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A Fractional Variational Iteration Approach for Solving Time-Fractional Navier-Stokes Equations

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Abstract: The fractional variational iteration technique (FVIM), a dependable semi-analytic approach for solving multi-dimensional Navier-Stokes equations, is explained in this article. The accuracy, efficiency, and convergence of the provided approach are tested using a variety of demonstrative instances.

Keywords: Navier-Stokes equations; Fractional variational iteration method; Atangana-Baleanu fractional derivative.
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1 Introduction

Many analytical and approximation approaches for solving fractional differential equations have been developed in recent years [1-7,9,11-35]. El-Shahed and Salem [8] added an approximate solution of order $\alpha,\ 0<\alpha\leq 1$ to the basic Navier-Stokes equations (NSEs) for the first time derivative. They used the Laplace transform, the Fourier sine transform, and the finite Hankel transform to produce precise answers in three different situations.
The NSEs, as well as the continuity equations, are provided by

$$\frac{\partial u}{\partial t}+(u.\Delta)u=-\frac{1}{\rho}\nabla\rho+\gamma\nabla^2 u, \tag{1}$$

$$\nabla.u=0, \tag{2}$$

where t is the time, u is the velocity vector, ρ is the pressure, γ is the kinematics viscosity and q is the density. This model may be extended by substituting a fractional derivative of order $\alpha,\ 0<\alpha\leq 1$ for the first-time derivative. The operator equation is then used to construct the time-fractional form of NSEs:

$${}_{a}^{AB}D_t^{\alpha}u(x,t)+(u.\Delta)u=-\frac{1}{\rho}\nabla\rho+\gamma\nabla^2 u, \tag{3}$$

where ${}_{a}^{AB}D_t^{\alpha}u(x,t)$ denotes ABFD of order α.

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Our objective is to illustrate the FVIM and show how to use it with ABDO to solve the Navier-Stokes problem. The remainder of this work is broken down into the sections below. In section 2, you'll find some fractional calculus definitions. The FVIM analysis is carried out in Section 3 utilizing ABDO. Section 4 shows how FVIM may be use. Section 5 is where this effort ends.

2 Preliminaries

Definition 2.1. The Atangana-Baleanu fractional derivative (ABFD) of order α defined as follows [10]:

$$ {}_{a}^{AB}D_{t}^{\alpha}\, u(t) = \frac{M(\alpha)}{1-\alpha}\int_{a}^{t} E_{\alpha}\left(\frac{-\alpha(t-x)^{\alpha}}{\alpha-1}\right) u'(x)dx \tag{2.1} $$

where $0<\alpha<1$ and $M(\alpha)$ is a normalization function, such that $M(0)=M(1)=1$.

(2.1)'s characteristics and Laplace transform are defined as follows:

1. ${}_{a}^{AB}D_{t}^{\alpha}\ c=0,$ where c is a constant.
2. $L\{{}_{a}^{AB}D_{t}^{\alpha}\ u\,(x\,,t\,)\} = \frac{s^{\alpha}L\,u\,(\,x\,,t\,)}{s^{\alpha}(1-\,\alpha\,)\,+\,\alpha} - \frac{s^{\alpha-1}\,u\,(\,x\,,0)}{s^{\alpha}(1-\,\alpha\,)\,+\,\alpha}.$

Definition 2.2. The Atangana-Baleanu fractional integral (ABFI) of order α defined as follows [10]:

$$ {}_{a}^{AB}I_{t}^{\alpha}\, u\,(\,t\,) = \frac{1-\,\alpha}{M\,(\,\alpha\,)} u\,(\,t\,) + \frac{\alpha}{M\,(\,\alpha\,)}\,\frac{1}{\Gamma\,(\alpha\,)}\int_{a}^{t}(\,t-x\,)^{\alpha-1}\,u\,(\,x\,)\,dx \tag{2.2} $$

The properties of (2.2) is defined as follows:

1. ${}_{a}^{AB}I_{t}^{\alpha}\ {}_{a}^{AB}D_{t}^{\alpha}\ \ u\,(\,t\,)\ =\ u\,(\,t\,).$
2. ${}_{a}^{AB}I_{t}^{\alpha}\ c\ =\ \frac{c}{M\,(\,\alpha\,)}\,(\,1-\,\alpha + \frac{t^{\alpha}}{\Gamma(\alpha\,)})\ .$
3. ${}_{a}^{AB}I_{t}^{\alpha}\ \ t^{k}\ =\ \frac{t^{k}}{M\,(\,\alpha\,)}\,(\,1-\,\alpha + \frac{\alpha\,\Gamma\,(\,k+1\,)\ \ t^{\alpha}}{\Gamma(\alpha+k+1\,)})$

3 Analysis of FVIM

Consider the following: partial differential equation with fractions

$$ {}^{AB}D_{t}^{\alpha}\, u(x,t\,) + R\,u(x,t\,) + N\,u(x\,,t\,) = g(x,t), \qquad 0\,<\,\alpha\,\le 1 \tag{3.1} $$

with the initial conditions

$$ u(x,0)=f(x), $$

where ${}^{AB}D_{t}^{\alpha}\,u(x,t\,)$ is ABFD, R is the linear differential operator, N denotes the nonlinear term, and g(x,t) denotes the source term.

