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Computational method for determining the bound state oscillator frequency

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Abstract: Creation and annihilation operator's method that is associated with the system was proposed to determine the oscillator frequency of the bound system which consists of two or more particles, as a function of the orbital quantum number, which is the main parameter to describe the interaction between particles that create new bounding systems like charmonium, hyperatoms, pentaquark, etc. Using quantum field theory and quantum electrodynamic methods, we are found that the creation of a bound state occurs if the coupling constant be small, and masses of gauge bosons also be very small in comparison with masses of constituent particles. The modified Hamiltonian (Schrödinger equation) based on the oscillator frequency parameter describes the bound state characteristic such as the mass spectrum, the constituent mass of particles, and binding energy. The method is typically used to solve the relativistic or nonrelativistic Schrödinger equation and to calculate the binding energy or energy eigenvalue of the system for a wide class of potentials allowing the existence of a bound state. The main purpose of this study is to investigate the relationship of the particle binding energy with the oscillator frequency of the Coulomb type potential (or other potentials) bound systems with the nonrelativistic Schrödinger equation.

Keywords: Bound states; Mass spectrum; Hypernuclei; Oscillator frequency; Wick-ordering.
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1 Introduction

The energy spectrum of a bound state can be determined by principles of relativistic/nonrelativistic quantum mechanics. Nonrelativistic Schrödinger equation which correctly describes a bound state needs relativistic corrections due to describe the modern experimental results obtained in both atomic [6] and hadron [2] physics. The actual relativistic corrections to the Schrödinger equation are small, so the theoretical problem is reduced

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to obtaining relativistic corrections to a nonrelativistic interaction's potential based on the quantum field theory methods. Breit [1], Caswell, and Lepage [3] presented new methods. All approaches and ideas use a scattering matrix and the kinematic part of the interaction as a source of the relativistic corrections. The interaction potential and a relativistic correction were determined within the quantum electrodynamics, renormalization of scattering matrix [3], and Feynman diagrams. Relativistic corrections are defined by asymptotic behavior of the polarization, which is formulated through quantum field Green's functions [4], is expressed in the form of the Feynman's functional path integral [4]. This method allows one to find the required behavior of the wave function of the bound state, and determine the Schrodinger equation based on the constituent masses of particles. Thus the relativistic corrections to the interaction Hamiltonian are taken into account in a form of the constituent particle masses, and then one can describe the bound state parameters, such as energy eigenvalue, mass spectrum, spins interaction. Therefore, one of the basic issues of relativistic and nonrelativistic quantum mechanics is to determine the binding energy and mass spectrum of an exotic hadronic system as a harmonic oscillator [11] with an appropriate potential model which is described by the Schrödinger equation. We use the Coulomb type potential and neglect, strong interaction between clusters $\sigma \approx 0$. Since the exact solutions of the Schrödinger equation for two or more different charged particles have already been found; These types of potentials are determining the nature of vibration or bonding states of quantum systems. As far as we know, the Schrödinger equation has long been recognized as an essential method for defining spectral behaviors of (micro/femto)-bounding systems like molecules, atoms, nuclei, quarks. We typically find the energy eigenvalue and the mass spectrum by the operator's algebraic method [11]. The Coulomb field is used for (+-), (++), (+0) charged, or uncharged clusters of bound states. Contrary to the general view of particles having a similar charge or the charged neutrality, due to their constructions that consist of charged quarks, there will occur electrostatic effects. Therefore, we start from the ideas and methods of the quantum theory of scalar fields and interactions of charged particles. In quantum mechanics, any ground state of a bound system can be near to an oscillator [11] that is described by the wave function, so based on the asymptotic behavior of the correlation functions of the Coulomb field currents, the computational and mathematical methods for the determination of the oscillator frequency is suggested. And the oscillator frequency of two-component system in the ground and orbital excited states is determined. We expect and predict that the interaction between clusters will affect the oscillator frequency of the bound system, and then we have to see very small fluctuations in the frequency chart which is drawn for (+-), (++), (+0) charged clusters. The frequency chart shows the formation of the bound state.

