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A numerical approach to solve the stochastic Allen-Cahn equation of fractional order

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Abstract: In this paper, we employ a collocation method based on Legendre polynomials (LPs) to solve the time-fractional stochastic Allen-Cahn equation. This method is applied to convert the solution of this stochastic equation to the solution of a nonlinear system of algebraic equations. The numerical approach is completely described. Finally, a test example is implemented to validate the robustness of the proposed scheme.

Keywords: Stochastic Allen-Cahn equation; Caputo's derivative; Legendre polynomials; Collocation scheme
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1 Introduction

There are many phenomena in physics, chemistry, and engineering that appear randomly and are explained by stochastic processes [1, 8, 9, 15, 17, 22]. Stochastic behaviour arises naturally in many phenomena where the effects of random "noise" perturbations to a system are being considered [5]. For this reason, in recent years, the theory of stochastic partial differential equations has attracted more attention of scholars.

In the other hand, fractional calculus provides some new aspects to introduce more complete models of real world phenomena. This somewhat new calculus is the generalization of the classical differential and integral calculus. Fractional order operators have a non-local nature. More precisely, the upcoming-state of a fractional system depends on all the historical states including its present state [14, 15, 18, 2]. Thus, researchers have used these useful tools to model inherited memory characteristics of real applications in different fields, such as finance [21], physics [3, 14, 18], biology [4, 7] and viscoelastic materials [11].

In this paper, we focus on the following time-fractional stochastic Allen-Cahn equation

$$\partial_t^\alpha u - \Delta u = -\mathbf{F}'(u) + \sigma u \dot{\mathrm{B}}, \qquad \text{in} \quad \Omega \times (0, \mathtt{T}), \tag{1.1}$$

with the initial and boundary conditions

$$u(x,t) = 0, \qquad \text{in} \quad \partial\Omega \times (0, \mathtt{T}), \tag{1.2}$$

$$u(x,0) = \zeta_0(x), \qquad \text{in} \quad \Omega, \tag{1.3}$$

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where the potential energy $\mathbf{F}$ given by the standard double-well function $\mathbf{F}(u) = \frac{1}{4}(u^2-1)^2$ [13]. Also, $\dot{\mathrm{B}}(t) := \frac{\mathrm{dB}(t)}{\mathrm{d}t}$ denotes a time white noise and the positive constant σ is the intensities of environmental oscillation. Moreover, $\mathtt{L}, \mathtt{T} \in \mathbb{R}^+$, $\Omega := [0, \mathtt{L}]$, $\zeta_0(x)$ is the continuous function and the operator $\partial_t^\alpha[\cdot]$ denotes Caputo fractional derivative of order $\alpha \in (0,1)$ [18],

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial u}{\partial s}(x,s)\mathrm{d}s, \tag{1.4}$$

where $\Gamma(\cdot)$ shows the Gamma function.

Usually, the analytical solutions of the problems in the form (1.1)-(1.3) are not in hand. Hence, numerical methods are employed to find suitable approximate solutions. The authors in [19] considered a new discontinuous Galerkin method to solve a stochastic adsorption-desorption problem. B-spline approach [16], finite difference methods [20] and finite element methods [12] are some other numerical algorithms that are applied for solving the stochastic problems. In this paper, our purpose is to use the collocation method based on the shifted LPs to solve the problem (1.1)-(1.3).

This paper is organised as follows. In Section 2, the fundamental definitions of the LPs and their properties are reviewed. In Section 3, we propose a collocation scheme to solve the considered problem. Numerical results are investigated in Section 4. Finally, in Section 5 the main conclusions are presented.

2 The LPs and Their Properties

In this section, some necessary definitions and fundamental properties that are used in the sequel, are reviewed briefly.

Definition 2.1. *([10]) The LPs are defined on $[-1,1]$ as:*

$$\theta_i(t) = \frac{(-1)^i}{2^i i!} \frac{d^i}{dt^i}\Big\{(1-t)^i(1+t)^i\Big\}. \tag{2.1}$$

This function satisfies the recurrence relation

$$\theta_{i+1}(t) = \frac{2i+1}{i+1} t\theta_i(t) - \frac{i}{i+1}\theta_{i-1}(t), \qquad i = 1, 2, ...$$

in which $\theta_0(t) = 1$ and $\theta_1(t) = t$.

