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Numerical simulation and optimal fishing effort for a fishery with total allowable catch (TAC)

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Abstract: Fish stocks in the developing world are often depleted as a result of the application of excessive fishing effort on the part of the fishermen. A bioeconomic model with logistic growth, proportional harvesting and quadratic costs is proposed to study the effect of overcapacity on a marine fishery. Also incorporated into the model is an isoperimetric constraint to account for the annual total allowable catch (TAC). Pontryagin's maximum principle is employed to determine the necessary conditions for optimality of the model. Additionally, the sufficiency conditions that guarantee the existence and uniqueness of the optimality system are discussed. Furthermore, the relationship between the shadow price of fish stock, the shadow price of the total allowable catch and the marginal net revenue as it relates to the optimal fishing effort is explored. Numerical simulation with empirical data on the Ghana sardinella fishery is performed to validate the theoretical results. The findings of the study indicate that for a TAC equal to the maximum sustainable yield (MSY), the average fishing effort should not exceed 95% of the MSY effort, provided that the initial stock size is exactly 55% of the carrying capacity.

Keywords: Optimal fishing effort; Total allowable catch (TAC); Isoperimetric constraint; Shadow price; Numerical simulation
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1 Introduction

The management of a fishery becomes necessary as a result of the undesirable consequences of open access exploitation of the resource. Oftentimes, the emphasis of management has been on preventing overfishing by capping annual catches so as to prevent stock depletion, or in the case of already depleted stock, to arrange for stock rehabilitation [2]. Diaz-de-Leon and Seijo [6] assert that one of the main problems encountered in fishery resource management is in estimating strategies that simultaneously satisfy biological, economic and social objectives. The concept of total allowable catch (TAC) is a means of controlling yield by limiting the tonnage of fish or the number of fish that may caught from a fishery within a specified time period [4]. Economic theory suggests that instituting a TAC with individual fishing quotas removes the incentive for fishermen to race to fish, as pertains in an open access fishery [19, 30]. This intervention has the tendency of preventing overexploitation of the resource with its attendant socio-economic consequences.

In recent times there has been a lot of studies applying optimal control techniques to the management of fisheries [1, 5, 7, 16, 17, 18, 20, 23, 28, 29]. Chakraborty et al. [1] studied a predatory-prey model with

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prey migration in a two-patch environment: free fishing zone and protected zone. Their results indicated that the migration of the prey between the zones plays a significant role towards stock assessment and harvesting efficiency of the fishery. In [16], a Lotka-Volterra type fishery model with marine reserve and incorporating a critical biomass level is investigated. Findings from the study indicate that when the initial fish population is at least 30% of the carrying capacity and 40% of the current stock is in a reserve, the fishery can sustain an effort rate that is more than twice the maximum sustainable yield (MSY) rate. The independent harvesting of both predator and prey in a predator-prey system with prey refuge was investigated by Kar [17]. It was discovered that, considering the fishing efforts as control mechanisms, the cyclic behavior of the system could be disrupted thereby driving it to a desired state. Ladino et al. [20] explored a bioeconomic model involving a two-stage species with recruitment. With fishing effort serving as a control parameter, simulations were performed on the model for validation as well as determining the optimal effort necessary for sustainability of the resource. A predator-prey model with reserve area and harvesting was analyzed by Lv et al. [23]. Their investigation revealed that as long as the prey population in the reserve area does not go extinct, both prey populations will be extant. Srinivas et al. [29] studied a predator-prey model with stage-structure in a two-patch marine environment. They discussed the local and global stability of the model using a variational matrix and an appropriate Lyapunov function respectively. In addition, numerical simulations were performed on the model to validate the theoretical results.

However, there are hardly any fishery models incorporating an isoperimetric constraint. The only few models encountered utilizing an isoperimetric constraint in the course of reviewing the literature were from a non-fishery setting [10, 11]. Therefore, a fishery model with logistic growth and proportional harvesting that also includes an isoperimetric constraint is proposed. To the best of this author's knowledge, this is the first time a fishery model incorporates an isoperimetric constraint to account for the TAC. Furthermore, another novelty in this study is the exploration of the relationship between the shadow price of resource, the shadow price of TAC and the marginal net revenue as it relates to the optimal fishing effort.

In Section 2 of the study, the optimal control model is formulated comprising the biomass dynamics as well as the complete bioeconomic model. Analysis of the control problem, which consists of the characterization of the optimal control, as well as the existence and uniqueness of the optimality system is discussed in Section 3. Numerical and graphical illustrations of the model are portrayed in Section 4, while the summary and conclusion are discussed in Section 5.

2 Model Formulation

The formulation of the model takes into account the biological considerations as well as the economic objectives of fisheries management. In the first place, the biological dynamics are modelled employing a logistic growth curve with harvesting. This model, also known as the Schaefer model, has been shown to adequately describe the dynamics of a fish population [27]. Also incorporated in the model is an isoperimetric constraint denoting the TAC. Secondly, the complete bioeconomic model is formulated incorporating a quadratic cost function of the fishing effort, as opposed to the typical linear cost function [3, 12].

