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Application of chebyshev polynomial basis function on the solution of volterra integro-differential equations using galerkin method

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Abstract: In this paper, the Galerkin Method (GM) is employed for finding the solution of Volterra integro-differential equations with the aid of Chebyshev polynomial basis function. Some examples are presented to demonstrate the effectiveness of the method and its reliability. The approximate solutions are in agreement with the exact solutions and are compared favorably with an existing methods available in the literature.

Keywords: Galerkin method, Chebyshev polynomial basis function, Volterra integro-differential equations.
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1 Introduction

Volterra integro-differential equations have been the hub of many studies due to its wide range of applications, such as in rheology, viscoelasticity and fluid mechanics. Volterra integral equations contrastingly arise in engineering such as heat transfer and neutron diffusion process, physics, chemistry and biological problems such as spread of epidemics, population dynamics and semi-conductor devices.
The volterra integro-differential equation appears in the form:

$$y^{(n)}(x) = f(x) + \lambda \int_0^x K(x,t)y(t)dt \quad (1)$$

Where $k(x,t)$ is the kernel of the integral equation, λ is a constant parameter and $y^{(n)}$ indicates the nth derivative of (x) , $n \geq 1$ inside and outside the integral sign. It is important to note that initial conditions should be given for volterra integro-differential equations to determine the particular solutions.
In recent years, various numerical methods have been presented to solve Volterra integro-differential equations. For example, modified Laplace Adomian decomposition method is used to solve nonlinear volterra integral and integro-differential equations as presented in [15]. In [20,21], application of Adomian decomposition method to solve integro-differential equations was also investigated. Legendre polynomial basis function was employed in

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the solution of volterra integro-differential equations using collocation method in [9]. In [8], the authour used an efficient Pseudo-spectral Legendre Galerkin method for solving a nonlinear partial integro-differential equation arising in population dynamics. [7] presented the numerical treatment of differential equations using collocation method for ordinary differential equation and presented its application with examples. Conclusion was drawn that the method presented gives desired result when compared with the exact solution. Also, [18] employed multiple perturbed collocation Tau method to solve higher order linear and nonlinear boundary value problems with the aid of Chebyshev basis functions. Other methods used by authors are He's Homotopy perturbation method [1, 2]. [3] Presented improved Adomian decomposition method. [19] Combined Laplace transform-Adomian decomposition method for handling nonlinear volterra integro-differential equations. In this study we presented the Galerkin method (GM) for the solution of volterra integro-differential equations with the aid of Chebyshev polynomial basis function as the approximate solution. Other studies which apply novel methods for solving fractional integro-differential equation and other forms of equation are found in (see for example, [4-6, 10-14, 16, 17]). In this paper, we employed the Galerkin Method (GM) to solve Volterra integro-differential equations with the aid of shifted Chebyshev polynomial basis functions as the approximate solution. The rest of the paper is organized as follows: Section 2 deals with the basis of the Chebyshev and shifted Chebyshev polynomials, section 3 involves construction of the method, section 4 deals with numerical application of the method and section 5 is the conclusion.

2.0 Chebyshev and shifted Chebyshev polynomials

Chebyshev polynomials are sequence of orthogonal polynomials which are related to de-Moivre's formula and which can be defined recursively. One usually distinguishes between Chebyshev polynomials of first kind which are denoted by T_n and Chebyshev polynomials of second kind which are denoted by U_n.

2.1 Chebyshev polynomials of first kind

Chebyshev polynomials of first kind $T_k(x)$ is defined as:

$$T_k(x) = \cos(k cos^{-1}x), -1 \leq x \leq 1, \quad (2)$$

or equivalently

$$T_k(x) = \cos n\theta, \quad where \quad \theta = cos^{-1}x . \quad (3)$$

The few Chebyshev polynomials of the first kind are;

n	$T_k(x)$
0	$T_0(x) = 1$
1	$T_1(x) = x$
2	$T_2(x) = 2x^2 - 1$
3	$T_3(x) = 4x^3 - 3x$
4	$T_4(x) = 8x^4 - 8x^2 + 1$
5	$T_5(x) = 16x^5 - 20x^3 + 5x$

2.2 The Shifted Chebyshev polynomials

For convenience and for the sake of problems that exist in intervals other than $-1 \leq x \leq 1$, $T_k(x)$ is in this subsection normalized to a general finite range

