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Legendre Wavelets Technique for Special Initial-Value Heat Transfer Problem in the Quarter Plane

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Abstract : In this work we have solved the heat transfer equation by means of the Volterra integral equation and Legendre Wavelets. Indeed, numerical facts show that resolving the related partial differential equation is difficult which motivates the choice of this approach. The integral equation corresponding to this system is a Volterra type of the first kind. These systems are ill posed, therefore an appropriate methods to solve this type of systems is wavelets approach, since wavelets can be generated in the space of solutions. In this work we use Legendre wavelets to solve the corresponding integral equation. Numerical implementation of the method is illustrated by benchmark problems originated from heat transfer. The time evolution of the initial heat function along with the $x-$axis is exhibited in a three dimensional plot.

Keywords : Volterra integral equation of the first kind; Heat equation; Numerical solution; Legendre Wavelets.
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1 Introduction

In this paper, we consider the following special initial-value problem describing the heat transfer on a quarter plane in one spatial dimension

$$u_t = u_{xx}, \qquad 0 < x < \infty, \quad 0 < t, \tag{1.1}$$

$$u(x,0) = f(x), \quad 0 < x < \infty, \tag{1.2}$$

$$\int_0^{s(t)} u(x,t)dx = g(t), \qquad 0 < t, \; 0 < s(t), \tag{1.3}$$

and

$$|u(x,t)| \leq C_1 \exp\left\{C_2 |x|^{1+\gamma}\right\}, \qquad \gamma < 1. \tag{1.4}$$

Where $u(x,t)$ is the unknown temperature function $C_i, i = 1, 2$, are positive constants and f, g and s are continuous functions.

In [6, 7] the authors have solved similar problems using product integration technique, which is a good method in the case of short time intervals and the second kind integral equation [15, 12, 1, 2, 3, 4, 5]. But the product integration is not efficient in the case of integral equations of the first kind therefore we solve this problem by wavelets in the interval $[0, 1]$ and show the efficiency of the method through two simple

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examples. More applications of Legendre wavelets for the integral equations of the first kind can be found in [9, 18].

2 Equivalent Integral Equation

This section includes some definitions, lemmas and theorems associated to equations (1.1)-(1.4).

Definition 2.1. *The fundamental solution of heat equation is denoted by $K(x,t)$, and the Green's function is denoted by $G(x,\xi,t)$,*

$$K(x,t) := \frac{1}{\sqrt{4\pi t}} \exp\left\{-\frac{x^2}{4t}\right\},\ G(x,\xi,t) := K(x-\xi,t) - K(x+\xi,t).$$

Here and throughout this paper, $lhs := rhs$, means that lhs is defined by the rhs, similarly $lhs =: rhs$, means that rhs is defined by the lhs.

Lemma 2.2. *For any integrable function f satisfies $|f(x)| \leq C_1 \exp\{C_2 x^2\}$, where C_1 and C_2 are positive constants, $\lim_{t\downarrow 0} \int_{-\infty}^{\infty} K(x-\xi,t) f(\xi) d\xi = f(x)$, and $0 < t$, at the point x of continuity of f.*

Proof. See Lemma 3.4.3 of [11]. $\square$

Theorem 2.3. *For continuous functions f, g and s with $g(0) = \int_0^{s(0)} f(\xi) d\xi$ the solution u of equations* (1.1)-(1.3)*, satisfying a growth condition of the form* (1.4) *has the following representation*

$$u(x,t) = \int_0^{\infty} G(x,\xi,t) f(\xi) d\xi - 2\int_0^t \frac{\partial K}{\partial x}(x, t-\tau)\phi(\tau) d\tau, \tag{2.1}$$

if and only if ϕ is a piecewise-continuous solution of integral equation,

$$2\int_0^t [K(0,t-\tau) - K(s(t),t-\tau)]\phi(\tau) d\tau = g(t) - \int_0^{s(t)} \int_0^{\infty} G(x,\xi,t) f(\xi) d\xi dx,\ 0 < t. \tag{2.2}$$