The state system consists of the logistic model with harvesting coupled with an isoperimetric constraint and is given by

$$\frac{dx(t)}{dt} = rx(t)\left(1 - \frac{x(t)}{K}\right) - qE(t)x(t), \quad x(0) = x_0\,, \tag{2.1}$$
$$\int_0^T qE(t)x(t)\,dt = B\,,$$

where $x(t)$ is the population size (or biomass) at time t, $E(t)$ denotes the rate of fishing effort at time t and x_0 is the initial biomass level. Additionally, r, q and K represent the intrinsic growth rate, the catchability coefficient and carrying capacity of the ecosystem, respectively. Finally, the constant B denotes the TAC at the finite time horizon, T. Note that the harvest (also known as catch or yield) is given by the *catch-effort relation*

$$h(t) = qE(t)x(t).$$

It has been established in [14] that the biomass at MSY, effort at MSY and the MSY of the Schaefer model (without the isoperimetric constraint) are respectively

$$\begin{aligned} x_{MSY} &= \frac{K}{2}, \\ E_{MSY} &= \frac{r}{2q}, \\ h_{MSY} &= \frac{rK}{4}. \end{aligned}$$

In this study the time horizon considered is one year, so the annual TAC is taken as a function of the MSY. Therefore,

$$\text{TAC} = \frac{\alpha r K}{4}, \qquad 0 < \alpha \leq 2.$$

It is worth noting that when the time horizon $T = 1$ year, $B = \dfrac{\alpha r K}{4}$.

One of the ways to convert the isoperimetric constraint in Equation (2.1) to a more familiar form is to introduce a second state variable $z(t)$ and let

$$z(t) = \int_0^t qE(s)x(s)\, ds.$$

Then, it follows from the fundamental theorem of calculus that

$$\frac{dz(t)}{dt} = qE(t)x(t),$$

$$z(0) = 0\,, \qquad z(T) = B.$$

Thus, Equation (2.1) is transformed into the state system

$$\begin{aligned} \frac{dx(t)}{dt} &= rx(t)\left(1 - \frac{x(t)}{K}\right) - qE(t)x(t), \quad x(0) = x_0\,, \\ \frac{dz(t)}{dt} &= qE(t)x(t), \quad z(0) = 0, \quad z(T) = B\,. \end{aligned} \tag{2.2}$$

Therefore, the state variable $z(t)$ denotes the TAC at any time t. Obviously, the TAC at the initial time is zero and the TAC at the final time is fixed (in this instance, the value is B). Incorporating economic parameters [9] into the system (2.2), the optimal control problem can be formulated as

$$\begin{aligned} \max_E \; J(E) &= \int_0^T e^{-\delta t}\left(pqx - c_1 - \frac{c_2}{2}E\right) E\, dt \\ \text{subject to} \quad \frac{dx}{dt} &= rx\left(1 - \frac{x}{K}\right) - qEx, \quad x(0) = x_0 \\ \frac{dz}{dt} &= qEx, \quad z(0) = 0, \quad z(T) = B \\ 0 &\leq E \leq E_{max}\,, \end{aligned} \tag{2.3}$$

where δ is the annual discount rate, p represents the price per unit harvest, c_1 and c_2 are the cost components relating to the fishing effort, and E_{max} is the maximum allowable fishing effort.

3 Analysis of Optimal Control Problem

The sufficiency conditions for the model are investigated and discussed. In particular, the existence of an optimal control is determined. Also, the characterization of the optimal control as well as the existence and uniqueness of the optimality system is analyzed.

3.1 Existence of optimal control

The goal, as stated in the objective functional, is to maximize the present-value of the net revenue. Thus, we seek an optimal control E^* such that

$$J(E^*) = \max\{J(E) \,\big|\, E \in U\},$$

where the control set is Lebesgue measurable for a finite time horizon, and defined by

$$U = \{E \,|\, 0 \le E(t) \le E_{max},\ t \in [0,T]\}.$$

As already intimated, in the solution of an optimal control problem, necessary and sufficient conditions of the problem need to be investigated and verified. Thus, conditions that are sufficient for the existence of an optimal control to the underlying problem are examined. To this end, a sufficiency result proposed by Fleming and Rishel [8] is invoked.

Theorem 3.1. *There exists an optimal control E^* that maximizes the objective functional J(E) over the control set U.*

Proof. To prove the given theorem, the following conditions must be established:

(i) The class of all initial conditions with a control E in the control set U together with the state system being satisfied is nonempty.

(ii) The control set is convex and closed.

(iii) The state system is bounded in the control and state variables.

(iv) The integrand of the objective functional is concave on U.

(v) There exist constants w_1, $w_2 > 0$ and $\eta > 1$ such that the integrand is bounded above by $w_1 - w_2|E|^\eta$.

To verify the first condition, the Picard-Lindelof existence theorem [22] guarantees the existence and uniqueness of a solution to a state system with bounded coefficients.

For verification of condition 2, by definition, the control set U is closed and convex. In order to verify condition 3, first, the boundedness of the solution to the state equation in $x(t)$ is determined using the comparison theory of differential equations and the theorem on differential inequalities. Since

$$x' = rx\left(1 - \frac{x}{K}\right) - qEx \le rx\left(1 - \frac{x}{K}\right)$$

for $0 \le t < \infty$ and $x_0 > 0$, then

$$x' \le rx\left(1 - \frac{x}{K}\right).$$

Thus, invoking Gronwall's inequality,

$$x(t) \le \frac{x_0 K}{x_0 + (K - x_0)e^{-rt}}.$$

Therefore, as $t \to \infty$,

$$0 \leq x \leq K.$$

Secondly, since $B = \dfrac{\alpha r K}{4}$, it implies that

$$0 \leq z \leq \frac{\alpha r K}{4}.$$

Therefore,

$$0 \leq x + z \leq \frac{K}{4}(\alpha r + 4).$$

Hence solutions to the state system are bounded and condition 3 is verified.