2 Frequency of Bond State

In the framework of a^{+}, a^{-} operators, we have considered the Hamiltonian of two oscillating charged clusters in a bound system with a potential interaction $V_c(r)$ and a classical harmonic potential $V_h(r)$ that is supposed to permit the existence of a bound state, and then one can determine the oscillator frequency and energy eigenvalue of the bound state. In this case, using the one-dimensional radial relativistic Schrodinger equation, that reads

$$\sqrt{m_1^2 + \hat{p}^2} + \sqrt{m_2^2 + \hat{p}^2} + V_c(\hat{x}) + V_h(\hat{r}) \;\; R(\hat{r}) = E_\ell R(\hat{r}) \tag{1.1}$$

Now, we explain the relativistic effect on the bound states. Two approximate methods usually are used for predicting the structure of the bound states ($\hbar = c = 1$):

$$I.\sqrt{m^2+\hat{p}_r^2}=m\sqrt{1+\frac{\hat{p}_r^2}{m^2}}\approx m+\frac{\hat{p}_r^2}{2m}-\frac{\hat{p}_r^4}{8m^3}+\frac{\hat{p}_r^6}{10m^5}+\ldots$$
$$II.\sqrt{m^2+\hat{p}_r^2}\approx\min_{\mu}(\mu+\frac{m^2+\hat{p}_r^2}{\mu}) \tag{1.2}$$

Using the Coulombic system and the second relation in (1.2), the Hamiltonian in (1.1) reads[2,7,8]

$$\left(\frac{\hat{p}_r^2}{2\mu}+z_1z_2\frac{\alpha_s}{\hat{r}}+\frac{1}{2}\mu\omega^2\hat{r}\right)R(\hat{r})=\left(M-\frac{1}{2}(\mu_1+\mu_2)-\frac{1}{2}\frac{m_1^2\mu_2+m_2^2\mu_1}{\mu_1\mu_2}\right)R(\hat{r})\Rightarrow$$
$$E_\ell(\mu_1,\mu_2)=M-\frac{1}{2}(\mu_1+\mu_2)-\frac{1}{2}\frac{m_1^2\mu_2+m_2^2\mu_1}{\mu_1\mu_2} \tag{1.3}$$
$$\frac{1}{\mu}=\frac{1}{\mu_1}+\frac{1}{\mu_2}$$

where μ is the parameter and it has the dimension of mass. In further calculations, we introduce readers with formulas. The Laplacian of (1.3) should change to the new variable $\hat{r}=\hat{q}^2\Rightarrow R_\ell(\hat{r})=\hat{q}^{2\ell}\Phi(\hat{q}^2)$ due to having a Gaussian solution for large distances which one can apply the quantum field theory methods to find binding energy, mass spectrum, and the oscillator frequency. Here ℓ is the orbital quantum number and q is considered to be a vector of new $4(1+\ell)$ -dimensional space. Therefore, the Laplacian of $4(1+\ell)$ -dimensional space and momentum read

$$\Delta_r=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{\ell(1+\ell)}{r^2}\rightarrow\Delta_q=\frac{\partial^2}{\partial q^2}+\frac{3+4\ell}{q}\frac{\partial}{\partial q}\Rightarrow\hat{p}_r^2=\frac{1}{4\hat{q}^2}\hat{p}_q^2 \tag{1.4}$$

then the modified radial Schrodinger equation is represented as the following term

$$\left(\frac{\hat{p}_q^2}{2\mu}+z_1z_2\frac{\alpha_s}{\hat{q}^2}+\frac{1}{2}\mu\omega^2\hat{q}^2\right)\Phi(\hat{q}^2)=E_\ell\Phi(\hat{q}^2)\Rightarrow\left(\frac{\hat{p}_q^2}{2}+4\mu\hat{q}^2(z_1z_2\frac{\alpha_s}{\hat{q}^2}-E_\ell)\right)\Phi(\hat{q}^2)=0 \tag{1.5}$$