Proof. We are going to search $u(x,t) = u_1(x,t) + u_2(x,t)$, in such a way that u_1, u_2 satisfy the heat equation and each of them obeys one of the equations (1.2), (1.3). For this aim, let $u_1(x,t) = -2\int_0^t \frac{\partial K}{\partial x}(x, t-\tau)\phi(\tau) d\tau$, $u_2(x,t) = \int_0^{\infty} G(x,\xi,t) f(\xi) d\xi$. Based on [11], chapter one, both of u_1 and u_2 are solution of equation (1.1). Lemma 2.2 yields $u(x,0) = u_2(x,0) = \lim_{t\downarrow 0} \int_0^{\infty} G(x,\xi,t) f(\xi) d\xi = \lim_{t\downarrow 0} \int_{-\infty}^{\infty} K(x-\xi,t) f_o(\xi) d\xi = f(x)$, where f_o is the odd extension of f to $-\infty < x < \infty$. Eq. (1.3) implies

$$\begin{aligned} g(t) &= \int_0^{s(t)} \left[\int_0^{\infty} G(x,\xi,t) f(\xi) d\xi - 2\int_0^t \frac{\partial K}{\partial x}(x,t-\tau)\phi(\tau) d\tau\right] dx \\ &= \int_0^{s(t)} \int_0^{\infty} G(x,\xi,t) f(\xi) d\xi dx - 2\int_0^t \int_0^{s(t)} \frac{\partial K}{\partial x}(x,t-\tau)\phi(\tau) dx d\tau \\ &= \int_0^{s(t)} \int_0^{\infty} G(x,\xi,t) f(\xi) d\xi dx - 2\int_0^t \phi(\tau) \left[K(x,t-\tau)\right]_{x=0}^{x=s(t)} d\tau \\ &= \int_0^{s(t)} \int_0^{\infty} G(x,\xi,t) f(\xi) d\xi dx - 2\int_0^t \phi(\tau) \left[K(s(t),t-\tau) - K(0,t-\tau)\right] d\tau, \end{aligned} \tag{2.3}$$

equivalent to the equation (2.2). Note that in row 2 we apply Fubini's theorem. By consideration of chapter3 of [11] the solution u is unique in the class (1.4) and hence the proof is completed. $\square$

3 Legendre Wavelet technique

Wavelets were first applied in geophysics to analyze data from seismic surveys, used in oil and mineral explorations to get "pictures" of layering in the subsurface rock [10]. There are several bases for wavelets, such as Haar wavelet, Daubechies wavelets, Chebyshev wavelets, Legendre wavelets, etc [10, 17, 8, 16, 13]. In this work we consider the Legendre wavelets, which belong to an orthonormal set of functions with respect to the weight function $w(t) = 1$, in the interval $[0, 1]$, as follow

$$\psi_{nm}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}2^{k/2}P_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}} \\ 0, & otherwise \end{cases}, \quad t \in [0, 1], \tag{3.1}$$

where $n = 1, ..., 2^{k-1}$, k is an integer, m is the degree of Legendre polynomial P_m, $m = 0, 1, ..., M-1$, for some positive integer M. A function $f \in L^2[0, 1]$, can be represented as series of Legendre wavelets

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} f_{nm}\psi_{nm}(t), \quad t \in [0, 1], \tag{3.2}$$

where $f_{nm} =< f, \psi_{nm} >$, is the inner product of f and ψ_{nm} in the Hilbert space $L^2[0, 1]$. In numerical processes, we consider the following approximation

$$f(t) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} f_{nm}\psi_{nm}(t) = F^T\Psi(t) =: P_{k-1}^M(f(t)), \quad t \in [0, 1], \tag{3.3}$$

where

$$\begin{aligned} F &= \left[f_{10}, f_{11}, ..., f_{1,M-1}, f_{20}, f_{21}, ..., f_{2,M-1}, ..., f_{2^{k-1},0}, ..., f_{2^{k-1},M-1}\right]^T \\ &= \left[f_1, f_2, ..., f_M, f_{M+1}, ..., f_{2^{k-1}M}\right]^T, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \Psi(t) &= \left[\psi_{10}(t), \psi_{11}(t), ..., \psi_{1,M-1}(t), \psi_{20}(t), \psi_{21}(t), ..., \psi_{2,M-1}(t), ..., \psi_{2^{k-1},0}(t), ..., \psi_{2^{k-1},M-1}(t)\right]^T \\ &= \left[\psi_1(t), \psi_2(t), ..., \psi_M(t), \psi_{M+1}(t), ..., \psi_{2^{k-1}M}(t)\right]^T, \end{aligned} \tag{3.5}$$

and for the simplicity of numerical evaluations, we rearrange indices in the second representation of vectors by the mapping $([\frac{i-1}{M}] + 1, i - M[\frac{i-1}{M}] - 1) \to i,\ 1 = 1, ..., 2^{k-1}M$, where $[x]$ denotes the greatest integer less than or equal to x. Now, we are going to give a theorem on convergence analysis of the approximated equation (3.3). For this aim, let us define $V_k^M := \left\{\psi_{nm} : n = 1, ..., 2^k, m = 0, 1, ..., M-1\right\}$, then

Theorem 3.1. *Let $f \in C^M[0, 1]$ and $P_k^M(f(t)) \in V_k^M$, then*

$$\left|f(t) - P_k^M(f(t))\right| \leq M_1 2^{-M(k+2)} \max_{\xi\in[0,1]} \left|f^{(M)}(\xi)\right|,$$

where M_1,is a constant, and P_k^M is defined by (3.3).

