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Positive fixed points of a nonlinear operators for a system of boundary value problems

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Abstract: In this paper, we study the existence of positive solutions for a class of multi points boundary value problems. We introduce a completely continuous operator such that, the fixed points of this operator are positive soultions of the problem. We establish some theorems to prove the existence of solutions for this system.

Keywords: Fixed point index; Boundary value problem; Positive solution; Jensen's inequality.
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1 Introduction

We study the existence of positive solutions for a class of multi point boundary value problems:

$$\left\{\begin{array}{l} A_{p_1}u + h_1(t)f_1(u) + k_1(t)g_1(v) = 0 \\ A_{p_2}v + h_2(t)f_2(u) + k_2(t)g_2(v) = 0 \end{array}\right. \quad \left\{\begin{array}{l} u(0) = \sum_{i=1}^{n} a_i u(\xi_i), u(1) = \sum_{i=1}^{n} a_i u(\eta_i) \\ v(0) = \sum_{i=1}^{n} b_i v(\xi_i), v(1) = \sum_{i=1}^{n} b_i v(\eta_i) \end{array}\right. \tag{1.1}$$

where

$$A_{p_i}s = \phi_{p_i}(s'), \phi_{p_i}(s) = |s|^{p_i-2}s, p_i > 1, \phi_{q_i} = (\phi_{p_i})^{-1}, \frac{1}{p_i} + \frac{1}{q_i} = 1, a_i \geq 0, b_i \geq 0,$$

$$0 \leq \sum_{i=1}^{n} a_i < 1, 0 \leq \sum_{i=1}^{n} b_i < 1, 0 < \xi_1 < \xi_2 < ... < \xi_n < \frac{1}{2}, \xi_i + \eta_i = 1, i = 1, 2, ..., n.$$

and $f_i, g_i \in C([0, +\infty), [0, +\infty)), h_i, k_i \in C([0, 1], [0, +\infty))$.
Many authors studied the existence of solutions for nonlinear boundary value problems. See, [8, 11, 1, 6, 4, 3] and the references therein.
In this paper, we extend the result in [10, 7]. In [10] authors, studied the existence of solution to the problem

$$\begin{array}{ll} x''(t) + a(t)x'(t) + b(t)x(t) + f(t, x(t), x'(t)) = 0, & t \in (0, 1) \\ x(0) = 0, x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), & \\ x'(0) = 0, x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i). & \end{array} .$$

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A completely continuous operator is defined and by using the fixed point theorem in cones, the existence of multiple positive solutions is proved. In [5] , by inroducing a cone and completely continuous operator, the authors proves the existence of positive solutions for this system:

$$\begin{array}{l}\varphi_{p_1}(u_1') + h_1(t)f_1(u_1,u_2) = 0\\ \varphi_{p_2}(u_2') + h_2(t)f_2(u_1,u_2) = 0\\ u_1(0) = u_1(1) = u_2(0) = u_2(1) = 0\end{array} \quad .$$

In [9] the authors define a cone and completely continuous operator and then by using the fixed point theorems in cone, they prove the existence of solutions to the problem :

$$\begin{array}{l}(\varphi_p(x'(t))' + q(t)f(t,x(t),x'(t)) = 0, t \in (0,1)\\ x(0) = 0 = x(1),\\ x'(0) = 0 = x'(1).\end{array}$$

2 The preliminary lemmas

Definitions([8]). Let E be a real Banach space. A nonempty convex closed set $K \subset E$ is said to be a cone provided that
$i)au \in K$ for all $u \in K$ and $a \geq 0$ and
$ii)u, -u \in K$ implies that $u = 0$.
The main tool of this paper is the following theorem.

Theorem 2.1. *([2]). Let E be a real Banach space and $K \subset E$ a cone. Suppose that $\Omega \subset E$ is a bounded open set and $T : \overline{\Omega} \bigcap K \to K$ is a completely continuous operator. Let $x_0 \in K \setminus \{0\}$*
(I) If $x - Tx \neq \eta x_0$ for $\eta \geq 0, x \in \delta\Omega \cap K$, then $i(T, \Omega \bigcap k, k) = 0$, where i indicates the fixed point index on K.
(II) Let E be a real Banach space and K a cone in E. Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and $T : \overline{\Omega} \bigcap K \to K$ is a completely continuous operator. If $x - \eta Tx \neq \eta x_0$ for $\eta \in [0, 1], x \in \delta\Omega \cap K$, then $i(T, \Omega \bigcap k, k) = 1$.

