - J^{\propto}[\textstyle\int_0^x k_i(x,t)(\sum_{r=1}^{n}[\sum_{j=0}^{m} Q^*{}_j(x)])\, dt]\right\} \tag{3.7}$$

Thus, Eq. (3.7) is then simplified for $j = 0,1, \dots n$ and collocated at equally spaced point $x_i = a + \frac{(b-a)i}{m}$, $(i = 1(1)m)$ to obtain $(m + 1)$ algebraic system of equations in $(m + 1)$ unknown $a^i{}_j$ which are then put in matrix form as follow:

$$A = \begin{pmatrix} R_i(x, a_0{}^i)h_0{}^i & R_i(x, a_1{}^i)h_0{}^i \cdots & R_i(x, a_m{}^i)h_0{}^i \\ R_i(x, a_0{}^i)h_1{}^i & R_i(x, a_1{}^i)h_1{}^i \cdots & R_i(x, a_m{}^i)h_1{}^i \\ \vdots & \vdots \quad \ddots & \vdots \\ R_i(x, a_m{}^i)h_m{}^i & R_i(x, a_m{}^i)h_m{}^i \dots & R_i(x, a_j{}^i)h_m{}^i \end{pmatrix}, \tag{3.8}$$

$$B = \begin{pmatrix} \left[J^{\propto} f_i(x) + \sum_{k=0}^{n-1} u_i{}^k(0)\ \frac{x^k}{k!}\right] h_0{}^i \\ \left[J^{\propto} f_i(x) + \sum_{k=0}^{n-1} u_i{}^k(0)\frac{x^k}{k!}\right] h_1{}^i \\ \vdots \\ \left[J^{\propto} f_i(x) + \sum_{k=0}^{n-1} u_i{}^k(0)\frac{x^k}{k!}\right] h_m{}^i \end{pmatrix} \tag{3.9}$$

Where

$$h_j{}^i = Q^*{}_j(x) - J^{\propto}\left[p(x)Q^*{}_j(x)\right] - J^{\propto}[\textstyle\int_0^x k_i(x,t)(\sum_{r=1}^{n}[\sum_{j=0}^{m} Q^*{}_j(t)])\, dt]\, , j = 0,1, \dots, m \tag{3.10}$$

$$R_i(x, a_j{}^i) = \textstyle\sum_{j=0}^{m} a_j{}^i Q^*{}_j(x) - \{J^{\alpha}\ [p(x)\sum_{j=0}^{m} a_j{}^i Q^*{}_j(x)\ + J^{\propto}[\int_0^x k_i(x,t)(\sum_{r=1}^{n}[\sum_{j=0}^{m} a_j{}^i Q^*{}_j(t)])\, dt], \tag{3.11}$$

$j = 0,1, \dots, m,\ \ I = 1,2, \dots, n$

In order to obtain the unknown constants $a_j(j = 0(1)m)$, the $(\mathrm{m} + 1)$ linear equations are then solved using Maple 18, which are then substituted back into the assumed approximate solution to give the required approximation solution.

Remark: The convergence and stability of the method were discussed in [20] while the existence and uniqueness of solution have been proved by [4].

4 Numerical Examples

In order to illustrate the computational cost accuracy and efficiency of the proposed method using Maple 18 for solving system of linear fractional integro-differential equations, we present some examples. All result are calculated using Maple 18.

Example 4.1: Consider the following fractional Integro-differential [18]

$$D^{\propto}u_1(x) = -\frac{1}{20} - \frac{x}{12} + \frac{4x^{\frac{1}{4}}(15-23x^2)}{15\Gamma\left(\frac{1}{3}\right)} + \int_0^1 (x+t)[u_1(t)+u_2(t)]dt, \tag{4.1}$$

$$D^{\propto}u_2(x) = -\frac{5x^3}{6} + \frac{9x^{\frac{4}{3}}}{2\Gamma\left(\frac{1}{3}\right)} + \int_0^1 x^{\frac{1}{2}}t^2[u_1(t)-u_2(t)]dt \tag{4.2}$$

Subject to initial conditions $u_1(0) = 0, u_2(0) = 0$ with the exact solution $u_1(x) = x - x^3, u_2(x) = x^2 - x$.