Definition 2.2. *([6]) The shifted LPs on $[a,b]$ are defined by explicit analytic form*

$$\theta_i^{a,b}(t) = \sum_{r=0}^{\left[\frac{i}{2}\right]} \sum_{m=0}^{i-2r} \sum_{j=0}^{m} \varpi_{r,m,j}^{a,b}\ t^j, \tag{2.2}$$

where

$$\varpi_{r,m,j}^{a,b} = \frac{(-1)^{r-j+m} 2^{m-i} b^{m-j} (2i-2r)!}{(b-a)^m (i-r)! r! (i-2r-m)! j! (m-j)!}.$$

Remark 2.3. *The shifted LPs $\theta_i^{a,b}(t)$ are orthogonal on $[a,b]$ according to the weight function $w(t) = 1$. Also, their orthogonality condition is satisfied*

$$\int_a^b \theta_i^{a,b}(t)\theta_r^{a,b}(t)\mathrm{d}t = \frac{2}{2i+1}\delta_{i,r},$$

where $\delta_{i,r}$ is the Kronecker function.

Let $\tilde{\Omega} := [0, \mathtt{L}] \times [0, \mathtt{T}]$ and $\mathrm{L}^2_w(\tilde{\Omega})$ be the space of all square integrable functions with the weight function w. A function $f \in \mathrm{L}^2_w(\tilde{\Omega})$ can be expanded based on the shifted LPs as:

$$f(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{i,j} \theta_i^{0,\mathtt{L}}(x) \theta_j^{0,\mathtt{T}}(t), \tag{2.3}$$

where the coefficients $c_{i,j}$ are given by

$$c_{i,j} = \left(\frac{2i+1}{2}\right)\left(\frac{2j+1}{2}\right) \int_0^{\mathtt{L}} \int_0^{\mathtt{T}} f(x,t) \theta_i^{0,\mathtt{L}}(x) \theta_j^{0,\mathtt{T}}(t) \mathrm{d}t\mathrm{d}x, \quad i,j = 0,1,2,....$$

By truncating the infinite series in (2.3), we can estimate $f(x,t)$ as follows:

$$f(x,t) \simeq f_{n,m}(x,t) = \sum_{i=0}^{n} \sum_{j=0}^{m} c_{i,j} \theta_i^{0,\mathtt{L}}(x) \theta_j^{0,\mathtt{T}}(t) = \Theta(x)^{\mathrm{T}} \mathbf{C} \tilde{\Theta}(t), \tag{2.4}$$

where

$$\Theta(x) := [\theta_0^{0,\mathtt{L}}(x), \cdots, \theta_n^{0,\mathtt{L}}(x)]^{\mathrm{T}}, \tag{2.5}$$

$$\tilde{\Theta}(t) := [\theta_0^{0,\mathtt{T}}(t), \cdots, \theta_m^{0,\mathtt{T}}(t)]^{\mathrm{T}}, \tag{2.6}$$

and

$$\mathbf{C} := \Big[c_{i,j}\Big]_{(n+1)\times(m+1)}, \qquad i = 0,1,...,n, \quad j = 0,1,...,m. \tag{2.7}$$