The next level in the determination of binding energy and the oscillator frequency in terms of Wick-ordering technique is to define the variable $\hat{q},\hat{p}_q$ as a function of creation and annihilation operators. Now, we use

$$\hat{a}^+=\sqrt{\frac{\omega}{2}}\left(\hat{q}-\frac{i}{\omega}\hat{p}_q\right),\quad\hat{a}^-=\sqrt{\frac{\omega}{2}}\left(\hat{q}+\frac{i}{\omega}\hat{p}_q\right)\Rightarrow\hat{q}=\frac{\hat{a}^-+\hat{a}^+}{\sqrt{2\omega}},\quad\hat{p}_q=\sqrt{\frac{\omega}{2}}\frac{\hat{a}^--\hat{a}^+}{2i} \tag{1.6}$$

and one presents $\hat{p}_q^2,\hat{q}^2$ in equation (1.5) with the using of Wick ordering of the operators $\hat{a}^+,\hat{a}^-$ [11] and very important requirement and conditions of interaction. By including this issue, the interaction Hamiltonian in the form of Wick ordering results should not contain any terms quadratic in $\hat{q}$. These are the oscillator conditions in the oscillator representation method (see [2,8] in more detail). Therefore, for quadratic terms based on the oscillator condition we obtain

$$p_q^2=2(1+\ell)\omega_\ell+:p^*:,\quad q^2=2(1+\ell)\frac{1}{\omega_\ell}+:q^*: \tag{1.7}$$

the condition of Hamiltonian interaction in Wick-ordering form includes in the term :*: , and contains all non-square parts, and then we can find the renormalization of the bound state parameters like wave function which let us introduce with the zero approximation in the oscillator method and then find the energy eigenvalue of the ground state $\varepsilon_0(E_\ell)$. Now to determine the bound state's oscillator frequency and the binding energy, from (1.5) and (1.7) the total modified radial Schrodinger's equation must be known. For simplicity of calculation, we choose the total Hamiltonian regardless of the spin-spin, spin-orbital, and tensor-tensor interactions and effects. Thus the equation (1.5) is written in the form (see [2,8] in more detail):

$$\varepsilon_0(E_\ell) = (1+\ell)\omega_\ell - 4\mu z_1 z_2 \alpha_s - 8\mu(1+\ell)E_\ell \omega_\ell^{-1} = A(\omega_\ell) - E_\ell B(\omega_\ell) = 0 \tag{1.8}$$

Therefore, the parameters are defined in the oscillator representation method using the following equations:

$$\varepsilon_0(E_\ell) = 0, \quad \frac{d\varepsilon_0(E_\ell)}{d\omega_\ell} = 0 \tag{1.9}$$

one can define the minimum ground state energy of the systems as a result of the zero approximation. Therefore, we define energy eigenvalue and oscillator frequency:

$$E_\ell = \frac{1}{8\mu}\omega_\ell^2 - \frac{1}{2(1+\ell)} z_1 z_2 \alpha_s \omega_\ell, \quad \omega_\ell = \frac{2}{(1+\ell)} \mu z_1 z_2 \alpha_s \tag{1.10}$$

The ground states energy eigenvalue and oscillator frequency of the bound state with reduced mass μ are given by (1.10) for $z_1 z_2 = 1$ with the different orbital quantum number ℓ are obtained and presented as follows (in units of MeV and MeV^{-1}):

$$E_0 = -2.6657.10^{-5}\mu, \quad E_1 = -1.065.10^{-4}\mu, \quad E_3 = -2.396.10^{-4}\mu \quad E_4 = -4.2601.10^{-4}\mu$$

$$\omega_0 = 0.57\mu, \quad \omega_1 = 0.288\mu, \quad \omega_3 = 0.19\mu \quad \omega_4 = 0.14\mu$$

then using (1.3) and (1.10) one can determine the mass spectrum of the predicted bound state and reads as follows:

$$M = \sqrt{m_1^2 - 2\mu^2 \dot{E}_\ell} + \sqrt{m_1^2 - 2\mu^2 \dot{E}_\ell} + \mu \dot{E}_\ell + E_\ell \tag{1.11}$$