$r \leq x \leq s$ as follows:

$$T^*_K(x) = \cos(K cos^{-1}x); \ -1 \leq x \leq 1 , \quad (4)$$

and the recurrence relation is given by
$T^*_{K+1}(x) = 2xT^*_K(x) - T^*_{K-1}(x), \ K \geq 1$.
Where N is the degree of the polynomial.
In general, Chebyshev polynomial valid in $r \leq x \leq s$ is given as

$$T^*_K(x) = \cos\left[Ncos^{-1}\left(\frac{2x-s-r}{s-r}\right)\right]; \ -1 \leq x \leq 1, \quad (5)$$

and the recurrence relation is given as

$$T^*_{K+1}(x) = 2\left(\frac{2x-s-r}{s-r}\right)T^*_K(x) - T^*_{K-1}(x). \quad (6)$$

Few terms of the shifted Chebyshev polynomials valid in the interval [0, 1] are given below:

$T^*_0(x) = 1$
$T^*_1(x) = 2x - 1$
$T^*_2(x) = 8x^2 - 8x + 1$
$T^*_3(x) = 32x^3 - 48x^2 + 18x - 1$
$T^*_4(x) = 128x^4 - 256x^3 + 100x^2 - 32x + 1$
$T^*_5(x) = 512x^5 - 128x^4 + 1120x^3 - 400x^2 + 50x - 1$
$T^*_6(x) = 204x^6 - 6144x^5 + 6912x^4 - 5484x^3 + 840x^2 - 72x + 1$
$T^*_7(x) = 8192x^7 - 286x^6 + 39424x^5 - 26990x^4 + 9408x^3 - 1568x^2 + 98x - 1$
$T^*_8(x) = 32765x^8 - 131072x^7 + 212992x^6 - 40224x^5 + 84480x^4 - 21 + 26868x^2 - 128x + 1$

3 Construction of the method

This section, we discussed the numerical application of Chebyshev polynomial basis function on the solution of Volterra Integro-Differential equations using the Galerkin method.

Consider the general Volterra integro-differential equations of the form:

$$u^{(n)}(x) = f(x) + \lambda\int_0^x (x-t)u(t)dt, \quad (7)$$

with the initial condition:

$$u_k(0) = \emptyset_k. \quad (8)$$

We assumed an approximate solution of the form:

$$u(x) = u_N(x) = \sum_{k=0}^{N} a_k T^*_k(x), \quad (9)$$

where a_k , $k = 0(1)N$ are unknown constants to be determined and $T^*_k(x)$ is the shifted Chebyshev polynomial basis function. Differentiating equation (9) n-times to obtain:

$$u'_N(x) = \frac{d}{dx}\sum_{k=0}^{N} a_k T^*_k(x) \quad (10)$$

$$u''_N(x) = \frac{d^2}{dx^2}\sum_{k=0}^{N} a_k\, T^*{}_k(x) \tag{11}$$

$$u'''_N(x) = \frac{d^3}{dx^3}\sum_{k=0}^{N} a_k\, T^*{}_k(x) \tag{12}$$

$$u^{'v}{}_N(x) = \frac{d^{'v}}{dx^{'v}}\sum_{k=0}^{N} a_k\, T^*{}_k(x) \tag{13}$$

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$$u^n{}_N(x) = \frac{d^n}{dx^n}\sum_{k=0}^{N} a_k\, T^*{}_k(x). \tag{14}$$

Substituting the assumed approximate solution equations (9) and (14) into equation (7) to obtain:

$$\frac{d^n}{dx^n}\sum_{k=0}^{N} a_k\, T^*{}_k(x) = f(x) + \lambda \int_0^x (x-t) \sum_{k=0}^{N} a_k\, T^*{}_k(x)\, dt, \tag{15}$$

and therefore, the residual $R(u,x)$ will be:

$$R(u,x) = \frac{d^n}{dx^n}\sum_{k=0}^{N} a_k\, T^*{}_k(x) - f(x) - \lambda \int_0^x (x-t) \sum_{k=0}^{N} a_k\, T^*{}_k(x)\, dt = 0. \tag{16}$$

To determine the constant coefficients, a_k , $k = 0,1,2,3, \dots N$ we find the inner product of (16) with the basis function $T^*{}_k(x)$, $k = 0,1,2,3, \dots N$ to get:

$$\int_0^1 \big(R(u,x)\big)\big(T^*{}_k(x)\big)dx = \ , \qquad k = 0,1,2, \dots N. \tag{17}$$

Equation (17) is further simplified to give rise to *N+1* linear algebraic system of equations with *N+1* number of constants for $T^*{}_k(x)$, $k = 0,1,2, \dots N$.

