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QOM UNIVERSITY OF TECHNOLOGY

# Mathematical computation of quantum optical control systems

J. Sharifi[1]

**Abstract:** Some models of linear control system schemas are developed here for quantum linear systems. The most important linear devices in quantum optics are introduced with their differential equations. These linear quantum systems are zero-order and first-**order** transfer functions with one pole and one zero. We mathematical compute transfer function of different interconnections by using zero-order and first-order systems. for instance, by designing series and feedback interconnection, we will obtain higher-order quantum linear systems. Also, we will analyze a closed-loop feedback of a first-order linear quantum system containing a gain in feedback path.

**Keywords:** Quantum Control; Optical Cavity; Beam Splitter; Electro-Optic Modulator.
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## 1 Introduction

Some important class of systems are linear control systems because they are fundamental for studying more complex systems. Several control strategies were developed for those systems both in frequency and time domain. This type of control systems has applications in magnifico technologies and engineering such as electrical engineering, aerospace engineering, chemical engineering and etc. furthermore recently, a few control engineers and scientists [3,7,13,18-21] are fascinated by innovating of control theory in quantum domain for design and realization of relevant technologies such as quantum communication [2], quantum computer [17], quantum networks analysis [9], spintronics [8] and etc. Some control mythologies have been developed for open nonlinear quantum systems, such as optimal control equations [10,14] and Lyapunov stability analysis for quantum trajectories [1,6,12,15] and open quantum systems [16]. You can read a great review about quantum control in the field of molecular systems in reference [5].
However, the aim of linear control system is to analyses and design for simple control system which is depicted in figure1. We must note here that for physical realization of such control system, each block and also the

[1]Electrical and Computer Engineering Department, Qom University of Technology, Qom, Iran, sharifi@qut.ac.ir, jv.sharifi@gmail.com

connections have its relevant physical elements. For example, in DC motor control, the linear system is DC motor, controller is an electrical circuit and connections are done by wires with electrical current flow. Similarly, there are some physical devices for quantum control realization of such sample block diagram. For instance, the components must be connecting by laser light and plane high-reflective mirrors. We will describe these components in context of the paper.
The organization of paper is as follows: in section2, gives a concise description of the most fundamental quantum linear systems. In Section3, we exhibit how we can derive higher-order linear quantum systems from zero-order and first-order quantum systems. In section4, we analyze the closed-loop feedback quantum linear control system.

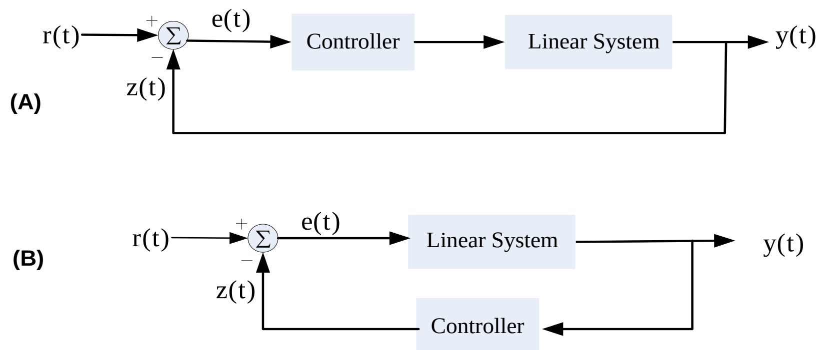


Figure 1: the schema of a linear control system: (A) control in forward path, (B) in feedback path

## 2 Mathematical model of Linear quantum optical elements

At first, we aim to introduce some important linear optical systems which are fundamental in many quantum control systems:

An **Electro-Optical Modulator** (EOM) is made from an electro-sensitive crystal. When a voltage applied on EOM, the incident light would modulate with some constant $k$ and then input-output relation for EOM are written as $a_{out}(t) = ka_{in}(t)$. Then an EOM can be considered as a gain controller in quantum linear control system.

A **beam splitter** (BS) transmits some portion of incident light beams and reflects the reminder portion of incident light beams. If the input light beams to BS be $a_{in}, b_{in}$ and output lights (reflected and transmitted) be $a_{out}, b_{out}$, then the input-output relation for BS written as:

$$\begin{bmatrix} a_{out} \\ b_{out} \end{bmatrix} = \begin{bmatrix} \beta & -\alpha \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} a_{in} \\ b_{in} \end{bmatrix} : \alpha, \beta \in \mathbf{R}, \ \alpha^2 + \beta^2 = 1 \quad (2.1)$$

really, the BS plays the role of summation and subtraction in quantum linear control system and it is counterpart of sigma notation in control block diagram (see figure1).

