Vol 1(1), 2020, pp:16 -24
DOI:10.30511/mcs.2020.44463
ISSN:2717-2708
http://mcs.qut.ac.ir/

Coupled fixed point results for mappings without mixed monotone property in partially ordered G-metric spaces

S. Moradi [1], E. Analouei Adegani [2], M. Moradipour [3]

Abstract : In this paper, we prove some coupled fixed point theorems for nonlinear contractive mappings which do not have the mixed monotone property in partially ordered G-metric spaces.

Keywords : G-metric space, Coupled fixed point, Mixed monotone property.
2010 Mathematics Subject Classification : 47H10; 54H25.
Receive: 8 March 2020, **Accepted:** 5 August 2020

1 Introduction and Preliminaries

The metric fixed point theory is very important and useful in Mathematics. It can be applied in various areas, for example: variational inequalities, optimization, approximation theory, etc. The fixed point theorems in partially ordered metric spaces play a major role to prove the existence and uniqueness of solutions for some differential and integral equations, see, for example, [1, 2, 4, 6, 7, 8, 9, 12, 13] and references therein. At first we need the following definitions and results.

Definition 1.1 (see [11]). *Let X be a non-empty set, $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:*
$(G1)$ $G(x, y, z) = 0$ *if* $x = y = z$.
$(G2)$ $0 < G(x, x, y)$ *for all* $x, y \in X$ *with* $x \neq y$.
$(G3)$ $G(x, x, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$ *with* $y \neq z$.
$(G4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$ *(symmetry in all three variables).*
$(G5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $x, y, z, a \in X$ *(rectangle inequality).*
Then the function G is called a generalized metric, or, more specially, a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 1.2 (see [11]). *Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ is G-convergent to $x \in X$ if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, that is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim x_n = x$.*

[1]Corresponding author: Department of Mathematics, Faculty of Sciences, Lorestan University, Khorramabad, 68151-4-4316, Iran, moradi.s@lu.ac.ir
[2]Department of Mathematics, Faculty of Sciences, Shahrood University, Shahrood, Iran, analoey.ebrahim@gmail.com
[3]Department of Mathematics, Faculty of Sciences, Lorestan University, Khorramabad, 68151-4-4316, Iran, moradipour.mo@lu.ac.ir

Definition 1.3 (see [11]). *Let (X, G) be a G-metric space. The following are equivalent:*
(1) $\{x_n\}$ *is G-convergent to* x.
(2) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$.
(3) $G(x_n, x, x) \to 0$ *as* $n \to \infty$.
(4) $G(x_n, x_m, x) \to 0$ *as* $n, m \to \infty$.

Lemma 1.1 (see [11]). *Let (X, G) be a G-metric space, A sequence $\{x_n\}$ is called a G-Cauchy sequence if, for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $m, n, l \in \mathbb{N}$, that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.*

Lemma 1.2 (see [10]). *Let (X, G) be a G-metric space. Then the following are equivalent:*
(1) *the sequence $\{x_n\}$ is G-Cauchy*
(2) *for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$ for all $m, n \geq \mathbb{N}$.*

Lemma 1.3 (see [3]). *Let (X, G) and (X', G') be two G-metric space. A mapping $f : X \to X$ is G-continuous at $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G-convergent to $f(x)$.*

Definition 1.4 (see [11]). *A G-metric space (X, G) is called G-complete if every G-Cauchy sequence is G-convergent in (X, G).*

Definition 1.5 (see [10]). *A G-metric space (X, G) is called a symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.*

Definition 1.6 (see [11]). *Let (X, G) be a G-metric space. A mapping $F : X \times X \to X$ is said to be continuous if for any two G-convergent sequences $\{x_n\}$ and $\{y_n\}$ converging to x and y respectively, $\{F(x_n, y_n)\}$ is G-convergent to $F(x, y)$.*

The concept of a mixed monotone property has been introduced by Bhaskar and Lakshmikantham in [2].