The system of equations obtained from (17) is solved to get values for the unknown constants. The values are now substituted in to the assumed solution given in equation (9) to get the approximate solution. It is important to note that when the problem contains some initial conditions, we first apply those conditions before executing the Galerkin procedure to obtain the remaining number of required equations.

4 Numerical Applications

Example 1: Consider the second order volterra integro-differential equation.

$$u''(x) = -x - \frac{x^3}{6} + \int_0^x (x-t)u(t)dt \tag{18}$$

With the initial conditions,

$$u(0) = 0, \qquad u'(0) = 2 \tag{19}$$

And the exact solution given by $u(x) = x + \sin(x)$

Solving equation (18) for N = 4, N=5, N=7 and N=10 as defined in equation (9), we get the approximate solutions:

$$u_4 = 2x + 0.01197749728x^2 - 0.1892421744x^3 + 0.02195105127x^4$$

$$u_5 = 2x + 0.0001907526892x^2 - 0.1675751856x^3 + 0.001698660276x^4 + 0.007174729989x^5$$
$$u_7 = 2x + -7.25 \times 10^{-7}x^2 - 0.1666586403x^3 - 0.00003333758509x^4 + 0.008400221986x^5 - 0.00006769643191x^6 - 0.0001688446053x^7$$

$$u_{10} = 2x - 2.721170410 \times 10^{-11}x^2 - 0.1666666659x^3 - 8.744722719 \times 10^{-9}x^4 + 0.008333381955x^5 - 1.57420440310^{-7}x^6 - 0.0001980979807x^7 - 3.948709504 \times 10^{-7}x^8 + 0.000003059527419x^9 - 1.317677404 \times 10^{-7}x^{10}$$

Table 1: Numerical Results for Example 1 (CASE *N=4* and *N=5*)

x	Exact	Approx. N=4	Error (GM) N=4	Olayiwola. et al. (2020). N=4	Approx. N=5	Error (GM) N=5	Olayiwola. et al. (2020). N=5
0.0	0.00000	0.00000	0.00000	4.2000e-11	0.00000	0.00000	0.00000
0.1	0.19983	0.19993	9.931e-05	1.1305e-02	0.19983	1.157e-06	5.7641e-03
0.2	0.39867	0.39900	3.310e-04	4.4645e-02	0.39867	2.712e-06	2.2430e-02
0.3	0.59552	0.59615	6.260e-04	9.9220e-02	0.59552	3.625e-06	4.9149e-02
0.4	0.78942	0.79037	9.485e-04	1.7431e-01	0.78942	4.321e-06	8.5189e-02
0.5	0.97943	0.98071	1.286e-03	2.6931e-01	0.97943	5.628e-06	1.2993e-01
0.6	1.16464	1.16628	1.638e-03	3.8369e-01	1.16465	8.011e-06	1.8286e-01
0.7	1.34422	1.34623	2.012e-03	5.1708e-01	1.34423	1.120e-05	2.4356e-01

0.8	1.51736	1.51976	2.409e-03	6.6918e-01	1.51737	1.428e-05	3.1171e-01
0.9	1.68333	1.68615	2.819e-03	8.3987e-01	1.68334	1.639e-05	3.8708e-01
1.0	1.84147	1.84469	3.215e-03	1.0291e+00	1.84149	1.797e-05	4.6949e-01

Table 2: Numerical Results for Example 1 (CASE *N*=7 and *N*=10)

x	Exact	Approx. *N*=7	Error (GM) *N*=7	Olayiwola. et al. (2020). *N*=7	Approx. *N*=10	Error (GM) *N*=10	Olayiwola. et al. (2020). *N*=10
0.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.1	0.19983	0.19983	1.960e-09	1.5841e-07	0.19983	3.223e-12	3.3315e-07
0.2	0.39867	0.39867	6.999e-10	1.0211e-05	0.39867	4.953e-12	1.0644e-05
0.3	0.59552	0.59552	9.560e-10	7.9922e-05	0.59552	3.893e-11	8.0615e-05
0.4	0.78942	0.78942	4.286e-10	3.3753e-04	0.78942	8.272e-12	3.3848e-04
0.5	0.97943	0.97943	3.497e-09	1.0270e-03	0.97943	6.073e-12	1.0283e-03
0.6	1.16464	1.16464	5.142e-09	2.5434e-03	1.16464	8.819e-12	2.5449e-03
0.7	1.34422	1.34422	4.802e-09	5.4654e-03	1.34422	3.144e-11	5.4667e-03
0.8	1.51736	1.51736	4.808e-09	1.0586e-02	1.51736	2.145e-11	1.0586e-02
0.9	1.68333	1.68333	6.220e-09	1.8943e-02	1.68333	6.429e-11	1.8937e-02
1.0	1.84147	1.84147	7.136e-09	3.1841e-02	1.84147	2.371e-12	3.1826e-02