Let $E := C([0, 1], \mathbf{R})$ and

$$K := \{u \in E : u(t) \geqslant 0, t \in [0, 1]\}, \|u\| := max\{|u(t)| : t \in [0, 1]\}$$

and

$$\|(u, v)\| := max\{\|u\|, \|v\|\}, (u, v) \in E \times E,$$

$$B_r = \{(u, v) \in E^2 : \|(u, v)\| < r\}$$

for $r > 0$. Then $(E, \| \,.\, \|)$ is a real Banach space and K, K^2 are cones.
We define the following operators:

$$L(u, v) = (L_1(u, v)(t), L_2(u, v)(t))$$

such that

$$L_1(u, v)(t) := \frac{1}{1 - \sum_{i=1}^{n} a_i} \sum_{i=1}^{n} a_i \int_0^t \varphi_{p_1}^{-1}(\int_s^1 (h_1(r)f_1(u(r)) + k_1(r)g_1(v(r)))dr)ds$$

$$L_2(u,v)(t) := \frac{1}{1-\sum_{i=1}^{n} b_i} \sum_{i=1}^{n} b_i \int_0^t \varphi_{p_2}^{-1}\Big(\int_s^1 (h_2(r)f_2(u(r)) + k_2(r)g_2(v(r)))dr\Big)ds$$

Hence $L : K^2 \to K^2$ is an completely continuous operator.

Remark 2.2. *Suppose that $x \in K$ is concave on [0,1] , $\| x \| = x(1)$. then $\| x \| \leq \frac{\pi^2}{4} \int_0^1 x(t) sin \frac{\pi}{2} t dt$.*

Lemma 2.3. *(Jensen's Integral Inequality for Nonnegative Concave Functions) Suppose that $u \in C([a,b], \boldsymbol{R}), \phi \in C(R^+, R^+)$. If ϕ is concave, then*

$$\phi\Big(\frac{1}{b-a}\int_a^b u(t)dt\Big) \geqslant \frac{1}{b-a}\int_a^b \phi(u(t))dt.$$

In particular, if $b-a \leqslant 1$, then we have

$$\Big(\int_a^b u(t)dt\Big)^{\alpha} \geqslant (b-a)^{\alpha-1}\int_a^b u^{\alpha}(t)dt \geqslant \int_a^b u^{\alpha}(t)dt$$

for $0 < \alpha \leq 1$. Let C is a cone in a Banach space $(K, \| \,.\, \|)$, the Bonsall cone spectral radius of T is defined by

$$R_C(T) := lim_{m\to\infty} \| T^m \|^{\frac{1}{m}} = inf_{m\geqslant 1} \| T^m \|^{\frac{1}{m}}$$

Lemma 2.4. *([10]), Let C is a cone in a Banach space $(K, \| \,.\, \|)$, and $T : C \to C$ is a countiuous homogeneous. If*

$$R_C(T) < 1, u, u_0 \in C$$

satisfy $u \leq Tu + u_0$, then $u \leq (I-T)_C^{-1}u_0$, where the Bonsall cone spectral radius of T is

$$R_C(T) := lim_{m\to\infty} \| T^m \|^{\frac{1}{m}} = inf_{m\geqslant 1} \| T^m \|^{\frac{1}{m}}$$

and $(I-T)_C^{-1}$ is the inverse operator of $I-T$ on C.

Let $\Theta > 1$, we define :

$$\Theta' := min\{2,\Theta\}, \Theta'' := max\{2,\Theta\}, \Theta_{.} = \frac{\Theta'-1}{\Theta-1}, \Theta^{\cdot} = \frac{\Theta''-1}{\Theta-1}$$

3 Main Results

Suppose that f, g satisfy:
$A_1) p_1, p_2 > 1, h_i, k_i \in C([0,1],[0,+\infty)), f_i, g_i \in C([0,+\infty),[0,+\infty))$
$A_2)$ There are two constants $\alpha > \frac{\pi^4}{16}, d > 0$ and two nonnegative functions $\chi_1, \Upsilon_1 \in C([0,+\infty),[0,+\infty))$ such that
$i) \chi_1^{p_{1\cdot}}$ is concave on $[0,+\infty)$
$ii) (h_1(t)f_1(u)+k_1(t)g_1(v)) \geqslant \chi_1(v^{p_2'-1}) - c, (h_2(t)f_2(u)+k_2(t)g_2(v)) \geqslant \Upsilon_1(u^{p_1'-1}) - c$ for all $u, v \in [0,+\infty)$,
$iii) \chi_1^{p_{1\cdot}}(\Upsilon_1^{p_{2\cdot}}(w)) \geqslant \iota w - d$ for all $w \in [0,+\infty)$
$A_3)$There are nonnegative constants $\alpha_1, \beta_1, \gamma_1, \delta_1, R$ such that $R_{K^2}(T_1) < 1$ and