Theorem 2.4. *([6]) Suppose that $C^{n+1,m+1}(\tilde{\Omega})$ is the space of functions with continuous partial derivatives and let $f(x,t) \in C^{n+1,m+1}(\tilde{\Omega})$ satisfies the conditions*

$$\max_{(x,t)\in\tilde{\Omega}} \left| \frac{\partial^{n+1} f}{\partial x^{n+1}}(x,t) \right| \le \beta_1,$$

$$\max_{(x,t)\in\tilde{\Omega}} \left| \frac{\partial^{m+1} f}{\partial t^{m+1}}(x,t) \right| \le \beta_2,$$

$$\max_{(x,t)\in\tilde{\Omega}} \left| \frac{\partial^{n+m+2} f}{\partial x^{n+1} \partial t^{m+1}}(x,t) \right| \le \beta_3,$$

where β_1, β_2 and β_3 are positive constants. Let $f_{n,m}(x,t)$ is an approximation of $f(x,t)$ defined by (2.4), then

$$\|f - f_{n,m}\|_2 \le \frac{\beta_1 \mathtt{L}^{n+1}}{(n+1)! 2^{2n+1}} + \frac{\beta_2 \mathtt{T}^{m+1}}{(m+1)! 2^{2m+1}} + \frac{\beta_3 \mathtt{L}^{n+1} \mathtt{T}^{m+1}}{(n+1)!(m+1)! 2^{2n+2m+2}}.$$

3 Description of the Collocation Approach

To find a numerical solution of Eq.(1.1), assume

$$u_{n,m}(x,t) = \sum_{i=0}^{n} \sum_{j=0}^{m} c_{i,j} \theta_i^{0,\mathtt{L}}(x) \theta_j^{0,\mathtt{T}}(t) = \Theta(x)^{\mathrm{T}} \mathbf{C} \tilde{\Theta}(t), \tag{3.1}$$

where $\Theta(x)$ and $\tilde{\Theta}(t)$ are defined by Eqs. (2.5)-(2.6) and the unknown coefficients matrix $\mathbf{C}$ is defined by (2.7) that must be determined. According to Eqs. (1.1) and (3.1), we have

$$\Theta(x)^{\mathrm{T}}\mathbf{C}\mathbf{D}^{\alpha}(t)-\Theta_{xx}(x)^{\mathrm{T}}\mathbf{C}\tilde{\Theta}(t)=-\left(\Theta(x)^{\mathrm{T}}\mathbf{C}\tilde{\Theta}(t)\right)^{3}+\Theta(x)^{\mathrm{T}}\mathbf{C}\tilde{\Theta}(t)+\sigma\Theta(x)^{\mathrm{T}}\mathbf{C}\tilde{\Theta}(t)\dot{\mathrm{B}}(t), \tag{3.2}$$

where $\mathbf{D}^{\alpha}(t)$ is the Caputo fractional derivative of the vector $\tilde{\Theta}(t)$ and is obtained by

$$\mathbf{D}^{\alpha}(t)=\partial_t^{\alpha}\tilde{\Theta}(t)=\left[\partial_t^{\alpha}\left(\theta_0^{0,\mathtt{T}}(t)\right),\cdots,\partial_t^{\alpha}\left(\theta_m^{0,\mathtt{T}}(t)\right)\right]^{\mathrm{T}}.$$

From Eq. (1.4) and the explicit form of the shifted LP (2.2), we have

$$\partial_t^{\alpha}\left(\theta_0^{0,\mathtt{T}}(t)\right)=\partial_t^{\alpha}(1)=0.$$

Also, we know that [5]

$$\partial_t^{\alpha}t^{j}=\frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}t^{j-\alpha}.$$

For $j=1,...,m$, leads to

$$\vartheta_j^{\alpha}(t):=\partial_t^{\alpha}\left(\theta_j^{0,\mathtt{T}}(t)\right)=\sum_{r=0}^{[\frac{j}{2}]}\sum_{k=0}^{j-2r}\sum_{j=0}^{k}\varpi_{r,k,j}^{0,\mathtt{T}}\ \partial_t^{\alpha}(t^{j})=\sum_{r=0}^{[\frac{j}{2}]}\sum_{k=0}^{j-2r}\sum_{j=0}^{k}\frac{\Gamma(j+1)\varpi_{r,k,j}^{0,\mathtt{T}}}{\Gamma(j+1-\alpha)}\ t^{j-\alpha},$$

thus

$$\mathbf{D}^{\alpha}(t):=\left[0,\vartheta_1^{\alpha}(t),...,\vartheta_m^{\alpha}(t)\right]^{T}. \tag{3.3}$$