Solving example 1, following the same procedure above, we take $\propto = \frac{3}{4}$ and $m = 3$. The following constants were obtained as: $a_0 = 0, a_1 = 0.333333333$, $a_2 = 0.666666666$, $a_3 = 0$ for Eq. (4.1) and $a_0 = 0, a_1 = -0.3333333333, a_2 = -0.3333333333, a_3 = 0$ for Eq. (4.2). Substituting the values into the assumed approximate solution hence, the approximate solution is obtained as:

$u_1(x) = -0.999999999x^3 + 0.999999999x$

$u_2(x) = 0.9999999999x^2 - 1.00000000t$. Comparing the result obtained by [18] with the new method, it suffices to say that the new method in this study is more accurate since the table of error found is smaller than [17] and the graph of the approximate solution is the same as the graph of the exact solution.

Example 4.2: Consider the following fractional Integro-differential [18]

$$D^{\propto}u_1(x) = \frac{83x}{80} + \frac{x}{12} + \frac{25x^{\frac{6}{5}}(11+15x)}{33\Gamma\left(\frac{1}{5}\right)} + \int_0^1 2xt[u_1(t)+u_2(t)]dt, \tag{4.3}$$

$$D^{\propto}u_2(x) = -\frac{5x^3}{6} + \frac{9x^{\frac{4}{3}}}{2\Gamma\left(\frac{1}{3}\right)} + \int_0^1 (x+t)[u_1(t)-u_2(t)]dt. \tag{4.4}$$

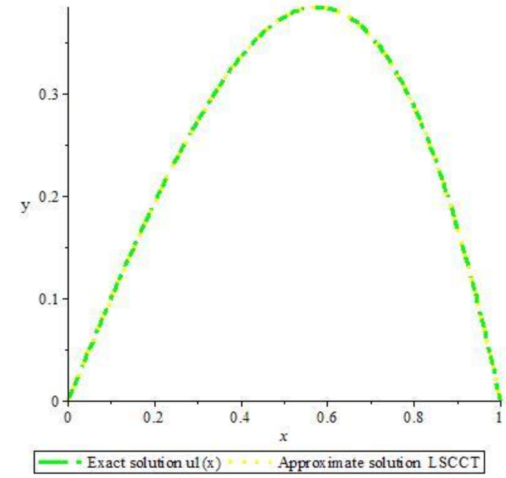

Figure 1: Showing the graph of approximation solution $u_1(x)$and exact of example 4.1

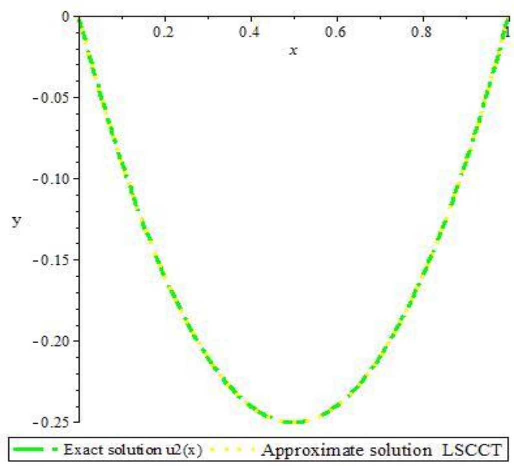

Figure 2: Showing the graph of approximation solution $u_2(x)$and exact of example 4.1

Subject to initial conditions $u_1(0) = 0, u_2(0) = 0$ with the exact solution $u_1(x) = x^3 - x^2, u_2(x) = \frac{15}{8}x^2$. Similarly solving Example 3, following the same procedure above, we take $\propto = \frac{4}{5}$ and $m = 2$. The following constants were obtained as: $a_0 = 0, a_1 = 0$, $a_2 = 0.33333333333$, $a_3 = 0$ for Eq. (4.3) and $a_0 = 0, a_1 = 0, a_2 = 0.6249999999, a_3 = 1.874999999$ for Eq. (4.4). Substituting the values back into the assumed approximate solution, the approximate solution is obtained as:

$u_1(x) = 0.999999999x^3 - 0.999999999x^3$,

$u_2(x) = -1 \times 10^{-9}x^3 + 1.8745000000x^2$. Comparing the result obtained by [18] with the new method, it suffices to say that new method in this study is more accurate since the table of error found is smaller than [18] and the graph of the approximate solution is the same as the graph of the exact solution.