An **optical cavity** made from two parallel concave mirrors. At least one of the mirrors is partially reflexive in order to pass inside and outside some light beams from outside (inside) of cavity. This type of cavity is called one-sided cavity. Consider an optical cavity as it shown in figure2. If the resonating light inside cavity be $a(t)$, the input light at left of cavity $a_{in}(t)$ and output light from right side of cavity $a_{out}(t)$, then input-output relation is described by the following differential equation [4,9]:

$$\begin{aligned} da(t)/dt &= (-i\Delta - \tfrac{1}{2}\gamma)a(t) - \sqrt{\gamma}a_{in}(t) \\ a_{out}(t) &= \sqrt{\gamma}a(t) + a_{in}(t) \end{aligned} \quad (2.2)$$

In which $\gamma > 0$ is the coupling strength of cavity and $\Delta$ is called the detuning, i.e. the difference between the nominal external field frequency and the cavity mode frequency and $i = \sqrt{-1}$. By Laplace transform from equation (2.2), the cavity transfer function is obtained as:

$$H_{cav}(s) = a_{out}(s)/a_{in}(s) = (s - \bar{\lambda})(s + \lambda)^{-1} : \lambda = i\Delta + \tfrac{1}{2}\gamma, \bar{\lambda} = -i\Delta + \tfrac{1}{2}\gamma \quad (2.3)$$

Then a cavity is a first order linear optical system with a pole at $s_p = -\lambda$ and a zero at $s_z = \bar{\lambda}$.

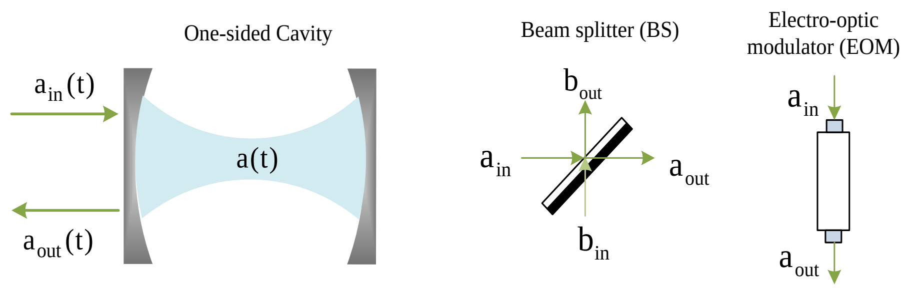


Figure 2: the schematics of an optical cavity, a beam splitter and an electro-optic modulator

# 3 Interconnections of quantum linear systems and higher-order

To construct higher order quantum linear systems, we can connect the zero-order and first-order quantum linear systems in different prototypes. For instance, the series connection of two optical cavities gives a two-order system with two poles and with two zeros. Also, feedback and parallel connections of two optical cavities gives a two-order system with two poles and two zeros. Two types of connections of linear system, i.e. the cascade (series) and feedback connections, are depicted in figure3. In cascade connection, the input laser light enters to cavity1 and the output light from cavity1 enters as an input light to cavity2 and the final output is the output light from cavity2. In feedback connection, the laser light splits by a BS and some portion of light enter to cavity1 and the output light from cavity1 enters as input light to cavity2, then the output light from cavity2 enters to BS to recombine with laser light to input to feedback to cavity1. Now, we will obtain the transfer function of these connections. In the case of cascade connection, the whole transfer function is obtained by multiplying of transfer function of each cavity, and then we will have:

$$H_{cascade}(s) = a_{out}(s)/a_{in}(s) = H_{cav1}(s)H_{cav2}(s) = (s - \bar{\lambda}_1)(s - \bar{\lambda}_2)(s + \lambda_1)^{-1}(s + \lambda_2)^{-1} \quad (3.1)$$

To obtain the transfer function of feedback connection in figure3-(B), by using the relations $\begin{bmatrix} a_{in1} \\ a_{out} \end{bmatrix} = \begin{bmatrix} \beta & -\alpha \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} a_{in} \\ a_{out2} \end{bmatrix}$ and $a_{out2}(s)/a_{in1}(s) = H_{cav1}(s)\,H_{cav2}(s) = (s-\bar{\lambda}_1)(s-\bar{\lambda}_2)(s+\lambda_1)^{-1}(s+\lambda_2)^{-1}$, we will obtain:

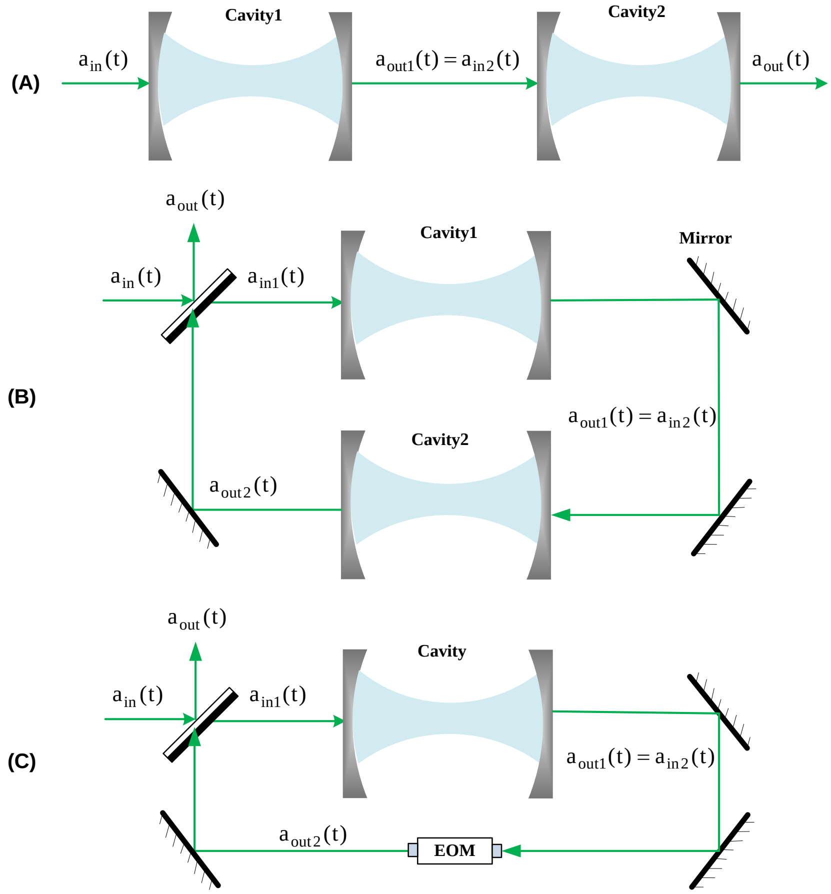


Figure 3: the schematics of (A) series or cascade, (B) feedback connections of two optical cavities and (C) feedback connections of an optical cavity to an EOM

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\begin{aligned}
H_{feedback}(s) = a_{out}(s)/a_{in}(s) &= \frac{\alpha(s+\lambda_1)(s+\lambda_2)+\beta^2(s-\bar{\lambda}_1)(s-\bar{\lambda}_2)}{(s+\lambda_1)(s+\lambda_2)+\alpha\beta(s-\bar{\lambda}_1)(s-\bar{\lambda}_2)} \\
&= \frac{(\alpha+\beta^2)s^2+\{\alpha(\lambda_1+\lambda_2)-\beta^2(\bar{\lambda}_1+\bar{\lambda}_2)\}s+\alpha\lambda_1\lambda_2+\beta^2\bar{\lambda}_1\bar{\lambda}_2}{(1+\alpha\beta)s^2+\{\lambda_1+\lambda_2-\alpha\beta(\bar{\lambda}_1+\bar{\lambda}_2)\}s+\lambda_1\lambda_2+\alpha\beta\bar{\lambda}_1\bar{\lambda}_2}
\end{aligned} \tag{3.2}
$$

which this transfer function has two poles and two zeros. Another feedback connection is by connecting of one optical cavity (first-order) to an EOM (gain). In this case, an EOM with gain $k$ would replace with cavity2. see figure3-(C). Then the transfer function is:

$$H_{feedback}(s) = a_{out}(s)/a_{in}(s) = \frac{(\alpha + k)s + \alpha\lambda - k\bar{\lambda}}{(1+\alpha k)s + \lambda - \alpha k\bar{\lambda}} \quad (3.3)$$

That is first-order system that we may set the gain in order to obtain a minimum phase system. For this system be minimum phase and stable the zero and pole real parts must be negative, i.e. $\Re(s_z) < 0$ , $\Re(s_p) < 0$ , then the inequality $\alpha - k < 0$ , $1-\alpha k < 0$ must be satisfied. But since $\alpha < 1$, thus for $k > \alpha^{-1}$ the closed loop system in figure3-(C) is minimum phase and stable.