Also

$$\Theta_{xx}(x)=\left[\frac{d^2}{dx^2}\theta_0^{0,\mathtt{L}}(x),\cdots,\frac{d^2}{dx^2}\theta_n^{0,\mathtt{L}}(x)\right]^{\mathrm{T}}.$$

Let $x_0=0$, $x_n=\mathtt{L}$ and $\{x_i;\ i=1,...,n-1\}$ are the roots of $\theta_{n-1}^{0,\mathtt{L}}(x)$. Also, suppose that $t_0=0$ and $\{t_j;\ j=1,...,m\}$ are the roots of $\theta_m^{0,\mathtt{T}}(t)$. By considering these collocation points, we define

$$\begin{aligned}
\Phi &:= [\Theta(x_1),...,\Theta(x_i),...,\Theta(x_{n-1})]^{\mathrm{T}}_{(n-1)\times(n+1)}, && (3.4)\\
\Phi_{xx} &:= [\Theta_{xx}(x_1),...,\Theta_{xx}(x_i),...,\Theta_{xx}(x_{n-1})]^{\mathrm{T}}_{(n-1)\times(n+1)}, && (3.5)\\
\Psi &:= [\tilde{\Theta}(t_1),...,\tilde{\Theta}(t_j),...,\tilde{\Theta}(t_m)]_{(m+1)\times m}, && (3.6)\\
\Psi_{\alpha} &:= [\mathbf{D}^{\alpha}(t_1),...,\mathbf{D}^{\alpha}(t_j),...,\mathbf{D}^{\alpha}(t_m)]_{(m+1)\times m}. && (3.7)
\end{aligned}$$

Now, by evaluating (3.2) at collocation points (x_i,t_j) for $i=1,...,n-1$, $j=1,...,m$ and by using the Kronecker product, we have

$$\mathbf{S}\mathbf{X}+\Lambda_{vec}=O, \tag{3.8}$$

where

$$\begin{aligned}
\mathbf{S} &:= \left(\Psi_{\alpha}-\sigma\Psi\mathcal{B}\right)^{\mathrm{T}}\otimes\Phi-\Psi^{\mathrm{T}}\otimes\left(\Phi_{xx}+\Phi\right),\\
\mathcal{B} &:= diag[\mathsf{b}_1,...,\mathsf{b}_j,...,\mathsf{b}_m],\quad \mathsf{b}_j:=\mathrm{B}(t_j)-\mathrm{B}(t_{j-1}),\\
\mathbf{X} &:= \mathrm{vec}(\mathbf{C}),\\
\Lambda_{vec} &:= \mathrm{vec}\left((\Phi\mathbf{C}\Psi)^{3}\right),
\end{aligned}$$

where $\otimes$ denotes the Kronecker product. Also, evaluating (1.2) at the collocation points t_j, $j = 1, ..., m$ and (1.3) at collocation points x_i, $i = 0, ..., n$, and using the Kronecker product, yield

$$\left(\Psi^{\mathrm{T}} \otimes \Theta(0)^{\mathrm{T}}\right) \mathbf{X} = 0, \tag{3.9}$$

$$\left(\Psi^{\mathrm{T}} \otimes \Theta(\mathtt{L})^{\mathrm{T}}\right) \mathbf{X} = 0, \tag{3.10}$$

$$\left(\tilde{\Theta}(0)^{\mathrm{T}} \otimes \hat{\Phi}\right) \mathbf{X} = \hat{\zeta}, \tag{3.11}$$

where

$$\hat{\Phi} := [\Theta(x_0), ..., \Theta(x_n)]^{\mathrm{T}}, \quad \hat{\zeta} := [\zeta_0(x_0), ..., \zeta_0(x_n)]^{\mathrm{T}}.$$