Definition 1.7 (see [2]). *Let $(X, \preceq)$ be a partially ordered set. A mapping $F : X \times X \to X$ is said to have mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y; that is, for any $x, y \in X$,*

$$x_1, x_2 \in X, x_1 \preceq x_2 \quad \textit{implies} \quad F(x_1, y) \preceq F(x_2, y),$$
$$y_1, y_2 \in X, y_1 \preceq y_2 \quad \textit{implies} \quad F(x, y_2) \preceq F(x, y_1).$$

Lakshmikantham and Ćirić in [7] introduced the concept of a g-mixed monotone mapping.

Definition 1.8 (see [7]). *Let $(X, \preceq)$ be a partially ordered set. Let us consider mappings $F : X \times X \to X$ and $g : X \to X$. The map F is said to have mixed g-monotone property if $F(x, y)$ is monotone g-non-decreasing in x and is monotone g-nonincreasing in y; that is, for any $x, y \in X$,*

$$x_1, x_2 \in X, gx_1 \preceq gx_2 \quad \textit{implies} \quad F(x_1, y) \preceq F(x_2, y),$$
$$y_1, y_2 \in X, gy_1 \preceq gy_2 \quad \textit{implies} \quad F(x, y_2) \preceq F(x, y_1).$$

Definition 1.9 (see [2]). *An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if*

$$F(x, y) = x \quad \textit{and} \quad F(y, x) = y.$$

Definition 1.10 (see [7]). *An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if*

$$F(x, y) = gx \quad \textit{and} \quad F(y, x) = gy.$$

Definition 1.11 (see [7]). *We say that mappings $F : X \times X \to X$ and $g : X \to X$ are commutative if*

$$g(F(x,y)) = F(gx, gy) \quad \forall x, y \in X.$$

Definition 1.12 (see [5]). *If elements x, y of a partially ordered set $(X, \preceq)$ are comparable (i.e.,$x \preceq y$ or $y \succeq x$ holds) we will write $x \asymp y$. Let $g : X \to X$ and $F : X \times X \to X$. We will consider the following condition, if $x, y, u, v \in X$ are such that*

$$gx \asymp F(x,y) = gu \ \ then \ \ F(x,y) \asymp F(u,v). \tag{1.1}$$

In particular, when $g = I_X$, it reduces to for all $x, y, v \in X$ if

$$x \asymp F(x,y) \ \ then \ \ F(x,y) \asymp F(F(x,y), v). \tag{1.2}$$

Dorić et al. [5, Eexample 2.2], has shown by a simple example that these conditions may be satisfied when F does not have the g-mixed monotone property.

The main purpose of this paper is to prove some coupled fixed point theorems for nonlinear contractive mappings which do not have the mixed monotone property in partially ordered G-metric spaces.

2 Main result

First, we state a definition and next prove our main results.

Definition 2.1. *Let X be a G-metric space let $g : X \to X$, $F : X \times X \to X$. The mappings g and F are said to be G-compatible if*

$$\lim_{n\to\infty} G(gF(x_n, y_n), F(gx_n, gy_n), F(gx_n, gy_n)) = 0 \ \ and$$
$$\lim_{n\to\infty} G(gF(y_n, x_n), F(gy_n, gx_n), F(gy_n, gx_n)) = 0,$$

hold whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n$.

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X, G) is a complete G-metric space. Suppose that there exist $k \in [0,1)$, $g : X \to X$ and $F : X \times X \to X$ such that*

$$G(F(x,y), F(u,v), F(z,w)) \leq k \ max\{G(gx, gu, gz), G(gy, gv, gw)\}, \tag{2.1}$$

for all $x, y, u, v, z, w \in X$ with $gx \asymp gu \asymp gz$ and $gy \asymp gv \asymp gw$. Suppose also that g is continuous and $g(X)$ is closed and $F(X \times X) \subseteq g(X)$ and g and F are compatible and F and g satisfy property (1.1). *Suppose that either*

- *(a) F is continuous,*
- *(b) if $x_n \to x$ when $n \to \infty$ in X, then $x_n \asymp x$ for n sufficiently large.*

If there exist $x_0, y_0 \in X$ such that $gx_0 \asymp F(x_0, y_0)$ and $gy_0 \asymp F(y_0, x_0)$, then g and F have a coupled coincidence point,that is, there exist $u, v \in X$ such that $gu = F(u,v)$ and $gv = F(v,u)$.