Next is to show that the integrand of the objective functional,

$$f(t,x,E) = e^{-\delta t} L(t,x,E) = e^{-\delta t}\left(pqxE - c_1 E - \frac{c_2}{2}E^2\right)$$

is concave on U. Then $f(t,x,E) \leq L(t,x,E)$, since $e^{-\delta t} > 0$ for $t \geq 0$.

Thus, the objective is to show that, for $0 \leq m \leq 1$,

$$mL(t,x,E_1) + (1-m)L(t,x,E_2) \leq L(t,x,mE_1 + (1-m)E_2).$$

The proof is commenced by noting that the difference of $mL(t,x,E_1) + (1-m)L(t,x,E_2)$ and $L(t,x,mE_1 + (1-m)E_2)$ is given by

$$\begin{aligned} & mL(t,x,E_1) + (1-m)L(t,x,E_2) - L(t,x,mE_1 + (1-m)E_2) \\ &\quad = mpqxE_1 - mc_1E_1 - m\frac{c_2}{2}E_1^2 + (1-m)pqxE_2 - (1-m)c_1E_2 - (1-m)\frac{c_2}{2}E_2^2 \\ &\quad - pqx[mE_1 + (1-m)E_2] + c_1[mE_1 + (1-m)E_2] + \frac{c_2}{2}[mE_1 + (1-m)E_2]^2 \end{aligned}.$$

The simplification of the right-hand-side gives

$$-\frac{c_2}{2}(m - m^2)(E_1 - E_2)^2 \leq 0,$$

since

$$(m - m^2) \geq 0 \quad \text{for} \quad 0 \leq m \leq 1\,.$$

Hence

$$mL(t,x,E_1) + (1-m)L(t,x,E_2) \leq L(t,x,mE_1 + (1-m)E_2).$$

This satisfies condition 4.

Finally, to verify condition 5, we note that x and z are bounded. So there exists a $G > 0$ such that $x \leq G$ and $E \leq G$ on $[0,T]$, where $G = \max(K, E_{max})$. Therefore,

$$\begin{aligned} e^{-\delta t}\left(pqxE - c_1E - \frac{c_2}{2}E^2\right) &\leq pqxE - c_1E - \frac{c_2}{2}E^2 \\ &\leq pqG^2 - \frac{c_2}{2}E^2 \\ &\leq w_1 - w_2E^2\,, \end{aligned}$$

where

$$w_1 = pqG^2, \quad w_2 = \frac{c_2}{2} \quad \text{and} \quad \eta = 2\,.$$ □

3.2 Characterization of optimal control

The optimal control will be characterized; that is, obtaining an explicit formulation for the optimal control level as well as for the optimality system. Since the existence of an optimal control to Problem (2.3) has already been established, to derive the necessary conditions for the optimal control, a version of Pontryagin's maximum principle [26] is employed.

Theorem 3.2. *Given an optimal control E^* and a solution to the corresponding state system, there exists adjoint variables λ_1 and λ_2 satisfying*

$$\lambda_1' = \left(\delta - r + \frac{2rx}{K}\right)\lambda_1 - (p - \lambda_1 + \lambda_2)qE\,, \tag{3.1}$$

$$\lambda_2' = \delta\lambda_2\,, \tag{3.2}$$

and the transversality condition,

$$\lambda_1(T) = 0\,.$$

Furthermore, E^* can be presented as

$$E^* = \min\left(E_{max}, \left(\frac{(p - \lambda_1 + \lambda_2)qx - c_1}{c_2}\right)^+\right)\,.$$

Proof. The current value Hamiltonian for the optimal control problem (2.3) is

$$H = \left(pqx - c_1 - \frac{c_2}{2}E\right)E + \lambda_1\left[rx\left(1 - \frac{x}{K}\right) - qEx\right] + \lambda_2 qEx\,. \tag{3.3}$$

Therefore, we obtain Equations (3.1) and (3.2) from the respective adjoint equations:

$$\lambda_1' = \delta\lambda_1 - \frac{\partial H}{\partial x}\,,$$
$$\lambda_2' = \delta\lambda_2 - \frac{\partial H}{\partial z}\,.$$

The optimality condition is given by

$$\frac{\partial H}{\partial E} = (p - \lambda_1 + \lambda_2)qx - c_1 - c_2E = 0\,.$$

Thus,

$$E^* = \frac{(p - \lambda_1 + \lambda_2)qx - c_1}{c_2}\,. \tag{3.4}$$

The characterization of the optimal control is

$$\begin{cases} E^* = 0 & \text{if} \quad \frac{\partial H}{\partial E} < 0\,, \\ 0 \le E^* \le E_{max} & \text{if} \quad \frac{\partial H}{\partial E} = 0\,, \\ E^* = E_{max} & \text{if} \quad \frac{\partial H}{\partial E} > 0\,. \end{cases}$$