Thus, from Eqs. (3.8)-(3.11), we obtain a system of nonlinear equations

$$\mathbf{AX} + \mathbf{E} = \mathbf{B}, \tag{3.12}$$

in which

$$\begin{aligned}
\mathbf{A} &:= \left[\mathbf{S}^{\mathrm{T}}, \left(\tilde{\Theta}(0)^{\mathrm{T}} \otimes \hat{\Phi}\right)^{\mathrm{T}}, \left(\Psi^{\mathrm{T}} \otimes \Theta(0)^{\mathrm{T}}\right)^{\mathrm{T}}, \left(\Psi^{\mathrm{T}} \otimes \Theta(\mathtt{L})^{\mathrm{T}}\right)^{\mathrm{T}}\right]^{\mathrm{T}}, \\
\mathbf{E} &:= \left[\Lambda_{vec}^{\mathrm{T}}, O, O, O\right]^{\mathrm{T}}, \\
\mathbf{B} &:= \left[O, \hat{\zeta}^{\mathrm{T}}, O, O\right]^{\mathrm{T}}.
\end{aligned}$$

The relation (3.12) gives a system of $(n+1) \times (m+1)$ nonlinear algebraic equations which can be solved, for the unknown coefficient $c_{i,j}$, $i = 0, 1, ..., n$, $j = 0, 1, ..., m$, using Newtons iterative method. As a result, an approximate solution $u_{n,m}(x,t)$ can be attained from (3.1).

4 Numerical test examples

In this section, we investigate our proposed approach for solving the stochastic Allen-Cahn equation of fractional order. We evaluate the numerical solution $u(x,t)$ along p discretized Brownian paths. Also, the arithmetic mean of $u(x,t)$ over these paths is considered. The codes are written in Matlab software and the computations are performed on a machine using a 1.70 GHz processor.

Example 4.1. *Consider the Eq.(1.1) with*

$$\partial_t^{\alpha} u(x,t) - \Delta u(x,t) = -\mathbf{F}'(u(x,t)) + \sigma u(x,t)\dot{\mathrm{B}}(t) + f(x,t),$$

where

$$f(x,t) = \frac{5\Gamma(5)}{\Gamma(5-\alpha)} t^{4-\alpha} e^{x} + 5t^4 e^x (25t^8 e^{2x} - 2) - \sigma 5t^4 e^x \dot{\mathrm{B}}(t).$$

Note that the exact solution of this example is $u(x,t) = 5t^4 e^x$.

Figure 1 shows the exact and numerical solution of $u(x,t)$ with $\mathsf{p} = 50$, $\sigma = 1$, $\alpha = 0.75$ and $n = m = 10$. Figure 2 displays the absolute error of $u(x,t)$ with different values of σ, when $\alpha = 0.5$, $\mathsf{p} = 80$ and $n = m = 7$.

Example 4.2. *Consider the Eq.(1.1) with $\zeta_0(x) = \sin(\pi x)$.*

Figures 3, 4, 5 and 6 show numerical solution of $u(x,t)$ (left) and contour plot (right) with $\mathsf{p} = 100$, $\sigma = 1$, $n = m = 7$ and $\alpha = 0.99, 0.75, 0.5, 0.2$. Figure 7 displays the numerical solution of $u(x,\mathtt{T})$ along $\mathsf{p} = 100$ different discretized Brownian paths (Blue) and their arithmetic mean (Red), when $\alpha = 0.99$, $\sigma = 1.2$ and $n = m = 6$. Figure 8 shows the numerical solution of $u(x,\mathtt{T})$ at $\mathtt{T} = 1$ for different values of α, when $\sigma = 1.2$, $\mathsf{p} = 50$ and $n = m = 8$.

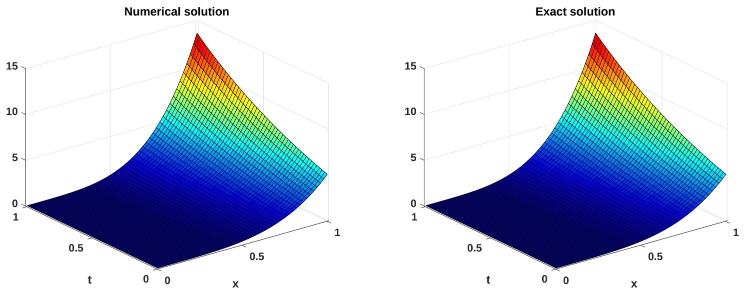

Figure 1: The exact and numerical solution of $u(x,t)$ with $\alpha = 0.75$ in Example 4.1.