Example 2: Consider the third order Volterra integro-differential equation.

$$u'''(x) = 1 + x + \frac{x^3}{6} + \int_0^x (x-t)u(t)dt\,, \quad (20)$$

with the initial conditions,

$$u(0) = 1, \qquad u'(0) = 0, \;\; u''(0) = 1, \quad (21)$$

and the exact solution given by $u(x) = e^x - x$.

Solving equation (20) for N = 4, N=5, N=7 and N=8 as defined in equation (9), we get the approximate solutions:

$$u_4 = 1 + 0.5x^2 + 0.15200023572x^3 + 0.06909920956x^4$$

$$u_5 = 1 + 0.5x^2 + 0.17589047519x^3 + 0.02896003899x^4 + 0.01605874189x^5$$

$$u_7 = 1 + 0.5x^2 + 0.1666806639x^3 + 0.04158644411x^4 + 0.008533085582x^5 + 0.001147090593x^6 + 0.0003354345246x^7$$

$$u_8 = 1 + 0.5x^2 + 0.1666662505x^3 + 0.04167127612x^4 + 0.008315581867x^5 + 0.001422640561x^6 + 0.0001645120583x^7 + 0.00004158368954x^8$$

Table 3: Numerical Results for Example 2 (CASE *N*=4 and *N*=5)

x	Exact	Approx. *N*=4	Error (GM) *N*=4	Olayiwola. et al. (2020). *N*=4	Approx. *N*=5	Error (GM) *N*=5	Olayiwola. et al. (2020). *N*=5
0.0	1.00000	1.00000	0.00000	2.0000e-10	1.00000	0.00000	1.0000e-10
0.1	1.00517	1.00516	1.201e-05	1.0532e-04	1.00518	8.029e-06	3.2687e-05
0.2	1.02140	1.02133	7.620e-05	7.8887e0-4	1.02146	5.584e-05	2.3002e-04
0.3	1.04986	1.04966	1.951e-04	2.4862e-03	1.05002	1.638e-04	6.8239e-04
0.4	1.09182	1.09150	3.277e-04	5.4881e-03	1.09216	3.381e-04	1.4235e-03
0.5	1.14872	1.14832	4.025e-04	9.9530e-03	1.14930	5.769e-04	2.4559e-03
0.6	1.22212	1.22179	3.315e-04	1.5922e-02	1.22299	8.755e-04	3.7752e-03

Table 4: Numerical Results for Example 2 (CASE *N*=7 and *N*=8)

x	Exact	Approximate		Error	
		N=7	N=8	N=7	N=8
0.0	1.00000	1.0000000000	1.00000	0.0000	0.00000
0.1	1.00517	1.0051709260	1.0051709170	7.820e-09	2.656e-11
0.2	1.02140	1.0214027910	1.0214027580	3.392e-08	2.932e-10
0.3	1.04986	1.0498588720	1.0498588090	6.511e-08	7.766e-10
0.4	1.09182	1.0918248020	1.0918247010	1.043e-07	2.182e-09
0.5	1.14872	1.1487214390	1.1487212740	1.675e-07	3.447e-09
0.6	1.22212	1.2221190680	1.2221188050	2.680e-07	5.590e-09
0.7	1.31375	1.3137531070	1.3137527140	4.002e-07	7.896e-09
0.8	1.42554	1.4255414780	1.4255409400	5.496e-07	1.078e-08
0.9	1.55960	1.5596038200	1.5596031250	7.090e-07	1.334e-08
1.0	1.71828	1.7182827200	1.7182818450	8.907e-07	1.680e-08

Example 3: Consider the fourth order volterra integro-differential equation.

$$u^{'v}(x) = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \int_0^x (x-t)u(t)dt\,, \tag{22}$$

with the initial conditions,

$$u(0) = 2, \quad u'(0) = 2, \ u''(0) = 1, \ u'''(0) = 1, \tag{23}$$

and the exact solution given by $u(x) = e^x + x + 1$.