Proof. See Theorem 2.4 of [14]. □

4 Numerical solution of Weakly singular Volterra integral equation

In this section in order to solve numerically the integral equation (2.2), we use Legendre wavelets. For given functions f, g and s, the right hand side function $rhs(t) = \sqrt{\pi}\left(g(t) - \int_0^{s(t)} \int_0^{\infty} G(x,\xi,t) f(\xi) d\xi dx\right)$ is known. So the integral equation (2.2) yields

$$\int_0^t \left[1 - \exp\left\{-\frac{s(t)^2}{4(t-\tau)}\right\}\right] \frac{\phi(\tau)}{\sqrt{t-\tau}} d\tau = rhs(t), \ 0 < t. \tag{4.1}$$

Using Eq. (3.3) for approximate $\phi(t) \simeq \Phi^T \Psi(t)$ and $rhs(t) \simeq R^T \Psi(t)$ in Eq. (4.1), implies

$$\Phi^T \int_0^t \left[1 - \exp\left\{-\frac{s(t)^2}{4(t-\tau)}\right\}\right] \frac{\Psi(\tau)}{\sqrt{t-\tau}} d\tau = R^T \Psi(t), \quad t \in [0,1], \tag{4.2}$$

where $\Phi^T = [c_1, c_2, ..., c_{2^{k-1}M}]^T$, is an unknown vector. Let $w(t) = \int_0^t \left[1 - \exp\left\{-\frac{s(t)^2}{4(t-\tau)}\right\}\right] \frac{\Psi(\tau)}{\sqrt{t-\tau}} d\tau$, then from Eq. (3.3) we obtain $w(t) \simeq W\Psi(t)$, where W is a $2^{k-1}M \times 2^{k-1}M$ known matrix. Substituting these quantities in (4.2) yields, $\Phi^T W \Psi(t) = R^T \Psi(t)$. Hence, the linear system $W^T \Phi = R$, must be solved.

5 Numerical results

Example 5.1. *In the problem* (1.1)-(1.4)*, for* $f(x) = 1, s(t) = \sqrt{t}$*,*

$$g(t) = \frac{t}{2} + \begin{cases} \left(1 - \exp\{\frac{t}{1-4t}\}\right)\sqrt{\frac{-1+4t}{\pi}} + \sqrt{t} Erfc\left(\sqrt{\frac{t}{-1+4t}}\right), & \frac{1}{4} < t \\ 0, & otherwise \end{cases},$$

and $rhs(t) = \begin{cases} \left(1 - \exp\{\frac{t}{1-4t}\}\right)\sqrt{-1+4t} + \sqrt{\pi t} Erfc\left(\sqrt{\frac{t}{-1+4t}}\right), & \frac{1}{4} < t \\ 0, & otherwise, \end{cases}$*, the integral equation associated to this problem is,*

$$\int_0^t \left[1 - \exp\left\{-\frac{t}{4(t-\tau)}\right\}\right] \frac{\phi(\tau)}{\sqrt{t-\tau}} d\tau = rhs(t),$$

which has the exact solution $\phi(t) = \begin{cases} 1, & \frac{1}{4} < t \\ 0, & otherwise \end{cases}$*. The column 2 of Table 1 shows absolute errors of* $\widetilde{\phi}$ *at* $t = 0.15i, i = 1,2,3,4,5,6$*,* ϕ *is exact solution and* $\widetilde{\phi}$ *is evaluated by Legendre Wavelets technique with* $M = 5, k = 3$*. Figure 1 shows the time variation of these solutions for the Example 5.1.*
Columns $3,4,5,6,7,8$ *of the Table 1 show absolute errors of* $\widetilde{u}$ *at* $(x,t) = (0.15i, 0.15j), i,j = 1,2,3,4,5,6$*,* u *is the exact solution and* $\widetilde{u}$ *is the approximated solution evaluated numerically substituting* ϕ *by* $\widetilde{\phi}$ *in* u *representation Eq.* (2.1)*. Here* $e_{ij}, i,j = 1,2,3,4,5,6$ *is the absolute error of* $\widetilde{u}$ *at* $(0.15i, 0.15j)$*.*