5 Conclusion

In this work, a numerical technique for solving the time-fractional stochastic Allen-Cahn equation was introduced. A collocation method based on the Legendre polynomials was employed to determine the approximate solution of the considered problem. In addition, two numerical examples were investigated to demonstrate the capability of the algorithm. The obtained results authenticate the feasibility and efficiency of this numerical approach. To our experience, the numerical results for the considered stochastic problem, obtained from the collocation approach based on the LPs, are more accurate than the collocation schemes using the other well-known basis functions such as Laguerre polynomials, Chebyshev polynomials or Hermite polynomials. For future research works, it may be possible to use this numerical scheme for solving the stochastic Allen-Cahn equation with fractional Brownian motion. Also, applying this approach for similar models with distributed order fractional derivative could be the subject of some future researches.

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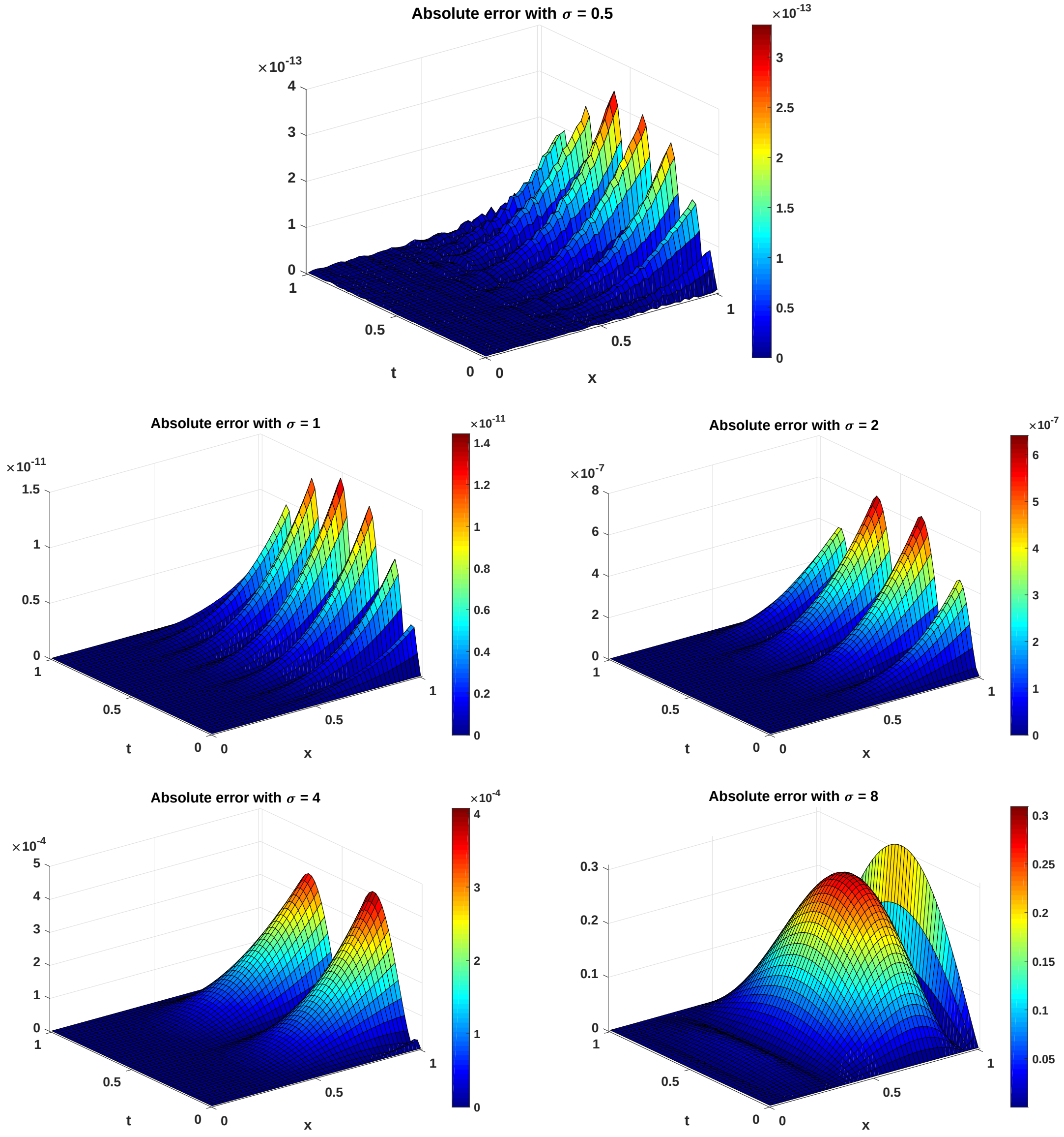