The parameter μ is determined by solving the equation

$$\frac{1}{\mu} = \frac{1}{\sqrt{m_1^2 - 2\mu^2 \dot{E}_\ell}} + \frac{1}{\sqrt{m_2^2 - 2\mu^2 \dot{E}_\ell}}. \tag{1.12}$$

In the presented method, the energy eigenvalue and the oscillator frequency of the bound state is determined. Now, we apply results to determine the formation and creation points of a bound state. The Computational method is presented for defining $_{\Sigma^-}Z^A$ hypernuclear frequency[5] with the rest masses that are given from Particle Data Group-2020[11]:

$$m_p = 938.272088 MeV, \quad m_n = 939.565413 MeV, \quad m_{\Sigma^-} = 1197.449 MeV$$
$$m_{\Xi^-} = 1321.71 MeV, \quad m_{\Omega^-} = 1672.45 MeV$$

and constant interaction range $0.0074 \leq \alpha_s \leq 0.5$ in the ground state $\ell = 0$ and without spins-orbital interactions. In the above equations, $m_1 = m_{\Sigma^-}$, $m_2 = m_p + m_n = m_{^AZ}$. As it is shown in Figure 1 and Figure 2, by computational method one can predict a new bound state, determine its mass spectrum, and also can exactly define whether it is stable or not.

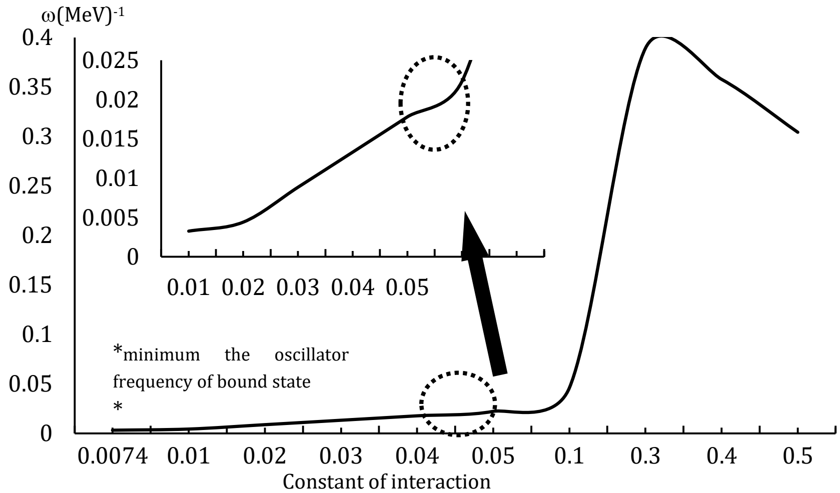

Figure 1: The curve of the oscillator frequency of the bound state as a function of α_s.

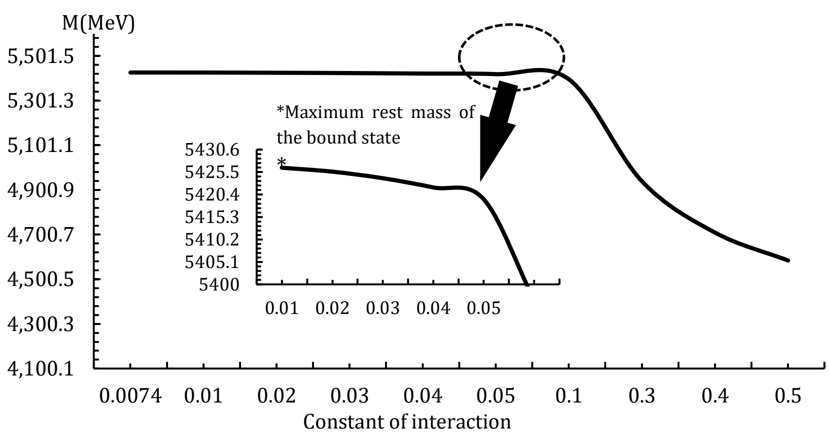

Figure 2: The curve of the predicted mass spectrum of the bound state as a function of α_s.