Example 5.2. *In the problem* (1.1)-(1.4)*, for* $f(x) = 1, s(t) = \sqrt{t}, g(t) = \frac{t}{2} + \frac{t^{5/2}}{60\sqrt{\pi}}\left(64 - \frac{102}{e^{1/4}} + 81\sqrt{\pi} Erfc\{\frac{1}{2}\}\right)$*, and* $rhs(t) = \frac{t^{5/2}}{60}\left(64 - \frac{102}{e^{1/4}} + 81\sqrt{\pi} Erfc\{\frac{1}{2}\}\right)$*, the integral equation associated to this problem is,*

$$\int_0^t \left[1 - \exp\left\{-\frac{s(t)^2}{4(t-\tau)}\right\}\right] \frac{\phi(\tau)}{\sqrt{t-\tau}} d\tau = rhs(t),$$

Table 1: Absolute errors of $\widetilde{\phi}$ and $\widetilde{u}$ for the Example 5.1.

i	$\lvert\phi-\widetilde{\phi}\rvert_i$	e_{i1}	e_{i2}	e_{i3}	e_{i4}	e_{i5}	e_{i6}
1	*negligible*	*negligible*	3.22×10^{-14}	2.18×10^{-14}	2.21×10^{-10}	3.75×10^{-10}	2.69×10^{-8}
2	*negligible*	*negligible*	1.22×10^{-15}	1.63×10^{-15}	8.66×10^{-11}	1.00×10^{-10}	4.76×10^{-9}
3	*negligible*	*negligible*	5.41×10^{-16}	1.48×10^{-16}	4.69×10^{-11}	3.73×10^{-11}	7.43×10^{-9}
4	1.33×10^{-13}	*negligible*	3.53×10^{-16}	1.22×10^{-17}	1.84×10^{-11}	8.60×10^{-12}	3.55×10^{-10}
5	5.17×10^{-11}	*negligible*	1.32×10^{-16}	1.10×10^{-17}	5.59×10^{-12}	1.52×10^{-13}	6.67×10^{-11}
6	6.64×10^{-12}	*negligible*	3.67×10^{-17}	1.08×10^{-18}	1.06×10^{-12}	1.29×10^{-12}	4.19×10^{-12}

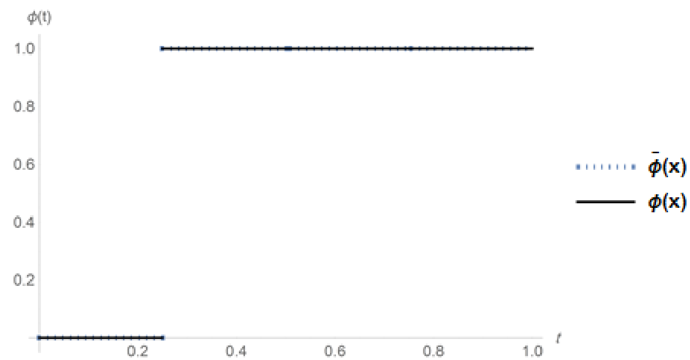

Figure 1: Variations of $\phi(t)$ and $\widetilde{\phi}(t)$ as functions of t for the Example 5.1.

which has the exact solution $\phi(t) = t^2$*. The column 2 of the Table 2 shows absolute errors of* $\widetilde{\phi}$ *at* $t = 0.15i, i = 1, 2, 3, 4, 5, 6$*,* ϕ *is the exact solution and* $\widetilde{\phi}$ *is evaluated by Legendre Wavelets technique with* $M = 5, k = 3$*. Figure 3 shows the time evolution of these solutions for the Example 5.2.*
Columns $3, 4, 5, 6, 7, 8$ *of the Table 2 show absolute errors of* $\widetilde{u}$ *at* $(x, t) = (0.15i, 0.15j), i, j = 1, 2, 3, 4, 5, 6$*,* u *is the exact solution and* $\widetilde{u}$ *is the approximated solution evaluated numerically substituting* ϕ *by* $\widetilde{\phi}$ *in* u *representation Eq.* (2.1)*. Here* $e_{ij}, i, j = 1, 2, 3, 4, 5, 6$ *is the absolute error of* $\widetilde{u}$ *at* $(0.15i, 0.15j)$*.*

Table 2: Absolute errors of $\widetilde{\phi}$ and $\widetilde{u}$ for the Example 5.2.