Solving equation (22) for N = 4, N=5, N=7 and N=8 as defined in equation (9), we get the approximate solutions:

$$u_4 = 2 + 2x + 0.5x^2 + 0.1666666667x^3 + 0.08978817772x^4$$

$$u_5 = 2 + 2x + 0.5x^2 + 0.1666666667x^3 + 0.02737260459x^4 + 0.01665619424x^5$$

$$u_7 = 2 + 2x + 0.5x^2 + 0.1666666667x^3 + 0.04147501000x^4 + 0.008692063589x^5 + 0.001040122369x^6 + 0.0003630772633x^7$$

$$u_8 = 2 + 2x + 0.5x^2 + 0.1666666667x^3 + 0.04167016598x^4 + 0.008317288838x^5 + 0.001422180888x^6 + 0.0001638701199x^7 + 0.00004192927492x^8$$

Table 5: Numerical Results for Example 3 (CASE N=4 and N=5)

x	Exact	Approx. N=4	Error (GM) N=4	Olayiwola. et al. (2020). N=4	Approx. N=5	Error (GM) N=5	Olayiwola. et al. (2020). N=5
0.0	2.00000	2.00000	0.0000	0.0000	2.00000	0.0000	1.0000e-09
0.1	2.20517	2.20518	4.727e-06	1.1352e-05	2.20517	1.348e-06	4.6971e-06
0.2	2.42140	2.42148	7.424e-05	1.8022e-04	2.42138	2.030e-05	7.1392e-05
0.3	2.64986	2.65023	3.685e-04	9.0502e-04	2.64976	9.662e-05	3.4269e-04
0.4	2.89182	2.89297	1.141e-03	2.8363e-03	2.89154	2.867e-04	1.0247e-03
0.5	3.14872	3.15145	2.724e-03	6.8638e-03	3.14806	6.566e-04	2.3611e-03
0.6	3.42212	3.42764	5.518e-03	1.4102e-02	3.42084	1.276e-03	4.6092e-03
0.7	3.71375	3.72372	9.972e-03	2.5876e-02	3.71154	2.214e-03	8.0165e-03
0.8	4.02554	4.04211	1.657e-02	4.3702e-02	4.02200	3.538e-03	1.2800e-02
0.9	4.35960	4.38541	2.581e-02	6.9267e-02	4.35429	5.309e-03	1.9125e-02
1.0	4.71828	4.75645	3.817e-02	1.0441e-01	4.71070	7.586e-03	2.7091e-02

Table 6: Numerical Results for Example 3 (CASE N=7 and N=8)

x	Exact	Approximate		Error	
		N=7	N=8	N=7	N=8
0.0	2.00000	2.0000000000	2.0000000000	0.00000	0.00000
0.1	2.20517	2.2051709030	2.2051709180	1.584e-08	2.950e-10
0.2	2.42140	2.4214025460	2.4214027600	2.120e-07	2.360e-09
0.3	2.64986	2.6498579070	2.6498588150	9.011e-07	6.710e-09
0.4	2.89182	2.8918222890	2.8918247120	2.409e-06	1.517e-08
0.5	3.14872	3.1487162370	3.1487212990	5.034e-06	2.860e-08
0.6	3.42212	3.4221097480	3.4221188500	9.052e-06	5.080e-08
0.7	3.71375	3.7137379620	3.7137527910	1.474e-05	8.400e-08

0.8	4.02554	4.0255185170	4.0255410590	2.241e-05	1.314e-07
0.9	4.35960	4.3595707540	4.3596033050	3.236e-05	1.938e-07
1.0	4.71828	4.7182369400	4.7182821020	4.489e-05	2.738e-07

5 Conclusion

In this study, we employed the Galerkin Method (GM) to solve Volterra integro-differential equations with the aid of shifted Chebyshev polynomial basis functions as the approximate solution. Three examples are considered with the method and we observed from the results obtained that as the degree of approximant N increases, the results obtained converges rapidly to the exact solutions. Also, the results obtained are better when compared to those from the existing literature. Thus, it is an efficient and reliable Numerical tools for the class of problem considered.

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