By standard control arguments involving the bounds on the control,

$$E^* = \begin{cases} 0 & \text{if} \quad \lambda_1 - \lambda_2 > p - \dfrac{c_1}{qx}\,, \\ \dfrac{(p - \lambda_1 + \lambda_2)qx - c_1}{c_2} & \text{if}\ p - \dfrac{(c_1 + c_2E_{max})}{qx} \le \lambda_1 - \lambda_2 \le p - \dfrac{c_1}{qx}\,, \\ E_{max} & \text{if} \quad \lambda_1 - \lambda_2 < p - \dfrac{(c_1 + c_2E_{max})}{qx}\,. \end{cases} \tag{3.5}$$

This implies that the optimal control comprises both the boundary solutions (or binding constraints) and the interior solution. The boundary solutions indicate that the resource should be harvested if and only if the marginal net revenue of harvest as a result of applying the maximum effort exceeds the difference of the shadow price of fish stock and the shadow price of the introduced state variable, TAC. The shadow price of fish stock is positive (only zero at the terminal time), implying an additional tonne of stock will increase net revenue. On the hand, the shadow price of TAC is negative, reflecting the fact that any additional tonne of catch (beyond the TAC) will depress net revenue (the costs will exceed revenues). See Ibrahim [13] for more information.

Furthermore, the interior solution is applicable if the difference of the shadow prices lies between the two marginal revenues. In compact notation,

$$E^* = \min\left(E_{max}, \left(\frac{(p-\lambda_1+\lambda_2)qx - c_1}{c_2}\right)^+\right).$$

The optimality system comprises the characterization of the optimal together with the state equations and the adjoint equations, with initial and transversality conditions for the stock variable whereas the TAC variable has initial and terminal conditions.

Therefore,

$$\begin{aligned}
x' &= rx\left(1-\frac{x}{K}\right) - q\min\left(E_{max}, \left(\frac{(p-\lambda_1+\lambda_2)qx - c_1}{c_2}\right)^+\right)x\,,\\
z' &= q\min\left(E_{max}, \left(\frac{(p-\lambda_1+\lambda_2)qx - c_1}{c_2}\right)^+\right)x\,,\\
\lambda_1' &= \left(\delta - r + \frac{2rx}{K}\right)\lambda_1 - (p-\lambda_1+\lambda_2)q\min\left(E_{max}, \left(\frac{(p-\lambda_1+\lambda_2)qx - c_1}{c_2}\right)^+\right),\\
\lambda_2' &= \delta\lambda_2\,,
\end{aligned} \tag{3.6}$$

$$\text{with}\quad x(0)=x_0,\quad z(0)=0,\quad z(T)=B \quad\text{and}\quad \lambda_1(T)=0\,.$$ □

For purposes of numerical simulation, a necessary condition is that

$$B = \int_0^T qEx\,dt \le \int_0^T qE_{max}K\,dt = qE_{max}KT\,.$$

So

$$0 \le B \le qE_{max}KT\,. \tag{3.7}$$

In other words, Equation (3.7) establishes bounds for the TAC in a given finite horizon T.

The existence of the optimal control has already been established by Theorem 3.1. Next is the determination of the uniqueness of the optimal control. Given that the state and adjoint systems are a priori bounded, and also the state system is continuously differentiable, then through the application of the mean value theorem the state system satisfies the Lipschitz condition with regard to the state variables. Thus, the uniqueness of the optimality system for small time intervals is guaranteed [15, 21].

4 Numerical Simulation

In this section, a brief description of the custom-built MATLAB code implemented to solve the optimality system and the simulation results are presented.

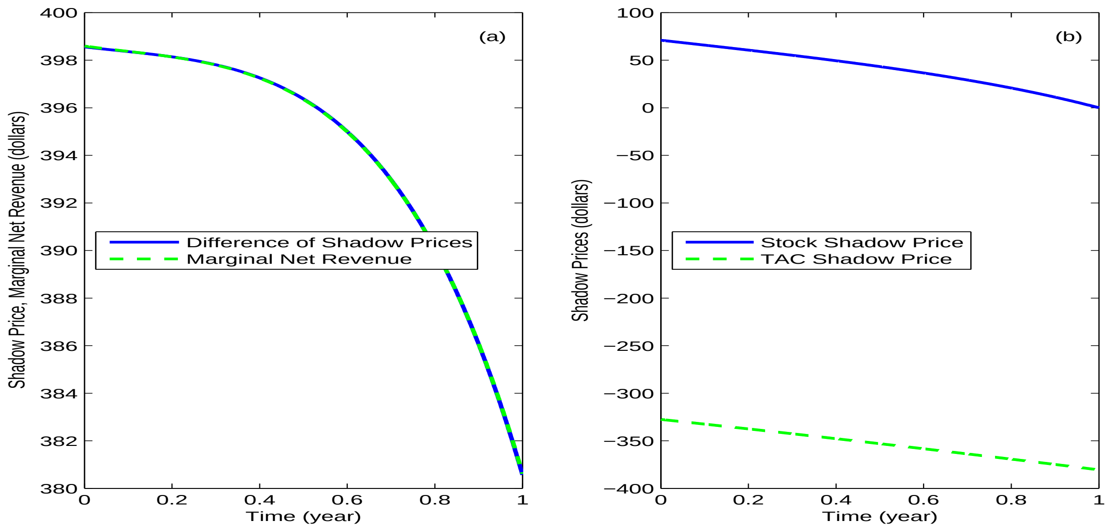

Figure 1: Shadow prices and marginal net revenue for $x_0 = 750,000$, $z(T) = 355,000$, $E_{max} = 800,000$ and $T = 1$

For validation of the model, empirical data on the Ghana sardinella fishery was employed [24, 25]. The biological parameter values used in the simulations are: $r = 1.42$/year, $q = 1.8 \times 10^{-6}$/trip/year and $K = 1 \times 10^6$ tonnes. On the other hand, the economic values are given by: $p = \$600$/tonne, $c_1 = \$195$/trip/year and $c_2 = \$2.47 \times 10^{-4}/\text{trip}^2$/year. In addition, the discount rate δ is estimated at 15% per year.