The minimum predicted mass of a bound state in which hypernuclei can be semi-stable is defined by the computational method equivalent to the value 5420.493MeV, as we know the rest mass of particles is 5426.83MeV. The result shows that our computational method for the binding energy and mass spectrum based on the oscillator representation model agrees well with the data of other researchers. Some of our predictions bound states shown in Table1 are presented as an exotic hadronic state. Maybe some new exotic hypernuclei bound states very soon could be experimentally discovered in the new generation facilities, such as J-PARC, MAMI, JLab, and FAIR.

Table 1: Mass spectrum and binding energy of exotic hypernuclei (in units of MeV)

$\ell = 0$	$-E_{\Sigma^-}$	M_{Σ^-}	$-E_{\Sigma^-}$	$-E_{\Sigma^-}$
${}^{5}_{\Sigma^-}Li$	3.742	5420.493	is predicted	is predicted
${}^{11}_{\Sigma^-}B$	12.37	10698.15	13.140[9]	9.2[10]
${}^{11}_{\Delta^-}B$	12.78	10608.40	is predicted	is predicted
${}^{13}_{\Omega^-}B$	15.30	12925.46	is predicted	is predicted
${}^{8}_{\Sigma^-}He$	5.36	7767.74	-	5.9[10]
${}^{16}_{\Xi^-}O$	19.27	16325.13	≈15.9[9]	16.0[10]

3 Conclusion

In the last few years, the experimental and theoretical physics of hadron is ongoing. There has been predictable potential in hypernuclei physics. We described the method for predicting the mass spectrum of exotic heavy hypernuclei based on relativistic phenomena of linked states and Coulomb type potential. The main subject of our interest is the formation conditions of the bound states. As far as know, if M and μ_i (-are the mass of the exotic hypernuclear bound state and constituent mass of core (m_2) or hyperon(m_1)), the following situation can be realized i) if $\delta \leq M < \infty$ and $M \neq m_1+m_2,\ 0 > \mu_i \geq \vartheta$, and then a hypernuclear with a mass M arises; δ and ϑ are dimensionless parameters and determine by the constant of interaction value and the rest mass of constituent particles of the bound system. For example, gluon at ultra-high energy interaction has $\vartheta \approx 500$. ii) if $M = m_1+m_2$, then the interaction is so weak that the hypernuclear cannot arise and the hyperon and core exist as two free particles. Hence, we can determine creation conditions and the connective mass range of hypernuclei in potential interactions such as van der Waals, Cornell, Yukawa, Lennard-Jones, and pseudoharmonic potentials. In the present work, the hyperon binding energy and the mass spectrum of hypernuclei have been estimated with the Coulomb type potential based on the computational harmonic oscillator model, the quantum field theory ideas and Wick-ordering method. We determined the best oscillator frequency of creating a bound system in the ground state and defined the mass of the bonding system that also has the best minimum value in the relativistic limit. In Table 1, we have predicted and reported the mass of hypernuclei obtained theoretically through our described method. For example, our results for the mass of heavy exotic hypernuclei such as ${}^{5}_{\Sigma^-}Li$, ${}^{11}_{\Delta^-}B$ and ${}^{13}_{\Omega^-}B$ are 5420.493, 10608.40, and 12925.46, respectively. We compared the binding energy of

${}^{8}_{\Sigma^-}He$:(5.36MeV) with theoretical data taken from[10], we can see good accordance between our predicted and theoretical results (5.9MeV) presented by Samanta and Roy. This computational method can be very useful in the hypernuclei spectroscopy at LHC, J-PARC, MAMI, JLab, and FAIR. Many new experimental results are coming, which need deep insight. We would like to investigate and predicate the mass spectra for higher bounding states in future works. We would also like to focus our new predictions on heavy hyperons which trapped in or injected into heavy nuclei.

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6 Conflict of interest I thank

The authors declare that there are no conflicts of interest regarding the publication of this manuscript.

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