$$(h_1(t)f_1(u) + k_1(t)g_1(v)) \leq \alpha_1 u^{p_1-1} + \beta_1 v^{p_1-1}, (h_2(t)f_2(u) + k_2(t)g_2(v)) \leq \gamma_1 u^{p_2-1} + \delta_1 v^{p_2-1}$$

for $u, v \in [0, R), t \in [0, 1]$ we define $T_1 : K^2 \to K^2$ by $T_1(u, v)(t) =$

$$(\frac{1}{1-\sum_{i=1}^{n} a_i}\sum_{i=1}^{n} a_i \int_0^t \varphi_{p_1}^{-1}(\int_s^1 (\alpha_1 u^{p_1-1}(r) + \beta_1 v^{p_1-1}(r))dr)ds,$$

$$\frac{1}{1-\sum_{i=1}^{n} b_i}\sum_{i=1}^{n} b_i \int_0^t \varphi_{p_2}^{-1}(\int_s^1 (\gamma_1 u^{p_2-1}(r) + \delta_1 v^{p_2-1}(r))dr)ds)$$

A_4) There are nonnegative constants $\alpha_2, \beta_2, \gamma_2, \delta_2, c$ such that $R_{K^2}(T_2) < 1$ and

$$(h_1(t)f_1(u) + k_1(t)g_1(v)) \leq \alpha_2 u^{p_1-1} + \beta_2 v^{p_1-1} + c, (h_2(t)f_2(u) + k_2(t)g_2(v)) \leq \gamma_2 u^{p_2-1} + \delta_2 v^{p_2-1} + c$$

for $u, v \in [0, +\infty), t \in [0, 1]$ we define $T_2 : K^2 \to K^2$ by $T_2(u, v)(t) =$

$$(\frac{1}{1-\sum_{i=1}^{n} a_i}\sum_{i=1}^{n} a_i \int_0^t \varphi_{p_1}^{-1}(\int_s^1 (\alpha_2 u^{p_1-1}(r) + \beta_2 v^{p_1-1}(r))dr)ds,$$

$$\frac{1}{1-\sum_{i=1}^{n} b_i}\sum_{i=1}^{n} b_i \int_0^t \varphi_{p_2}^{-1}(\int_s^1 (\gamma_2 u^{p_2-1}(r) + \delta_2 v^{p_2-1}(r))dr)ds)$$

A_5)There are two continuous nonnegative functions χ_2, Υ_2 and two constants $\tau > \dfrac{\pi^4}{16}, \rho > 0$ satisfying
$i)\chi_2^{p_1 \cdot}$ is concave on $[0, +\infty)$
$ii)(h_1(t)f_1(u) + k_1(t)g_1(v)) \geqslant \chi_2(v^{p_2'-1}), (h_2(t)f_2(u) + k_2(t)g_2(v)) \geqslant \Upsilon_2(u^{p_1'-1})$ for all $u, v \in [0, +\infty)$,
$iii)\chi_2^{p_1 \cdot}(\Upsilon_2^{p_2 \cdot}(w)) \geqslant \iota' w$ for all $w \in [0, +\infty)$
A_6)There is a positive constant $l > o$ such that for every $u, v \in [0, l]$

$$(h_1 f_1(u) + k_1 g_1(v)) \leq (h_1 f_1(l) + k_1 g_1(l)), (h_2 f_2(u) + k_2 g_2(v)) \leq (h_2 f_2(l) + k_2 g_2(l)),$$

$$\frac{1}{1-\sum_{i=1}^{n} a_i}\sum_{i=1}^{n} a_i \int_0^1 \varphi_{p_1}^{-1}(\int_s^1 (h_1 f_1(l) + k_1 g_1(l))dr)ds < l,$$

$$\frac{1}{1-\sum_{i=1}^{n} b_i}\sum_{i=1}^{n} b_i \int_0^1 \varphi_{p_2}^{-1}(\int_s^1 (h_2 f_2(l) + k_2 g_2(l))dr)ds < l$$

Theorem 3.1. *Let assumptions A_1, A_4, A_5 be satisfied. Then the problem (1.1) has at least one positive solution.*