4.1 Numerical method

Simulations were carried out using an adapted version of the Forward-Backward Sweep Method (FBSM), which is an iterative method based on the Runge- Kutta Four order scheme. This method is utilized because the standard FBSM cannot solve problems containing an isoperimetric constraint, just as the constraint had to be amended before Pontryagin's maximum principle could be used to find the necessary conditions for the model.

From the optimality system (3.6), $\lambda_1(T) = 0$ and $\lambda_2(T)$ is unknown. Therefore, it is assumed that $\lambda_2(T) = \theta$ and the optimality system computed by solving the state equations forward in time and the adjoint equations backward in time (ignoring for the time being the fact that $z(T) = B$) until convergence occurs at the terminal time with $z(T) = \mu$. Thus, a new function $g(\theta) = \mu - B$ is defined and the problem now boils down to finding the zeros of $g(\theta)$ using the secant root-finding method. Finally, the adapted FBSM is employed to solve the optimality system when the zeros of $g(\theta)$ have been attained [11, 21].

4.2 Results and discussion

First of all, a number of simulations are performed with the TAC (represented by $z(T)$) set at the level of maximum sustainable yield (MSY), 355,000 tonnes, and the maximum rate of fishing effort at more than twice the MSY level of 394,444 fishing trips, while varying the initial population size. This high rate of effort, which is beyond the bifurcation point of model, is chosen to reflect the open-access nature

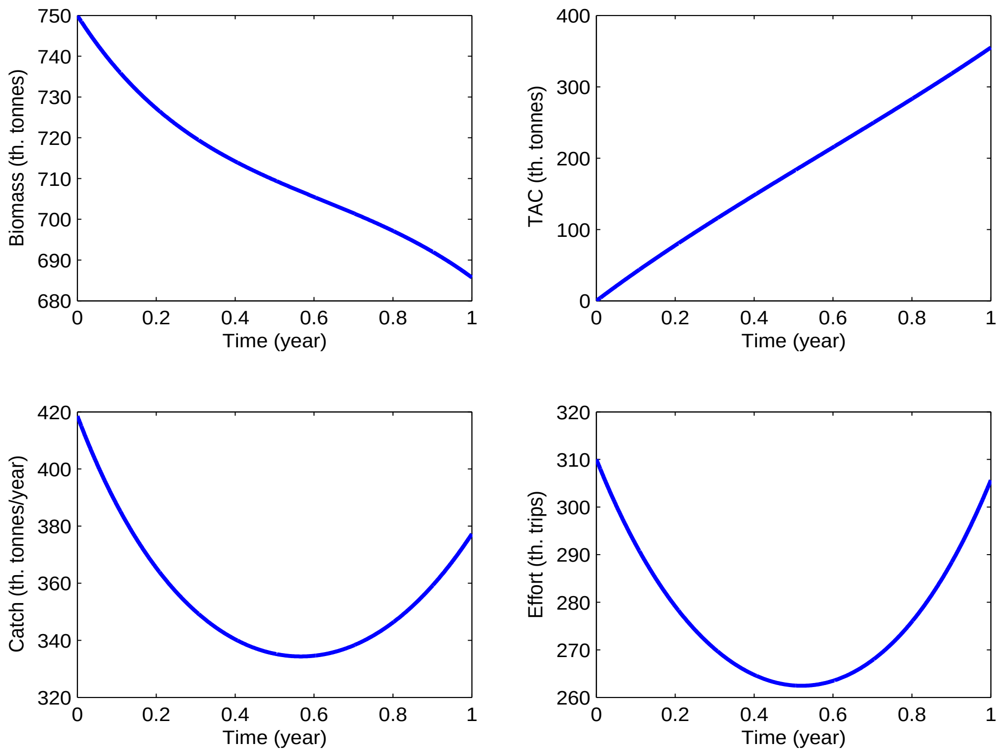

Figure 2: Effort, Catch, Biomass and TAC levels for $x_0 = 750,000$, $z(T) = 355,000$, $E_{max} = 800,000$ and $T = 1$

of the fishery. For further details, see Ibrahim [14]. Subsequently, the maximum rate of fishing effort is maintained at almost twice the MSY level, while simulations are carried out varying the TAC for the given time horizon of one year.