Figure 2: The absolute error of $u(x,t)$ for different values of σ in Example 4.1.

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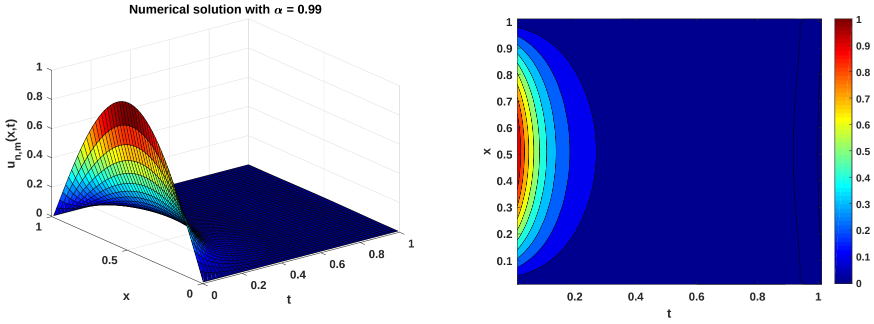

Figure 3: The numerical solution of $u(x,t)$ (left) and contour plot (right) with $\alpha = 0.99$ in Example 4.2.

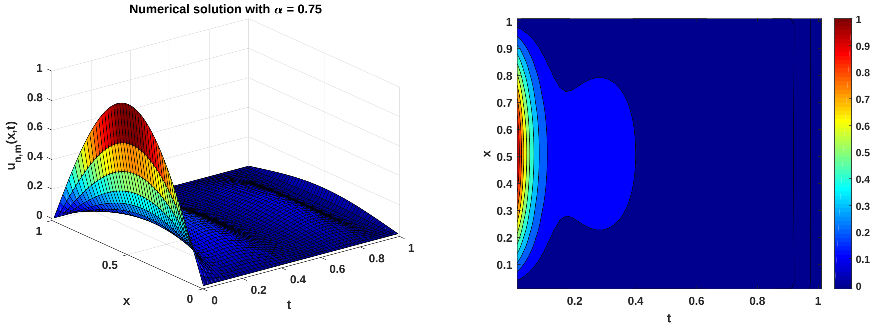

Figure 4: The numerical solution of $u(x,t)$ (left) and contour plot (right) with $\alpha = 0.75$ in Example 4.2.

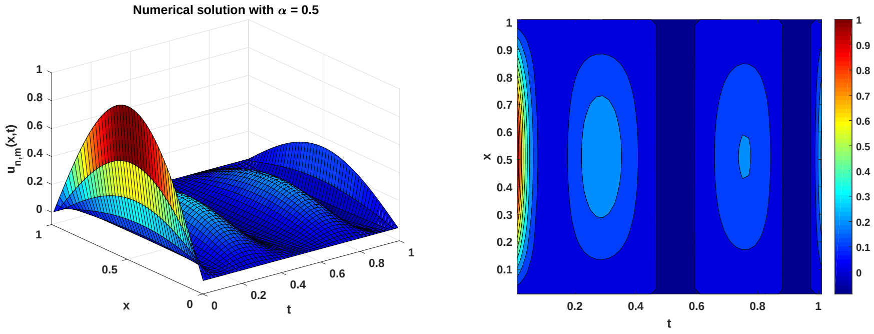

Figure 5: The numerical solution of $u(x,t)$ (left) and contour plot (right) with $\alpha = 0.5$ in Example 4.2.