Proof. Let $x_0, y_0 \in X$ such that $gx_0 \asymp F(x_0, y_0)$ and $gy_0 \asymp F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Again since $F(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Continuing in this way we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $n \geq 0$,

$$g(x_{n+1}) = F(x_n, y_n) \quad and \quad g(y_{n+1}) = F(y_n, x_n). \tag{2.2}$$

Now we prove that for all $n \geq 0$

$$g(x_n) \asymp g(x_{n+1}) \quad and \quad g(y_n) \asymp g(y_{n+1}). \tag{2.3}$$

We shall use the mathematical induction. Let $n = 0$. Since $gx_0 \asymp F(x_0, y_0)$ and $gy_0 \asymp F(y_0, x_0)$, in view of $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$, we have $g(x_0) \asymp g(x_1)$ and $g(y_0) \asymp g(y_1)$, that is, (2.3) hold for $n = 0$. We presume that (2.3) hold for some $n > 0$. As F and g have property (1.1) and $g(x_n) \asymp g(x_{n+1})$, $g(y_n) \asymp g(y_{n+1})$, from (2.2), we get

$$\begin{aligned} g(x_{n+1}) &= F(x_n, y_n) \asymp F(x_{n+1}, y_{n+1}) = g(x_{n+2}), \\ g(y_{n+1}) &= F(y_n, x_n) \asymp F(y_{n+1}, x_{n+1}) = g(y_{n+2}). \end{aligned} \tag{2.4}$$

Then from (2.4) we obtain

$$g(x_{n+1}) \asymp g(x_{n+2}) \quad and \quad g(y_{n+1}) \asymp (y_{n+2}). \tag{2.5}$$

Thus by the mathematical induction, we conclude that (2.5) hold for all $n \geq 0$.

If for some n, we have $(gx_{n+1}, gy_{n+1}) = (gx_n, gy_n)$, then $F(x_n, y_n) = gx_n$ and $F(y_n, x_n) = gy_n$, that is, F and g have a coincidence point. So from now on, we assume $(gx_{n+1}, gy_{n+1}) \neq (gx_n, gy_n)$, for all $n \in N$, that is, we assume that either $gx_{n+1} = F(x_n, y_n) \neq gx_n$ or $gy_{n+1} = F(y_n, x_n) \neq gy_n$.
Since $g(x_n) \asymp g(x_{n-1})$ and $g(y_n) \asymp g(y_{n-1})$, from contractive condition (2.1), we have

$$\begin{aligned} G(gx_{n+1}, gx_{n+1}, gx_n) &= G(F(x_n, y_n), F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\leq k\ max\{G(gx_n, gx_n, gx_{n-1}), G(gy_n, gy_n, gy_{n-1})\}, \\ G(gy_{n+1}, gy_{n+1}, gy_n) &= G(F(y_n, x_n), F(y_n, x_n), F(y_{n-1}, x_{n-1})) \\ &\leq k\ max\{G(gy_n, gy_n, gy_{n-1}), G(gx_n, gx_n, gx_{n-1})\}, \end{aligned}$$

and hence

$$\begin{aligned} max\ \{G(gx_{n+1}, gx_{n+1}, gx_n), G(gy_{n+1}, gy_{n+1}, gy_n)\} \leq\ & k\ max\{G(gx_n, gx_n, gx_{n-1}), \\ & G(gy_n, gy_n, gy_{n-1})\}, \end{aligned}$$

for each $n \in \mathbb{N}$. By induction, we get that

$$\begin{aligned} max\ \{G(gx_{n+1}, gx_{n+1}, gx_n), G(gy_{n+1}, gy_{n+1}, gy_n)\} \leq\ & k^n\ max\{G(gx_1, gx_1, gx_0), \\ & G(gy_1, gy_1, gy_0)\}, \end{aligned}$$