Proof. Let

$$\Im := \{(u, v) \in K^2 | (u, v) = \eta T(u, v), 0 \leq \eta \leq 1\},$$

Now, we suppose that $(u_0, v_0) \in \Im$ so, $\exists \eta_0 \in [0, 1]$ such that $(u_0, v_0) = \eta_0 T(u_0, v_0)$ So we have,

$$u_0(t) \leq \frac{1}{1-\sum_{i=1}^{n} a_i}\sum_{i=1}^{n} a_i \int_0^t \varphi_{p_1}^{-1}(\int_s^1 (h_1(r)f_1(u_0(r)) + k_1(r)g_1(v_0(r)))dr)ds,$$

$$v_0(t) \leq \frac{1}{1-\sum_{i=1}^{n} b_i}\sum_{i=1}^{n} b_i \int_0^t \varphi_{p_2}^{-1}(\int_s^1 (h_2(r)f_2(u_0(r)) + k_2(r)g_2(v_0(r)))dr)ds$$

Suppose that $\varepsilon, c_0 > 0, e \geq 0$ such that

$$(1+\varepsilon)R_{K^2}(T_2) < 1, \varphi_{p_1}^{-1}(e + c) \leq (1+\varepsilon)\varphi_{p_1}^{-1}(e) + c_0, \varphi_{p_2}^{-1}(e + c) \leq (1+\varepsilon)\varphi_{p_2}^{-1}(e) + c_0,$$

from A_4 we have,

$$u_0(t) \leq \frac{1}{1-\sum_{i=1}^{n} a_i} \sum_{i=1}^{n} a_i \int_0^t \varphi_{p_1}^{-1}(\int_s^1 (\alpha_2(u_0(r))^{p_1-1} + \beta_2(v_0(r))^{p_1-1} + c)dr)ds,$$

$$u_0(t) \leq \frac{(1+\varepsilon)}{1-\sum_{i=1}^{n} a_i} \sum_{i=1}^{n} a_i \int_0^t \varphi_{p_1}^{-1}(\int_s^1 (\alpha_2(u_0(r))^{p_1-1} + \beta_2(v_0(r))^{p_1-1})dr)ds + c_0,$$

$$v_0(t) \leq \frac{1}{1-\sum_{i=1}^{n} b_i} \sum_{i=1}^{n} b_i \int_0^t \varphi_{p_2}^{-1}(\int_s^1 (\gamma_2 u^{p_2-1}(r) + \delta_2 v^{p_2-1}(r) + c)dr)ds$$

$$v_0(t) \leq \frac{(1+\varepsilon)}{1-\sum_{i=1}^{n} b_i} \sum_{i=1}^{n} b_i \int_0^t \varphi_{p_2}^{-1}(\int_s^1 (\gamma_2 u^{p_2-1}(r) + \delta_2 v^{p_2-1}(r)dr)ds + c_0,$$

lemma 2.4 implies that

$$(u_0, v_0) \leq (1+\varepsilon)T_2(u_0, v_0) + (c_0, c_0), (u_0, v_0) \leq (I - (1+\varepsilon)T_2)_{K^2}^{-1}(c_0, c_0).$$

So, $\Im$ is bounded. For

$$R > sup\{\| (u,v) \| | (u,v) \in \Im\}$$

we have

$$(u,v) \neq \eta T(u,v), \forall (u,v) \in \partial B_R \cap K^2,$$

theorem 2.1 implies that

$$i(T, B_R \cap K^2, K^2) = 1 \tag{3.1}$$

We define

$$\Im' := \{(u,v) \in \overline{B_r} \cap K^2 | (u,v) = T(u,v) + \eta(z_0, z_0), \eta \geqslant 0\}, where \;\; z_0 = 2t - t^2,$$

Now, we suppose that $(u_0, v_0) \in \Im'$ this implies that $\eta_0 \geqslant 0$, $(u_0, v_0) = T(u_0, v_0) + \eta(z_0, z_0)$,

$$u_0(t) \geq \frac{1}{1-\sum_{i=1}^{n} a_i} \sum_{i=1}^{n} a_i \int_0^t \varphi_{p_1}^{-1}(\int_s^1 (h_1(r)f_1(u_0(r)) + k_1(r)g_1(v_0(r)))dr)ds,$$

$$v_0(t) \geq \frac{1}{1-\sum_{i=1}^{n} b_i} \sum_{i=1}^{n} b_i \int_0^t \varphi_{p_2}^{-1}(\int_s^1 (h_2(r)f_2(u_0(r)) + k_2(r)g_2(v_0(r)))dr)ds$$