Figure 1 (a) portrays the situation where the effort is $800,000$ fishing trips and the TAC is $355,000$ tonnes, with an initial biomass level of $750,000$ tonnes. It shows the effect of the difference of the shadow prices of stock and TAC, as depicted in Equation (3.5), to be monotonically decreasing and concave. This scenario is also true for the marginal net revenue, which exactly intersects with the difference of the shadow prices. At the start of the harvesting process, the two curves start at a common value of \$398.56 and at the end of the one-year horizon, the value is \$380.56. This signifies that the revenue due to the effect of the difference of the shadow prices exactly matches the net revenue from harvesting a tonne of fish, so it is optimal to harvest at the interior rate of fishing effort. Since the difference of the shadow prices is neither lower nor higher than the net revenue for the entire horizon, the recommendation is to harvest, but not at the maximum rate of effort.

In Figure 1 (b), the shadow price of stock is strictly positive, except at the final horizon, indicating that adding an extra tonne of fish to the biomass will always increase the net revenue. The shadow price starts at \$71.01 and ends at zero at the final horizon. On the other hand, the shadow price of TAC is always negative, starting with a value of \$-327.55 and ending at \$-380.56. The negative value indicates that each additional tonne of fish, beyond the TAC, leads to a decrease in net revenue.

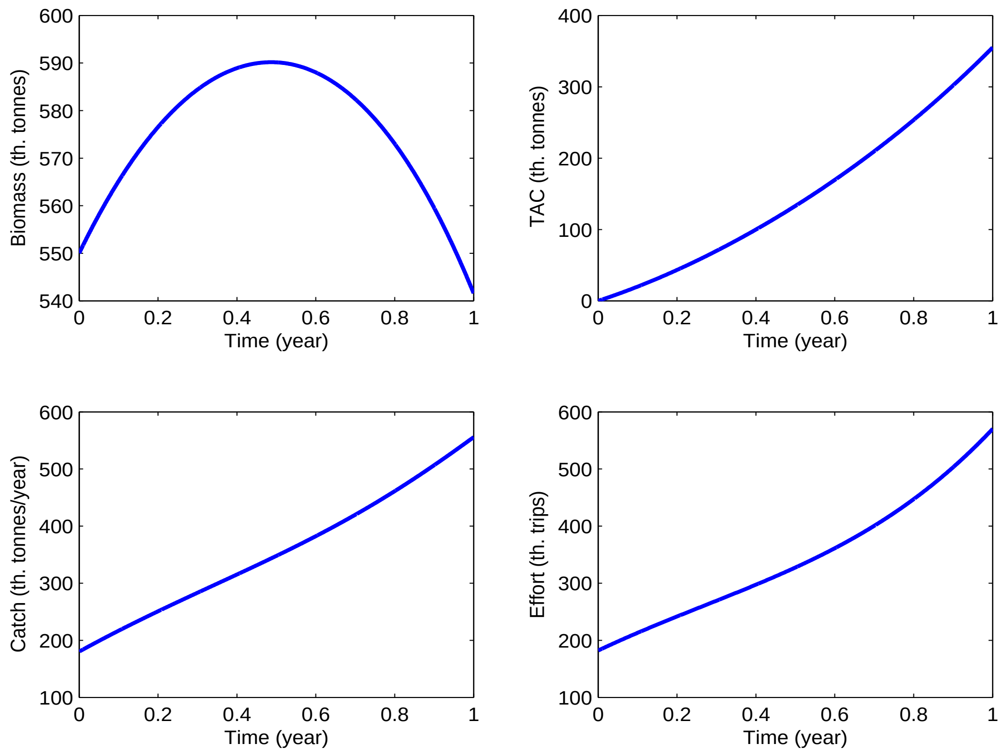

Figure 3: Effort, Catch, Biomass and TAC levels for $x_0 = 550,000$, $z(T) = 355,000$, $E_{max} = 800,000$ and $T = 1$

Figure 2 shows that when the maximum effort level E_{max} is set at more than twice the MSY level, the optimal effort level appears convex (see effort plot), starting from $309,443$ trips to $305,628$ trips, with a minimum value of $263,255$ trips for the one-year horizon. The catch plot also shows the catch rate to be convex, starting from an initial value of about $417,747$ tonnes per year to around $377,242$ tonnes per year, with the minimum being $335,236$ tonnes per year. Furthermore, the fish biomass decreases to a value of around $685,733$ tonnes (see biomass plot). The TAC plot shows that the TAC is almost linear, going from an initial value of zero to $355,000$ tonnes, which is the annual catch at the MSY level. A point worthy of note is that the annual catch corresponds to the area under the curve of the catch plot and is computed as $355,000$ tonnes. This value is exactly the TAC. The total net revenue over the one-year horizon corresponding to a TAC value equal to the MSY is computed as $\$139,040,000$.

In Figure 3, it is observed that when the maximum effort level E_{max} is set beyond the bifurcation point ($E = r/q$) of the Schaefer model, the optimal effort starts at $181,980$ trips and increases to $569,964$ trips at the end of the horizon. Thus, the average effort is $375,972$ trips, which translates into 95% of the effort at MSY. The catch rate increases to about $555,658$ tonnes per year, from an initial value of around $180,161$ tonnes per year. However, the fish biomass is concave, starting from $550,000$ tonnes to around $541,611$ tonnes, with a maximum of $590,207$ tonnes. On the other hand, the TAC is convex, going from an initial value of zero to $355,000$ tonnes. Given an initial population size of 55% of the carrying capacity and a TAC at the MSY, the total net revenue over the horizon with the effort rate is $\$119,500,000$.