Now, we shall show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. For all positive integeres $n, m \in \mathbb{N}$, $n < m$ we have by the rectangele inqulity ($G5$ of Definition 1.1) that

$$\begin{aligned} G(gx_m, gx_m, gx_n) &\leq G(gx_m, gx_m, gx_{m-1}) + G(gx_{m-1}, gx_{m-1}, gx_{m-2}) \\ &\quad +... + G(gx_{n+1}, gx_{n+1}, gx_n) \\ &\leq k^{m-1}\ max\{G(gx_1, gx_1, gx_0), G(gy_1, gy_1, gy_0)\} \\ &\quad k^{m-2}\ max\{G(gx_1, gx_1, gx_0), G(gy_1, gy_1, gy_0)\} \\ &\quad +... + k^n\ max\{G(gx_1, gx_1, gx_0), G(gy_1, gy_1, gy_0)\} \\ &= (k^{m-1} + k^{m-2} + ... + k^n) max\{G(gx_1, gx_1, gx_0), G(gy_1, gy_1, gy_0)\} \\ &= k^n(1 + k + ... + k^{m-n-1}) max\{G(gx_1, gx_1, gx_0), G(gy_1, gy_1, gy_0)\} \\ &< \frac{k^n}{1-k} max\{G(gx_1, gx_1, gx_0), G(gy_1, gy_1, gy_0)\}, \end{aligned}$$

that is, $G(gx_m, gx_m, gx_n) < \frac{k^n}{1-k} max\{G(gx_1, gx_1, gx_0), G(gy_1, gy_1, gy_0)\}$.
Thus $\lim_{m,n\to\infty} G(gx_m, gx_m, gx_n) = 0$, and similarly $\lim_{m,n\to\infty} G(gy_m, gy_m, gy_n) = 0$. Therefore, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences and, since $g(X)$ is closed in a complete metric space there exists $u, v \in g(X)$ such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} F(x_n, y_n) = u \quad and \quad \lim_{n\to\infty} gy_n = \lim_{n\to\infty} F(y_n, x_n) = v. \tag{2.6}$$

Compatibility of g and F implies that

$$\begin{aligned} &\lim_{n\to\infty} G(gF(x_n, y_n), F(gx_n, gy_n), F(gx_n, gy_n)) = 0, \\ &\lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n), F(gy_n, gx_n)) = 0. \end{aligned} \tag{2.7}$$

Suppose that assumption (a) holds. Using triangle inequality, we have

$$\begin{aligned} G(gu, F(gx_n, gy_n), F(gx_n, gy_n)) \quad &\leq \quad G(gu, gF(x_n, y_n), gF(gx_n, y_n)) \\ &\quad + G(gF(x_n, y_n), F(gx_n, gy_n), F(gx_n, gy_n)). \end{aligned} \tag{2.8}$$

Taking the limit when $n \to \infty$ and using (2.9) and continuity of g and F we get that $G(gu, F(u, v), F(u, v)) = 0$, that is, $gu = F(u, v)$. Similarly, we can show that $F(v, u) = gv$.

Finally, suppose that (b) holds. Since $gx_n \to u$ and $gy_n \to v$ and $u, v \in g(X)$. we get that $gx_n \asymp u = gx$ and $gy_n \asymp v = gy$ for some $x, y \in X$ and n sufficiently large. For such n, using (2.1) we have

$$\begin{aligned} G(F(x, y), gx, gx) \quad &\leq \quad G(F(x, y), gx_{n+1}, gx_{n+1}) \\ &\quad + G(gx_{n+1}, gx, gx) \\ &= \quad G(F(x, y), F(x_n, y_n), F(x_n, y_n)) \\ &\quad + G(gx_{n+1}, gx, gx) \\ &\leq \quad k\ max\{G(gx, gx_n, gx_n), G(gy, gy_n, gy_n)\} \\ &\quad + G(gx_{n+1}, gx, gx) \end{aligned} \tag{2.9}$$