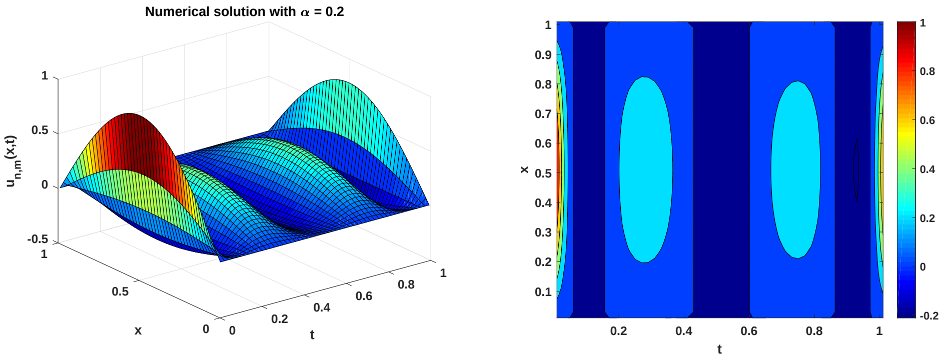

Figure 6: The numerical solution of $u(x,t)$ (left) and contour plot (right) with $\alpha = 0.2$ in Example 4.2.

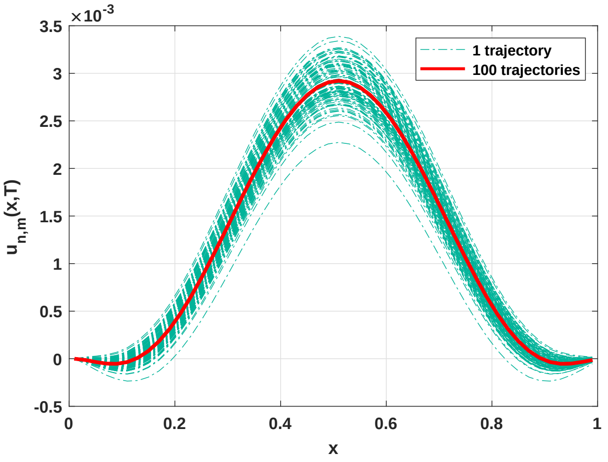

Figure 7: The numerical solution of $u(x,\mathtt{T})$ along $\mathtt{p} = 100$ different discretized Brownian paths (Blue) and their arithmetic mean (Red) with $\alpha = 0.99$ in Example 4.2.

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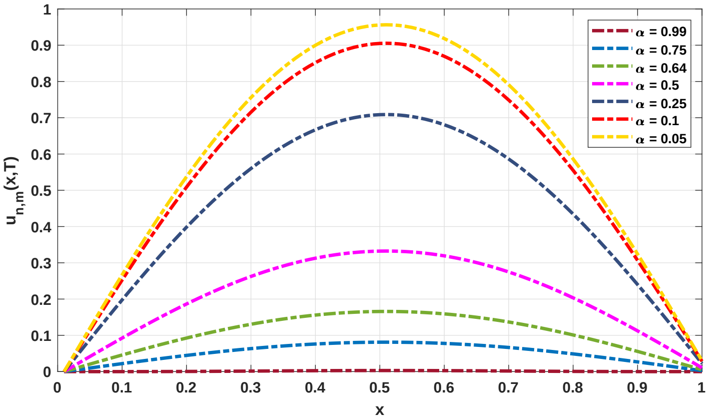

Figure 8: The numerical solution of $u(x,\mathtt{T})$ at $\mathtt{T}=1$ for different values of α in Example 4.2.

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