The correctional functional for (3.7) is approximately expressed as follows:

$$ u_{n+1}(x,t\,) = u_{n}(x,t\,) + {}_{a}^{AB}I_{t}^{\alpha}\ \left[\lambda(\xi\,)\left({}^{AB}D_{\xi}^{\alpha}u_{n}(x,\xi) + R\,\tilde{u}_{n}(x,\xi) + N\tilde{u}_{n}(x,\xi) - g(x,\xi)\right)\right], \tag{3.2} $$

where $\lambda(\xi\,)$ is general Lagrange's multiplier. $\tilde{u}_{n}$ and g are considered as restricted variations. Putting the relevant adjustment in place and making it functioning and noticing $\delta\tilde{u}_{n}=0$ and $g=0,$ we obtain

$$ \delta u_{n+1}(x,t\,) = \delta u_{n}(x,t\,) + {}_{a}^{AB}I_{t}^{\alpha}\ \left[\delta\lambda(\xi\,)\left({}^{AB}D_{\xi}^{\alpha}u_{n}(x,\xi)\right)\right], $$

or

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \lambda(\xi)\, \delta u_n(x,t) - {}^{AB}_{a}I^{\alpha}_{t}\ [\lambda'(\xi)\, \delta u_n(x,\xi)],$$

which produces the stationary conditions

$$\lambda'(\xi) = 0,$$
$$1 + \lambda(\xi) = 0$$

Therefore, we identified $\lambda = -1$ and obtain the following variational iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - {}^{AB}_{a}I^{\alpha}_{t}\ \left[{}^{AB}D^{\alpha}_{\xi} u_n(x,\xi) + R\, u_n(x,\xi) + N u_n(x,\xi) - g(x,\xi)\right]. \tag{3.3}$$

Finally, we obtain the solution of (3.1) as follows:

$$u(x,t) = \lim_{n\to\infty} u_n.$$

4 Applications of FVIM

Example 4.1 Consider the fractional NSE below:

$$^{AB}D^{\alpha}_{t} u = p + u_{xx} + \frac{1}{x}\, u_x, \quad 0 < \alpha \le 1 \tag{4.1}$$

with initial condition

$u(x,0) = 1 - x^2$

In view of (3.3) and (4.1), we get

$$u_{n+1}(x,t) = u_n(x,t) - {}^{AB}I^{\alpha}\left({}^{AB}D^{\alpha}_{t}\, u_n(x,t) - p - u_{n_{xx}} - \frac{1}{x}\, u_{n_x}\right).$$

Therefore, we obtain the successive approximations as follows:

$$u_0(x,t) = 1 - x^2$$
$$u_1(x,t) = 1 - x^2 - {}^{AB}I^{\alpha}[-p+2+2]$$
$$= 1 - x^2 + (p-4)\left[1 - \alpha + \frac{t^{\alpha}}{\Gamma(\alpha)}\right]$$
$$u_2(x,t) = 1 - x^2 - (p-4)\left(1 - \alpha + \frac{t^{\alpha}}{\Gamma(\alpha)}\right) - {}^{AB}I^{\alpha}[p-4-p+4]$$
$$= 1 - x^2 + (p-4)\left(1 - \alpha + \frac{t^{\alpha}}{\Gamma(\alpha)}\right)$$
$$\vdots$$
$$u_n(x,t) = 1 - x^2 + (p-4)\left(1 - \alpha + \frac{t^{\alpha}}{\Gamma(\alpha)}\right)$$

The solution of (4.1) is

$$u(x,t) = 1 - x^2 + (p-4)\left(1 - \alpha + \frac{t^{\alpha}}{\Gamma(\alpha)}\right)$$

If $\alpha = 1$, then the closed form solution of (4.1) is

$$u(x,t) = 1 - x^2 + (p-4)t$$

Example 4.2 Consider the fractional NSE below:

$$^{AB}D^{\alpha}_{t} u = u_{xx} + \frac{1}{x} u_x, \quad 0 < \alpha \le 1 \tag{4.2}$$

with initial condition

$u(x,0) = x$

From (3.3) and (4.2), we obtain

$$u_{n+1}(x,t) = u_n(x,t) - {}^{AB}I^{\alpha}\left[{}^{AB}D^{\alpha}_{t} u_n - u_{nxx} - \frac{1}{x} u_{nxx}\right].$$