Jensen's inequality for concave functions implies that

$$u_0^{p_1'-1}(t) \geqslant \frac{1}{1-\sum_{i=1}^{n} a_i} \sum_{i=1}^{n} a_i \int_0^t (\int_s^1 (h_1(r)f_1(u_0(r)) + k_1(r)g_1(v_0(r)))^{p_1'} dr)ds =$$

$$\frac{1}{1-\sum_{i=1}^{n} a_i} \sum_{i=1}^{n} a_i \int_0^1 (min\{t,s\})(h_1(r)f_1(u_0(r)) + k_1(r)g_1(v_0(r)))^{p_1'} ds,$$

$$v_0^{p_2'-1}(t) \geqslant \frac{1}{1-\sum_{i=1}^{n} b_i} \sum_{i=1}^{n} b_i \int_0^t (\int_s^1 (h_2(r)f_2(u_0(r)) + k_2(r)g_2(v_0(r)))^{p_2'} dr)ds =$$

$$\frac{1}{1-\sum_{i=1}^{n} b_i} \sum_{i=1}^{n} b_i \int_0^1 (min\{t,s\})((h_2(r)f_2(u_0(r)) + k_2(r)g_2(v_0(r)))^{p_2'})ds$$

From A_5 we find

$$u_0^{p'_1-1}(t) \geqslant \int_0^1 (min\{t,s\})m_1^{p_1\cdot}(v_0^{p'_2-1}(s))ds, v_0^{p'_2-1}(t) \geqslant \int_0^1 (min\{t,s\})n_1^{p_2\cdot}(u_0^{p'_1-1}(s))ds, \tag{3.2}$$

then, we have

$$u_0^{p'_1-1}(t) \geqslant \int_0^1 (min\{t,s\})m_1^{p_1\cdot}(\int_0^1 (min\{r,s\})n_1^{p_2\cdot}(u_0^{p'_1-1}(r))dr)ds$$
$$\geqslant \iota' \int_0^1 \int_0^1 (min\{t,s\})(min\{r,s\})u_0^{p'_1-1}(r)drds$$

so,

$$\int_0^1 u_0^{p'_1-1}(t)sin\frac{\pi}{2}tdt \geqslant \frac{16\iota'}{\pi^4}\int_0^1 u_0^{p'_1-1}(t)sin\frac{\pi}{2}tdt, so \int_0^1 u_0^{p'_1-1}(t)sin\frac{\pi}{2}tdt = 0, then\ u_0 = 0.$$

From (2) we have $m_1^{p_1\cdot}(v_0^{p'_2-1})(t) = 0$ so, $v_0 = 0$, then $\Im' = 0$. Therefor, we have

$$(u,v) \neq T(u,v) + \eta(z_0,z_0), \eta \geqslant 0.$$

Theorem 2.1, implies

$$i(T, B_R \cap K^2, K^2) = 0 \tag{3.3}$$

from

$$i(T, (B_R \setminus \overline{B_r}) \cap K^2, K^2) = 1 - 0.$$

Consequently, T has at least one positive solution. □

Theorem 3.2. *Let assumptions A_1, A_2, A_5, A_6 be satisfied. Then the problem (1.1) has at least two positive solutions.*

Proof. FromA_1, A_2, A_5 and Theorem 3.1, we obtain

$$i(T, B_R \cap K^2, K^2) = 0$$

suppose that $r < z < R$. From A_6 for all $(u,v) \in \overline{B}_z \cap K^2$, we have

$$\|T_1(u,v)\| = T_1(u,v)(1) \leq \frac{1}{1-\sum_{i=1}^n a_i}\sum_{i=1}^n a_i \int_0^t \varphi_{p_1}^{-1}(\int_s^1 (h_1 f_1(z) + k_1 g_1(z))dr)ds < z,$$

$$\|T_2(u,v)\| = T_2(u,v)(1) \leq \frac{1}{1-\sum_{i=1}^n b_i}\sum_{i=1}^n b_i \int_0^t \varphi_{p_2}^{-1}(\int_s^1 (h_2 f_2(z) + k_2 g_2(z))dr)ds < z,$$

then, for $(u,v) \in \partial B_z \cap K^2$, we conclude that

$$\|T(u,v)\| < \|(u,v)\|, (u,v) \neq \eta T(u,v), 0 \leqslant \eta \leqslant 1.$$

So, by Theorem 2.1 we have,

$$i(T, B_z \cap K^2, K^2) = 1. \tag{3.4}$$

By (3),(4) we have,

$$i(T, (B_R \setminus \overline{B_z}) \cap K^2, K^2) = 0 - 1, i(T, (B_z \setminus \overline{B_r}) \cap K^2, K^2) = 1 - 0 = 1.$$

Therefore, T has at least two fixed points on $B_R \setminus \overline{B}_z) \cap K^2$. Then the proof is completed. □

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