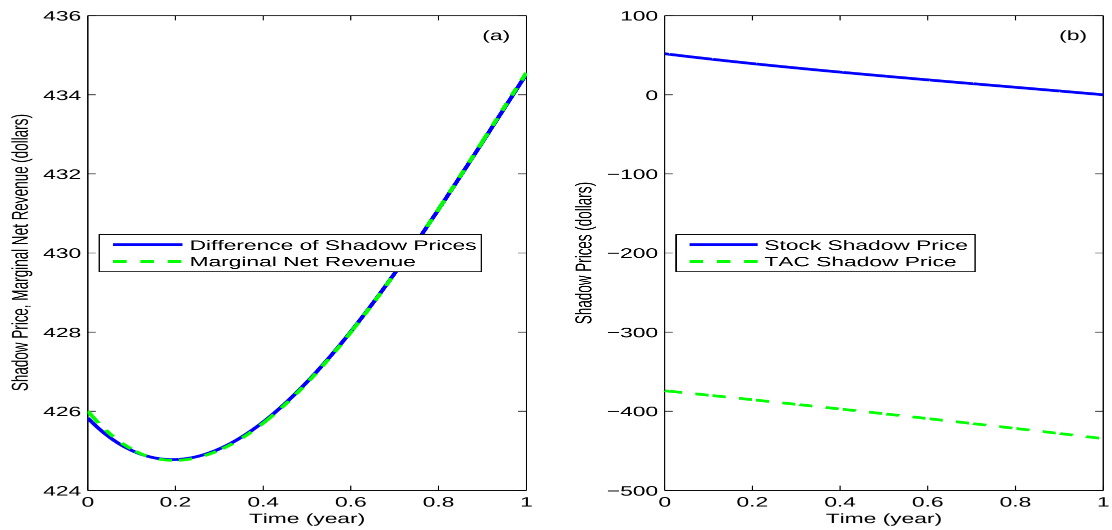

Figure 4: Shadow prices and marginal net revenue for $x_0 = 1,000,000$, $z(T) = 400,000$, $E_{max} = 800,000$ and $T = 1$

Figure 4 (a) portrays the scenario where the TAC is 13% higher than the MSY and the initial biomass level is at the carrying capacity. It shows the difference of the shadow prices as well as the marginal net revenue to be convex. Initially, the net revenue and the difference of the shadow prices have a common value of around \$425.83 and finally end at \$434.52. The shadow price of stock, as shown in Figure 4 (b), is nonnegative. It starts at \$51.84 and goes to zero at the final horizon. However, the shadow price of TAC is negative, starting with a value of \$-373.99 and ending at \$-434.52.

In Figure 5, it is observed that the optimal effort starts at $475,400$ trips and decreases monotonically to $161,700$ trips for the one-year horizon. The catch rate decreasing to around $229,800$ tonnes per year, from an initial value of about $855,800$ tonnes per year. Furthermore, the fish biomass decreases to about $789,500$ tonnes. In the TAC plot, the curve is increasing concave, starting from zero to $400,000$ tonnes. It is instructive to note that, even at the full carrying capacity, the optimal effort is not at the maximum. With an initial population at the carrying capacity and a TAC equal to 113% of the MSY, the total net revenue over the horizon is found to be $\$169,970,000$.

Figure 6 presents results for the scenario where the initial biomass is at half the carrying capacity and the TAC is pegged at about 70% of the MSY. It is seen that the optimal effort starts at $71,647$ trips and increases to $406,874$ trips over the horizon. As observed in the catch plot, the rate increases to about $435,660$ tonnes per year, from an initial value of around $64,483$ tonnes per year. However, the fish biomass is concave, starting from $500,000$ tonnes to around $594,862$ tonnes, with a maximum of $605,600$ tonnes. The TAC is convex, starting from an initial value of zero to a final value of $250,000$ tonnes. For an initial population size at half the carrying capacity and a TAC at 70% of the MSY, the total net revenue regarding this scenario is $\$87,667,000$.