i	$\lvert\phi-\widetilde{\phi}\rvert_i$	e_{i1}	e_{i2}	e_{i3}	e_{i4}	e_{i5}	e_{i6}
1	6.94×10^{-18}	2.33×10^{-15}	1.96×10^{-12}	1.07×10^{-12}	1.09×10^{-9}	8.65×10^{-9}	6.92×10^{-9}
2	7.22×10^{-16}	3.70×10^{-16}	1.72×10^{-13}	7.88×10^{-14}	5.41×10^{-11}	7.41×10^{-10}	1.36×10^{-9}
3	3.89×10^{-16}	1.19×10^{-16}	1.63×10^{-14}	4.27×10^{-15}	9.72×10^{-12}	1.62×10^{-11}	4.02×10^{-10}
4	5.98×10^{-13}	4.91×10^{-17}	2.26×10^{-15}	2.83×10^{-15}	9.34×10^{-12}	1.40×10^{-11}	1.14×10^{-10}
5	1.63×10^{-11}	2.34×10^{-17}	1.98×10^{-15}	1.66×10^{-15}	4.47×10^{-12}	7.93×10^{-12}	2.46×10^{-11}
6	1.46×10^{-12}	1.21×10^{-17}	6.86×10^{-16}	3.89×10^{-16}	1.63×10^{-12}	3.01×10^{-12}	1.05×10^{-12}

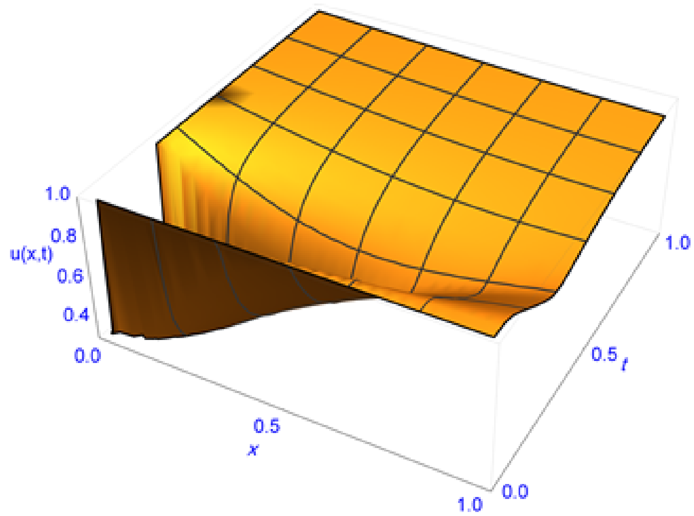

Figure 2: Variation of the $\widetilde{u}(x,t)$ as a function of (x,t) for the Example 5.1.

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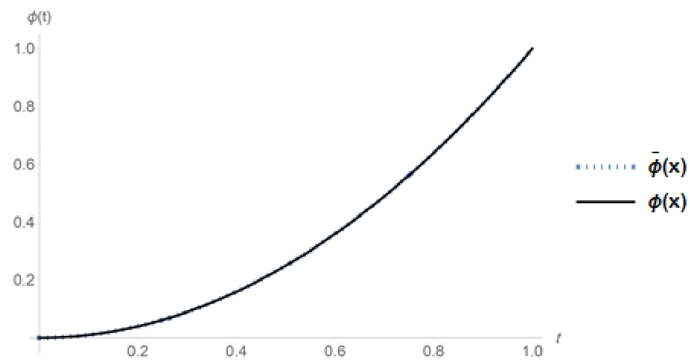

Figure 3: Variations of $\phi(t)$ and $\widetilde{\phi}(t)$ as functions of t for the Example 5.2.

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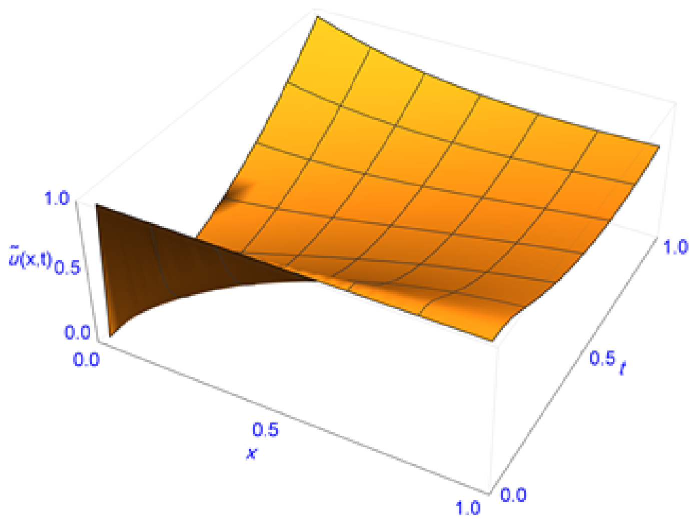

Figure 4: Variation of the $\widetilde{u}(x,t)$ as a function of (x,t) for the Example 5.2.