Taking the limit when $n \to \infty$, we get that $G(F(x, y), gx, gx) = 0$. Hence $gx = F(x, y)$ and similarly $gy = F(y, x)$.
Note that in this case continuity (g) and compatibility (g, F) assumptions were not needed in the proof. □

Corollary 2.2. *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X, G) is a complete G-metric space. Suppose that there exist $k \in [0, 1)$ and $F : X \times X \to X$ such that*

$$G(F(x, y), F(u, v), F(z, w)) \leq k\ max\{G(x, u, z), G(y, v, w)\}, \tag{2.10}$$

for all $x, y, u, v, z, w \in X$ with $x \asymp u \asymp z$ and $y \asymp v \asymp w$. Suppose also F satisfy property (1.2). Suppose that either

(a) F is continuous,

(b) if $x_n \to x$ when $n \to \infty$ in X, then $x_n \asymp x$ for n sufficiently large.

If there exist $x_0, y_0 \in X$ such that $x_0 \asymp F(x_0, y_0)$ and $y_0 \asymp F(y_0, x_0)$, then F has a coupled coincidence point,that is, there exist $u, v \in X$ such that $u = F(u, v)$ and $v = F(v, u)$.

Now we shall prove the existence and uniqueness theorem of a coupled common fixed point. If $(X, \preceq)$ is a partially ordered set, we endow the product set $X \times X$ with the partial order $\preceq$ defined by

$$(x, y) \rhd (u, v) \Leftrightarrow x \preceq u \ \ and \ \ v \preceq y. \tag{2.11}$$

Theorem 2.3. *In addition to the hypotheses of Theorem 2.1, suppose that,*
(c) for every $(x,y),(u,v)\in X\times X$, there exists $(w,z)\in X\times X$ such that $(F(w,z),F(z,w))$ is comparable to both $(F(x,y),F(y,x))$ and $(F(u,v),F(v,u))$.
Then F and g have a unique common coupled fixed point, that is, there exists a unique $(p,q)\in X\times X$ such that $p=gp=F(p,q)$ and $q=gq=F(q,p)$.

Proof. From Theorem 2.1, the set of coupled coincidences is non-empty. We shall show that if (x,y) and (u,v) are coupled coincidence points, that is, if $gx=F(x,y)$, $gy=F(y,x)$, $gu=F(u,v)$ and $gv=F(v,u)$, then

$$gx=gu \quad and \quad gy=gv. \tag{2.12}$$

By assumption, there exists $(w,z)\in X\times X$ such that $(F(w,z),F(z,w))$ is comparable to both $(F(x,y),F(y,x))$ and $(F(u,v),F(v,u))$. Without restriction to the generality, we can assume that

$$\begin{aligned}(F(x,y),F(y,x)) &\quad\triangleright\quad (F(w,z),F(z,w))\\ (F(u,v),F(v,u)) &\quad\triangleright\quad (F(w,z),F(z,w)).\end{aligned}$$

Put $w_0=w$, $z_0=z$ and choose $w_1,z_1\in X$ such that $gw_1=F(w_0,z_0)$ and $gz_1=F(z_0,w_0)$. Then, similarly as in the proof of Theorem 2.1, we can inductively define sequences $\{gw_n\}$ and $\{gz_n\}$ in X by

$$gw_{n+1}=F(w_n,z_n) \quad and \quad gz_{n+1}=F(z_n,w_n) \tag{2.13}$$

for $n\in\mathbb{N}$. Starting from $x_0=x$, $y_0=y$ and $u_0=u$, $v_0=v$, choose sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$, $\{v_n\}$, satisfying $gx_n=F(x_{n-1},y_{n-1})$, $gy_n=F(y_{n-1},x_{n-1})$, $gu_n=F(u_{n-1},v_{n-1})$ and $gv_n=F(v_{n-1},u_{n-1})$ for $n\in\mathbb{N}$, taking into account properties of coincidence points, it is easy to see that it can be done so that $x_n=x$, $y_n=y$ and $u_n=u$, $v_n=v$, that is,

$$gx_n=F(x,y),\quad gy_n=F(y,x) \;\; and \;\; gu_n=F(u,v),\quad gv_n=F(v,u) \;\; for\ n\in\mathbb{N}. \tag{2.14}$$