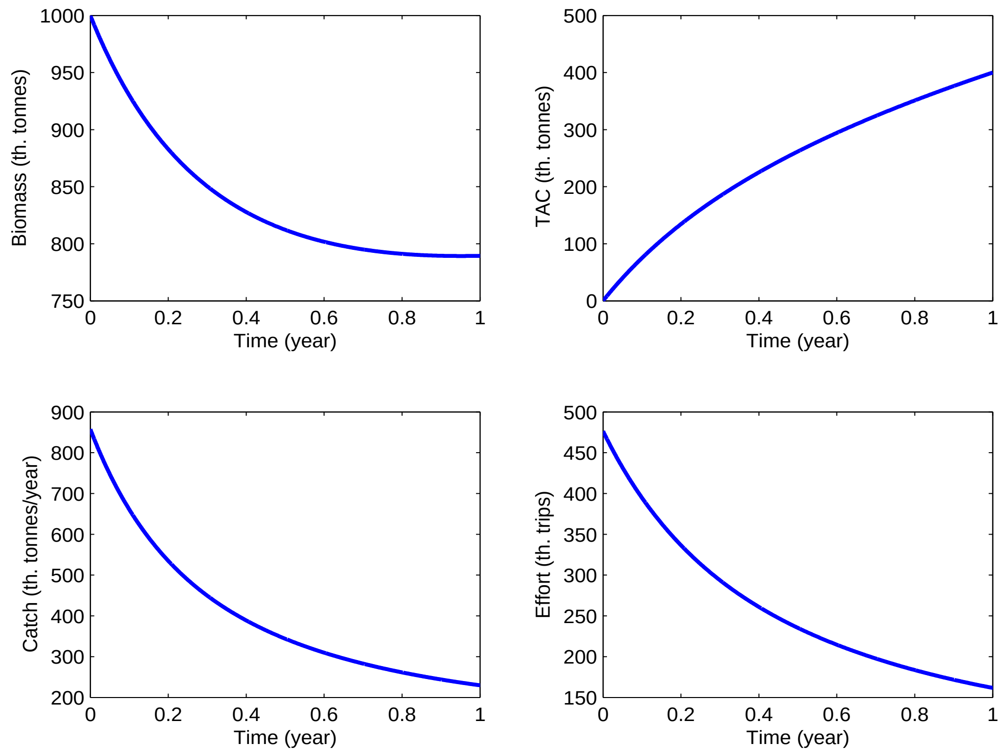

Figure 5: Effort, Catch, Biomass and TAC levels for $x_0 = 1,000,000$, $z(T) = 400,000$, $E_{max} = 800,000$ and $T = 1$

5 Conclusion

This work has focused on the fishing effort strategies of the sardinella fishery under a model incorporating TAC in order to determine the optimal strategy. Dynamics of the biomass were modeled using the Schaefer equation subject to an annual quota. However, the objective functional of the canonical Gordon-Schaefer model was subjected to a modification. Instead of the linear costs in that model, the objective functional in the model employed assumed quadratic costs. The necessary and sufficient conditions of the model were determined and discussed using Pontryagin's maximum principle and a result from Fleming and Rishel, respectively.

Numerical simulations were performed on the model to further highlight some useful insights. As depicted in the model, an isoperimetric constraint was present giving rise to one of the state variables having a fixed endpoint as well as a fixed initial point. Therefore, convergence of the iterates for a lot of parameter values was not achievable. For those that achieved convergence, the model recommended a relatively small effort rate out of the maximum allowable in order to realize a TAC set around the MSY. Even assuming an initial population size of the full carrying capacity did not seem to affect the magnitude of the optimal fishing rate in relation to the maximum rate available. Thus, there was no instance during the simulation when the boundary solution was achieved for the optimal control; only the interior solution was attained. This implies that, in order to ensure sustainability of the resource, there needs to be a

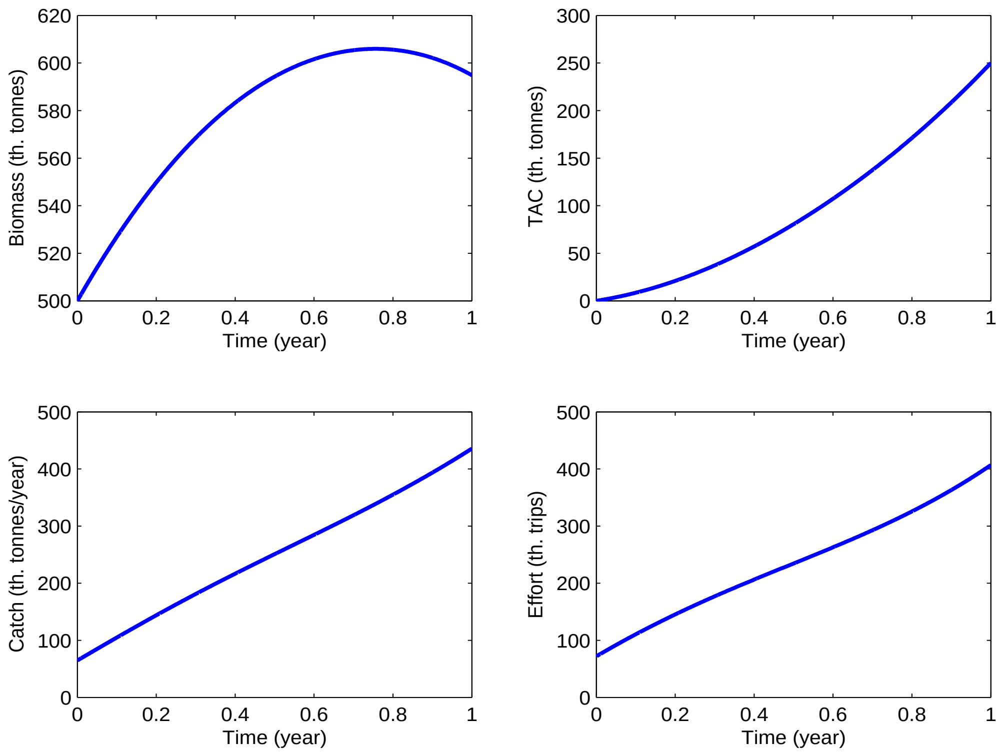

Figure 6: Effort, Catch, Biomass and TAC levels for $x_0 = 500,000$, $z(T) = 250,000$, $E_{max} = 800,000$ and $T = 1$

drastic reduction in the fishing effort. This can be achieved by downsizing the capacity of the fishing fleet or reducing the number of fishing days. It was further observed that for the same TAC, a lower initial biomass level requires a higher effort rate at the final horizon than for a higher initial biomass level. Also, for nearly identical initial biomass levels, a higher TAC requires a higher average effort rate.

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