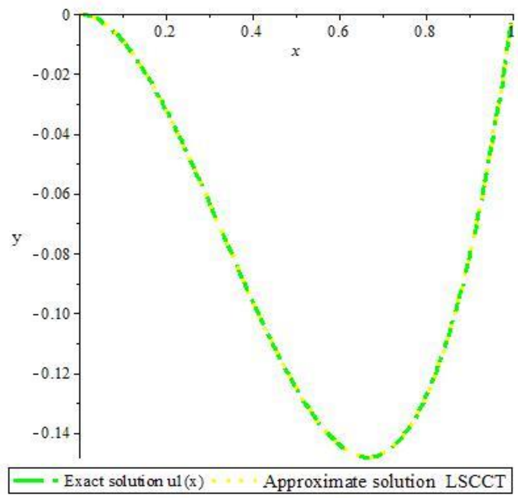

Figure 3: Showing the graph of approximation solution $u_1(x)$and exact of example 4.2

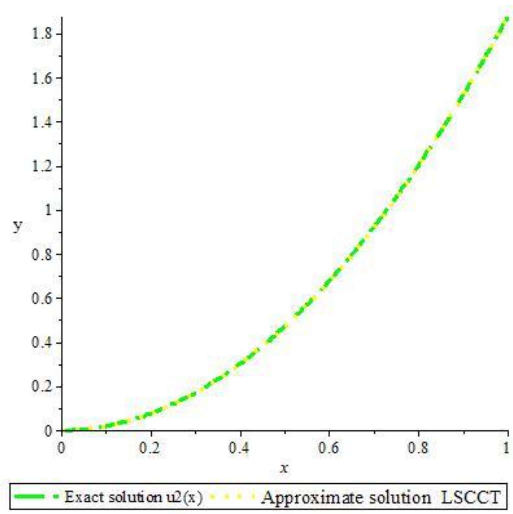

Figure 4: Showing the graph of approximation solution $u_2(x)$and exact of example 4.2

Example 4.3: Consider the following fractional Integro-differential [18]

$$D^{\propto}u_1(x) = \frac{3x^{\frac{1}{3}}}{2}\frac{\sqrt{3}\,\Gamma(\frac{2}{3})}{\pi} - \frac{1}{6}x + \int_0^1 2xt\big(u_1(t) + u_2(t)\big)dt, \quad (4.5)$$

$$D^{\propto}u_2(x) = \frac{9x^{\frac{4}{3}}}{4}\frac{\sqrt{3}\,\Gamma(\frac{2}{3})}{\pi} - \frac{5}{6}x^3 + \int_0^1 x^3\big(u_1(t) - u_2(t)\big)dt \quad (4.6)$$

Subject to $u_1(0) = -1, u_2(0) = 0$ with the exact solution $u_1(x) = x - 1, u_2(x) = x^2$

Similarly, solving example 3, following the same procedure above, we take $\propto = \frac{2}{3}$ and $m = 2$, we obtained the approximation solution which is the same as the exact solution.

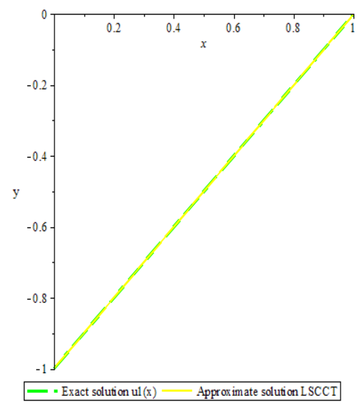

Figure 5: Showing the graph of approximation solution $u_1(x)$and exact of example 4.3

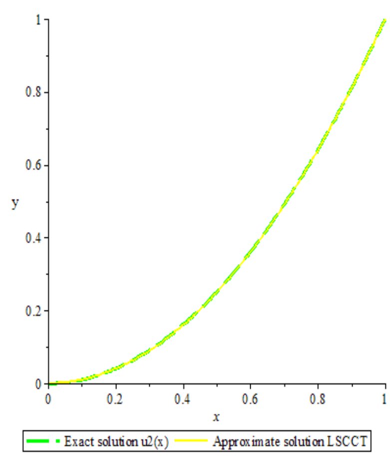

Figure 6: Showing the graph of approximation solution $u_2(x)$and exact of example 4.3