Since

$$(F(x,y),F(y,x))=(gx_1,gy_1)=(gx,gy)\triangleright(F(w,z),F(z,w))=(gw_1,gz_1),$$

then $gx\asymp gw_1$ and $gy\asymp gz_1$. Using that F satisfy property (1.2), one can show easily that $gx\asymp gw_n$ and $gy\asymp gz_n$ for all $n\geq 1$. Thus, from (2.1), we get

$$\begin{aligned}G(gw_{n+1},gx,gx) &= G(F(w_n,z_n),F(x,y),F(x,y))\\ &\leq k\ max\{G(gw_n,gx,gx),G(gz_n,gy,gy)\},\\ G(gz_{n+1},gy,gy) &= G(F(z_n,w_n),F(y,x),F(y,x))\\ &\leq k\ max\{G(gz_n,gy,gy),G(gw_n,gx,gx)\}.\end{aligned}$$

Therefore,

$$\max\{G(gw_{n+1},gx,gx),G(gz_{n+1},gy,gy)\} \quad\leq\quad k\ max\{G(gw_n,gx,gx),G(gz_n,gy,gy)\},$$

and by induction

$$\max\{G(gw_{n+1},gx,gx),G(gz_{n+1},gy,gy)\} \quad\leq\quad k^n\ max\{G(gw_1,gx,gx),G(gz_1,gy,gy)\},$$

Letting $n\to\infty$, in the above inequality we obtain

$$\lim_{n\to\infty}G(gw_{n+1},gx,gx)=0 \;\; and \;\; \lim_{n\to\infty}G(gz_{n+1},gy,gy)=0. \tag{2.15}$$

Similarly, one can show that

$$\lim_{n\to\infty} G(gw_{n+1}, gu, gu) = 0 \quad and \quad \lim_{n\to\infty} G(gz_{n+1}, gv, gv) = 0. \tag{2.16}$$

Therefore, from (2.15), (2.16) and the uniqueness of the limit, we get $gx = gu$ and $gy = gv$. So (2.12) holds.

Denote now $gx = p$ and $gy = q$, so we have that

$$gp = g(gx) = gF(x, y) \quad and \quad gq = g(gy) = gF(y, x). \tag{2.17}$$

By definition of the sequences $\{x_n\}$ and $\{y_n\}$ we have

$$gx_n = F(x, y) = F(x_{n-1}, y_{n-1}) \quad and \quad gy_n = F(y, x) = F(y_{n-1}, x_{n-1}),$$

and so

$$F(x_{n-1}, y_{n-1}) \to F(x, y) \quad and \quad gx_n \to F(x, y),$$
$$F(y_{n-1}, x_{n-1}) \to F(y, x) \quad and \quad gy_n \to F(y, x).$$

Compatibility of g and F implies that

$$G(gF(x_n, y_n), F(gx_n, gy_n), F(gx_n, gy_n)) \to 0, \quad n \to \infty,$$

so $gF(x, y) = F(gx, gy)$. From (2.17) we get that

$$gp = g(gx) = gF(x, y) = F(gx, gy) = F(p, q), \tag{2.18}$$

in a similar way,

$$gq = g(gy) = gF(y, x) = F(gy, gx) = F(q, p), \tag{2.19}$$

so $gp = F(p, q)$ and $gq = F(q, p)$. Thus, (p, q) is a coincidence point. Then, from (2.12) with $u = p$ and $v = q$, we have $gx = gp = p$ and $gy = gq = q$, that is,

$$gp = p \quad and \quad gq = q. \tag{2.20}$$

From (2.18), (2.19) and (2.20), we get

$$p = gp = F(p, q) \quad and \quad q = gq = F(q, p),$$

and (p, q) is a common coupled fixed point of g and F. To prove the uniqueness, assume that (x_1, x_2) is another coupled common fixed point. Then by (2.12), we have $x_1 = gx_1 = gp = p$ and $x_2 = gx_2 = gq = q$. □