We want to know that, may it possible to produce an oscillator from figure3-(B)? For this purpose, consider the transfer function of equation (3.2), then we must have $\alpha(\lambda_1 + \lambda_2) - \beta^2(\bar{\lambda}_1 + \bar{\lambda}_2) = 0 = \lambda_1 + \lambda_2 - \alpha\beta(\bar{\lambda}_1 + \bar{\lambda}_2)$ , thus the following result must be hold:

$$\alpha - \beta^2 = 0, 1-\alpha\beta = 0, \alpha + \beta^2 = 0, 1+\alpha\beta = 0 \quad (3.4)$$

Consequently, the solution of above relation is $\alpha = \beta = 0$ , which it is a contradiction. Then the oscillatory is not possible by configuration of figure3-(B).

# 4 Quantum Optical Linear Feedback System

In this section, we will analyze the quantum linear feedback of a cavity system. For this purpose, consider the schematic of control system which is illustrated in figure4. The equations of system are:

$$\begin{aligned} &a_{in1}(s) = \beta_1 a_{in}(s) - \alpha_1 a_{out3}(s) \text{ , } a_{out3}(s) = k a_{out2}(s) \text{ , } a_{out2}(s) = -\alpha_2 a_{out1}(s) \\ &a_{out}(s) = \beta_2 a_{out1}(s) \text{ , } a_{out1}(s) = (s - \bar{\lambda})(s+\lambda)^{-1} a_{in1}(s) \end{aligned} \quad (4.1)$$

Then by this Laplace domain equation, the closed-loop transfer function simply computes as:

$$H_{cl}(s) = a_{out}(s)/a_{in}(s) = \tfrac{\beta_1\beta_2}{1-k\alpha_1\alpha_2}(s - \bar{\lambda})\left\{s + \lambda(1+k\alpha_1\alpha_2)(1-k\alpha_1\alpha_2)\right\}^{-1} \quad (4.2)$$

Since the open-loop system has the transfer function $H_{op}(s) = a_{out1}(s)/a_{in1}(s) = (s-\bar{\lambda})(s+\lambda)^{-1}$, then by comparison with closed loop transfer function, someone will find that the electro-optical (gain) feedback, only can displace the pole of system from $s_p = \lambda$ to $s_p = -\lambda(1 + k\alpha_1\alpha_2)(1 - k\alpha_1\alpha_2)$ and the system zero is unchanged in $s_z = \bar{\lambda}$. If we want to displace the zero of this non-minimum phase system, we can add a forward path. To do this, we must use another EOM (gain) from second output of BS1, i.e. $a_{out4}$, to second input of BS2. In this situation, we will rename the EOMs to EOM1 and EOM2 with two gains $k_1$ , $k_2$ respectively. Then by a similar procedure someone can obtain the closed-loop transfer function for new system.

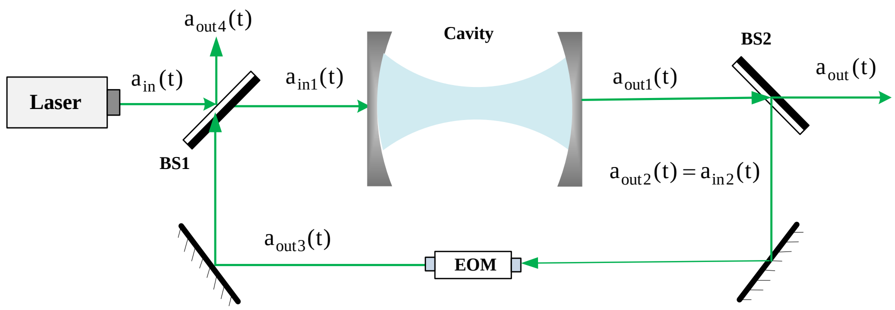


Figure 4: quantum linear feedback control schematic of an optical cavity with a gain in feedback path

## 5 Conclusion

We introduced some fundamental linear quantum systems. These systems have zero-order and first-order transfer functions in which in first order the pole and zero are conjugate of each other. It observed that the input-output transfer function of an optical cavity has a stable pole and unstable zero, and then the optical cavity system is a non-minimum phase system. Also, we found out that the series and feedback connections of two optical cavities can displace the poles of system but the zeroes remain unchanged. However, the feedback connection of an optical gain (EOM) in feedback path to an optical cavity in forward path can displace both the pole and the zero. In last section of the paper, we analyzed the closed-loop transfer function of a simple linear quantum system and observed that the closed loop system is non-minimum phase and as a future work we can design linear electro-optical feedback system to control non-minimum phase quantum linear systems.

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