5 Table of Results

Table 1. Comparison of the absolute errors for Example 4.1

X	LSCCT $u_1(x)$	ADM [18]
0.0	$0.000 \times 10^{+00}$	$0.000 \times 10^{+00}$
0.2	1.920×10^{-10}	3.381×10^{-3}
0.4	3.360×10^{-10}	6.510×10^{-3}
0.6	3.840×10^{-10}	9.942×10^{-3}
0.8	2.880×10^{-10}	1.372×10^{-2}
1.0	$0.000 \times 10^{+00}$	1.786×10^{-2}

Table 2. Comparison of the absolute errors for Example 4.1

X	LSCCT $u_2(x)$	ADM [18]
0.0	$0.000 \times 10^{+00}$	$0.000 \times 10^{+00}$
0.2	$0.000 \times 10^{+10}$	4.753×10^{-4}
0.4	$0.000 \times 10^{+10}$	1.130×10^{-3}
0.6	6.000×10^{-11}	1.876×10^{-3}
0.8	8.000×10^{-11}	2.689×10^{-3}
1.0	$0.000 \times 10^{+00}$	3.534×10^{-3}

Table 3. Comparison of the absolute errors for Example 4.1

X	LSCCT $u_1(x)$	ADM [18]
0.0	$0.000 \times 10^{+00}$	$0.000 \times 10^{+00}$
0.2	$0.000 \times 10^{+10}$	6.852×10^{-4}
0.4	$0.000 \times 10^{+10}$	2.386×10^{-3}
0.6	$0.000 \times 10^{+10}$	4.950×10^{-3}
0.8	$0.000 \times 10^{+10}$	8.309×10^{-3}
1.0	$0.000 \times 10^{+00}$	1.241×10^{-2}

Table 4. Comparison of the absolute errors for Example 4.2

X	LSCCT $u_1(x)$	ADM [18]
0.0	$0.000 \times 10^{+00}$	$0.000 \times 10^{+00}$
0.2	8.000×10^{-12}	2.400×10^{-3}
0.4	6.400×10^{-11}	3.307×10^{-3}
0.6	2.160×10^{-10}	5.331×10^{-3}
0.8	5.120×10^{-10}	7.662×10^{-3}
1.0	1.000×10^{-9}	1.029×10^{-2}

Table 5. Comparison of the absolute errors for Example 4.3

X	LSCCT $u_1(x)$	ADM [18]
0.0	$0.000 \times 10^{+00}$	7.776×10^{-6}
0.2	$0.000 \times 10^{+00}$	4.852×10^{-4}
0.4	$0.000 \times 10^{+00}$	7.838×10^{-4}
0.6	$0.000 \times 10^{+00}$	1.540×10^{-3}
0.8	$0.000 \times 10^{+00}$	2.488×10^{-3}
1.0	$0.000 \times 10^{+00}$	3.609×10^{-3}

Table 6. Comparison of the absolute errors for Example 4.3

X	LSCCT $u_1(x)$	ADM [18]
0.0	$0.000 \times 10^{+00}$	7.858×10^{-8}
0.2	$0.000 \times 10^{+00}$	4.413×10^{-5}
0.4	$0.000 \times 10^{+00}$	1.267×10^{-5}
0.6	$0.000 \times 10^{+00}$	5.604×10^{-5}
0.8	$0.000 \times 10^{+00}$	1.609×10^{-4}
1.0	$0.000 \times 10^{+00}$	3.647×10^{-4}

6 Conclusion

All the work presented in this study were solved using Maple 18, It can be seen from Figures 1-6 that the required approximate solutions obtained and the exact solutions are in good agreement, also tables 1 – 6 show that the new method performed more accurately than that of [18] with less rigorous work in terms of computational cost. Findings have shown that there is no other place in literature where the table of results is presented other than [18]. Many researchers have proposed numerical approach for solving fractional differential equations, whereas numerical approach for solving Integro-differential equations is relatively new. Therefore, the proposed method can be adopted for solving FIDEs since they performed favorably well.

Acknowledgement: Authors thank those who contributed to write this article and give some valuable comments.

Funding Statement: The authors received no specific funding for this study.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

References

[1] L. Adam, Fractional Calculus: history, Definition and application for the engineer, Department of Aerospace and Mechanical Engineering University of Notre Dame, IN 46556, U.S.A., 2004.