Corollary 2.4. *In addition to the hypotheses of Corollary* 2.2 *we let condition* (*c*) *of Theorem* 2.3 *be satisfied. Then the coupled fixed point of* F *is unique. Moreover, if for the terms of sequences* $\{x_n\}$, $\{y_n\}$ *defined by* $x_n = F(x_{n-1}, y_{n-1})$ *and* $y_n = F(y_{n-1}, x_{n-1})$, $x_n \asymp y_n$ *holds for* n *sufficiently large, then the coupled fixed point of* F *has the form* (x, x).

Proof. Following the proof of Theorem 2.3 with $g = I_X$, we only have to show that $x = F(x, x)$. Suppose that for n sufficiently large, $x_n \asymp y_n$. Then, by (2.1), it follows that $G(F(x_n, y_n), F(x_n, y_n), F(y_n, x_n)) \leq k\ G(x_n, x_n, y_n)$. The triangle inequality ($G5$ of Definition 1.1) implies that

$$\begin{aligned}
G(x, x, y) &\leq G(x, x, x_{n+1}) + G(x_{n+1}, x_{n+1}, y) \\
&\leq G(x, x, x_{n+1}) + G(x_{n+1}, x_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+1}, y) \\
&= G(x, x, x_{n+1}) + G(F(x_n, y_n), F(x_n, y_n), F(y_n, x_n)) + G(y_{n+1}, y_{n+1}, y) \\
&\leq G(x, x, x_{n+1}) + k\ G(x_n, x_n, y_n) + G(y_{n+1}, y_{n+1}, y).
\end{aligned}$$

Passing to the limit when $n \to \infty$, since $x_n \to x$ and $y_n \to y$, we get that $(1 - k)G(x, x, y) < 0$ and so $G(x, x, y) = 0$ and hence $x = y$. □

Example 2.2. *Let $X = [0,1]$, Then $(X, \preceq)$ is a partially ordered set with the natural ordering of real numbers. let G be the G- metric on $X \times X \times X$ defined as follows:*

$$G(x,y,z) = max\{|x-y|, |y-z|, |z-x|\} \quad \forall \quad x,y,z \in X.$$

Define $g : X \to X$ by $gx = x^2$ and $F : X \times X :\to X$ by $F(x,y) = \frac{x^2+2y^2}{4}$.
All the conditions of Theorems 2.1 and 2.3 are satisfied. In particular, we will check that g and F are compatible. Let $\{x_n\}$ and $\{x_n\}$ be two sequences in X such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} F(x_n, y_n) = a \quad and \quad \lim_{n\to\infty} gy_n = \lim_{n\to\infty} F(y_n, x_n) = b.$$

Then $\frac{a+2b}{4} = a$ and $\frac{b+2a}{4} = b$, wherefrom it follows that $a = b = 0$. Then

$$\begin{aligned} & G(gF(x_n,y_n), F(gx_n,gy_n), F(gx_n,gy_n)) \\ = \;& G\Big((\frac{{x_n}^2+2{y_n}^2}{4})^2, \frac{{x_n}^4+2{y_n}^4}{4}, \frac{{x_n}^4+2{y_n}^4}{4} \Big) \\ = \;& max\Big\{ |(\frac{{x_n}^2+2{y_n}^2}{4})^2 - \frac{{x_n}^4+2{y_n}^4}{4}|, |\frac{{x_n}^4+2{y_n}^4}{4} - \frac{{x_n}^4+2{y_n}^4}{4}|, \\ & |\frac{{x_n}^4+2{y_n}^4}{4} - (\frac{{x_n}^2+2{y_n}^2}{4})^2| \Big\} \to 0 \ (n\to\infty), \end{aligned}$$

and similarly, $G(gF(y_n,x_n), F(gy_n,gx_n), F(gy_n,gx_n)) \to 0$.

$$\begin{aligned} & d(F(x,y), F(u,v), F(z,w)) \\ = \;& max\Big\{ |\frac{x^2+2y^2}{4}) - \frac{u^2+2v^2}{4}|, |\frac{u^2+2v^2}{4} - \frac{z^2+2w^2}{4}|, |\frac{z^2+2w^2}{4} - \frac{x^2+2y^2}{4}| \Big\} \\ \leq \;& max\Big\{ \frac{1}{4}|x^2-u^2| + \frac{2}{4}|y^2-v^2|, \frac{1}{4}|u^2-z^2| + \frac{2}{4}|v^2-w^2|, \frac{1}{4}|z^2-x^2| + \frac{2}{4}|w^2-y^2| \Big\} \\ \leq \;& max\Big\{ \frac{3}{4}max\{|x^2-u^2|, |y^2-v^2|\}, \frac{3}{4}max\{|u^2-z^2|, v^2-w^2|\}, \frac{3}{4}max\{|z^2-x^2|, |w^2-y^2|\} \Big\} \\ = \;& \frac{3}{4}max\Big\{ max\{|x^2-u^2|, |u^2-z^2|, |z^2-x^2|\}, max\{|y^2-v^2|, |v^2-w^2|, |w^2-x^2|\Big\} \\ = \;& \frac{3}{4}max\{G(gx,gu,gz), G(gy,gv,gw)\}. \end{aligned}$$

There exists a unique common coupled fixed point $(0,0)$ of the mappings g and F. Note that F does not satisfy the g-mixed monotone property. Also, g and F do not commute.

References

[1] R.P. Agarwal, M.A. El-Gebeily and D. O'Regan, Generalized contractions in partially ordered metric spaces, Applicable Analysis, 87 (2008), 1-8.

[2] T.G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Analysis, 65 (2006), 13791393.

[3] B.S. Choudhury and P. Maity, Coupled fixed point results in generalized metric spaces, Mathematical and Computer Modelling, 54(12) (2011), 73-79.

[4] S. Dalal and S. Chauhan, Coupled fixed points results for W-Compatible Maps in symmetric G-Metric spaces, Functional Analysis, Approximation and Computation, 7(1) (2018), 39-46.

[5] D. Doric, Z. Kadelburg and S. Radenovic, Coupled fixed point results for mappings without mixed monotone property, Applied Mathematics Letters, 25(11) (2012), 1803-1808.

[6] B. Hazarika, R. Arab and P. Kumam, Coupled fixed point theorems in partially ordered metric spaces via mixed g-monotone property, Journal of Fixed Point Theory and Applications, 21(1) (2019).

[7] V. Lakshmikantham and L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Analysis, 70 (2009), 43414349.

[8] S. Moradi, E. A. Adegani and D. Alimohammadi, Common fixed point theorems for maps under a new contractive condition, International Journal of Nonlinear Analysis and Applications, 4(2) (2013), 15-25.

[9] S. Moradi, Z. Fathi, and E. A. Adegani, The common fixed point of single-valued generalized φ_f-weakly contractive mappings, Applied Mathematics Letters, 24(5) (2011), 771-776.

[10] Z. Mustafa and B. Sims, Some remarks concerning D-metric spaces, in: Proceedings of the International Conference on Fixed Point Theory and Applications, Yokohama, Japan, (2004), 189198.

[11] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, Journal of Nonlinear and convex Analysis, 7(2) (2006), 289-297.

[12] M. Shoaib, M. Sarwar and T. Abdeljawad, Hybrid Coupled Fixed Point Theorems in Metric Spaces with Applications, Journal of Function Spaces, 2019 (2019), Article ID 6870308, 15 pages.

[13] G. Wang, C. Zhu, Z. Wu, Some new coupled fixed point theorems in partially ordered complete Menger probabilistic G-metric spaces, Journal of Computational Analysis and Applications, 27(2) (2019).