[2] S. Alkan, V.F. Hatipoglu, Approximate solutions of Volterra- Fredholm integro-differential Equations of fractional order, Tbilisi Mathematical Journal, 10 (2) 2017, 1-13.

[3] Y.A. Amr, M. S. Mahdy, E. S. M Youssef, Solving fractional integro-differential equations by using Sumudu transform method and Hermite spectral collocation method, Computers, Materials and Continua, 54 (2) 2018, 161-180.

[4] R. B. Adeniyi, On Tau method for the Numerical solution of ordinary differential equation, Ph.D. Thesis, University of Ilorin, Ilorin, Nigeria, 1991.

[5] D. Aysegul , V. B. Dilek, Solving fractional Fredholm integro-differential equations by Laguerre polynomials, Sains Malaysiana, 48(1) 2019, 251-257.

[6] F. Awawdeh, E.A. Rawashdeh, H.M. Jaradat, Analytic solution of fractional integro-differential Equations, Annals of the University of Craiova-Mathematics and Computer Science Series, 38 (1), 2011, 1-10.

[7] M. Caputo, Linear models of dissipation whose Q is almost frequency Independent, Geophysical Journal International, 13 (5) 1967, 529 –539.

[8] V. B. Dilkel, D. Aysegül, Applied collocation method using Laguerre polynomials as the basis Functions, Advances in difference equations a Springer Open Journal, 2018, 1-11.

[9] C. Edwards, Math 312 Fractional calculus final presentation, Accessed 20 Sep. 2018.

[10] X. Ma, C. Huang, Numerical solution of fractional integro- differential equations by a Hybrid collocation method, Applied Mathematics and Computation , 219 (12) 2013, 6750–6760.

[11] X. Ma, C. Huang, Spectral collocation method for linear fractional integro- differential Equations, Applied Mathematical Modelling, 38 (4) 2014, 1434-144.

[12] A. M. S. Mahdy, R. M. H. Mohamed, Numerical studies for solving system of linear fractional integro-differential equations by using least squares method and shifted Chebyshev polynomials, Journal of Abstract and Computational Mathematics, 1 (24) 2016, 24-32.

[13] D. Sh. Mohammed, Numerical solution of fractional integro- differential equations by least squares method and shifted Chebyshev polynomial, Mathematical Problems in Engineering, Article ID 431965, (1) 2014.

[14] D. Sh. Mohammed, Numerical solution of fractional singular integro-differential equations by using Taylor series expansion and Galerkin method, Journal of Pure and Applied Mathematics: Advances and Applications, 12 (2) 2014, 129-143.

[15] S. Nemati, S. Sedaghatb, I. Mohammadi, A fast numerical algorithm based on the second Kind Chebyshev polynomials for fractional integro-differential equations with weakly singular Kernels, Journal of Computational and Applied Mathematics, 308, 2016, 231-242.

[16] T. Oyedepo, A. F. Adebisi, M. T. Raji, M. O. Ajisope, J. A. Adedeji, J. O. Lawal, O. A. Uwaheren, Bernstein modified Homotopy perturbation method for the solution of Volterra Fractional integro- differential equations, Pasifi Journal of Science and Technology, 22(1) 2021, 30-36.

[17] H.M. Osama, A. A. Sarmad, Approximate solution of Fractional integro- differential equations by using Bernstein polynomials, Engineering and Technology Journal, 30 (8) 2012, 1362-1373.

[18] M.H. Saleh, S. H. Mohamed, M. H. Ahmed, and M. K. Marjan, System of linear fractional integro-differential equations by using Adomian decomposition method, International Journal of Computer Applications, 121 (24) 2015, 9–19.

[19] A. Setia, Y. Liu, A. S. Vatsala, Numerical solution of Fredholm- Volterra Fractional integro- differential equation with nonlocal boundary conditions", Journal of Fractional Calculus and Applications, 5 (2) 2014, 155-165.

[20] O.A. Taiwo, Collocation approximation for singularly perturbed boundary value problems, Ph.D. Thesis, University of Ilorin, Nigeria, 1991.

[21] M. Yi, J. Huang, CAS wavelet method for solving the fractional integro-differential equation with a weakly singular kernel, International Journal of Computer Mathematics, 92(8), 2015, 1715-1728.