Therefore, we obtain the successive approximations as follows:

$$u_0(x,t) = x$$
$$u_1(x,t) = x - {}^{AB}I^{\alpha}\left[-\frac{1}{x}\right]$$
$$= x + \frac{1}{x}\left(1 - \alpha + \frac{t^{\alpha}}{\Gamma(\alpha)}\right)$$
$$u_2(x,t) = x + \frac{1}{x}\left(1 - \alpha + \frac{t^{\alpha}}{\Gamma(\alpha)}\right) - {}^{AB}I^{\alpha}\left(\frac{1}{x} - \frac{2}{x^3}\left(1 - \alpha + \frac{\xi^{\alpha}}{\Gamma(\alpha)}\right) - \frac{1}{x} + \frac{1}{x^3}\left(1 - \alpha + \frac{\xi^{\alpha}}{\Gamma(\alpha)}\right)\right)$$
$$= x + \frac{1}{x}\left(1 - \alpha + \frac{t^{\alpha}}{\Gamma(\alpha)}\right) + \frac{1}{x^3}\left((1-\alpha)^2 + 2(1-\alpha)\frac{t^{\alpha}}{\Gamma(\alpha)} + \frac{\alpha^2 t^{2\alpha}}{\Gamma(2\alpha+1)}\right)$$
$$\vdots$$

The solution of (4.2) is

$$u(x,t) = x + \frac{1}{x}\left(1 - \alpha + \frac{t^{\alpha}}{\Gamma(\alpha)}\right) + \frac{1}{x^3}\left((1-\alpha)^2 + 2(1-\alpha)\frac{t^{\alpha}}{\Gamma(\alpha)} + \frac{\alpha^2 t^{2\alpha}}{\Gamma(2\alpha+1)}\right) + \cdots$$

If $\alpha = 1$, then the closed form solution of (4.2) is

$$u(x,t) = x + \sum_{n=1}^{\infty} \frac{1^2 \times 2^2 \times 3^2 \times \ldots\ldots(2n-3)^2}{x^{2n-1}} \cdot \frac{t^n}{n!}$$

Example 4.3 Consider the two-dimensional Navier-Stokes equations in time fractional-order:

$${}^{AB}D_t^{\alpha}u + uu_x + vu_y = p[u_{xx} + u_{yy}] + q$$
$${}^{AB}v_t^{\alpha}v + uv_x + vv_y = p[v_{xx} + v_{yy}] - q \quad (4.3)$$

with initial condition

$$u(0) = -sin(x+y)$$
$$v(0) = sin(x+y).$$

Using (3.3) and (4.3), we get

$$u_{n+1}(x,y,t) = u_n - {}^{AB}I^{\alpha}\left[{}^{AB}D_t^{\alpha}u_n + (u_n)(u_n)_x + (v_n)(u_n)_y - p[u_{nxx} + u_{nyy}] - q\right]$$
$$v_{n+1}(x,y,t) = v_n - {}^{AB}I^{\alpha}\left[{}^{AB}D_t^{\alpha}u_n + (u_n)(v_n)_x + (v_n)(v_n)_y - p[(v_n)_{xx} + v_{nyy}] + q\right]$$

Therefore, we obtain the successive approximations as follows:

$$u_0 = -sin\,x\,(x+y)$$
$$v_0 = sin\,x\,(x+y)$$
$$u_1 = -sin(x+y) - {}^{AB}I^{\alpha}[-2p\,sin\,(x+y) - q]$$
$$= -sin(x+y) + 2p\,sin\,(x+y)\left[1 - \alpha + \frac{t^{\alpha}}{\Gamma(\alpha)}\right] + q\left(1 - \alpha + \frac{t^{\alpha}}{\Gamma(\alpha)}\right)$$
$$v_1 = sin(x+y) - {}^{AB}I^{\alpha}[\,p[-2\,sin(x+y)] + q]]$$
$$= sin(x+y) - 2p\,sin\,(x+y)\left[1 - \alpha + \frac{t^{\alpha}}{\Gamma(\alpha)}\right] - q\left(1 - \alpha + \frac{t^{\alpha}}{\Gamma(\alpha)}\right)$$
$$u_2 = -sin(x+y) + 2p\,sin(x+y)\left(1 - \alpha + \frac{t^{\alpha}}{\Gamma(\alpha)}\right) + q(1 - \alpha + \frac{t^{\alpha}}{\Gamma(\alpha)})$$

$$-{}^{AB}I^{\alpha}\left[2p\,sin(x+y)+q-p\left[2\,sin(x+y)-4p\,sin(x+y)\left[1-\alpha+\frac{\xi^{\alpha}}{\Gamma(\alpha)}\right]-q\right]\right]$$

$$= -sin(x+y)+2p\,(sin\,x+y)\left(1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right)-4p^2\,sin(x+y)\{\,(1-\alpha)^2+\,2(1-\alpha)\frac{t^{\alpha}}{\Gamma(\alpha)}$$

$$+\frac{\alpha^2\,t^2}{\Gamma(2\alpha+1)}\ \}+q\left(1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right)$$

$$v_2) = sin(x+y)$$

$$-\,2p\,sin(x+y)\left(1-\alpha+\frac{t^{\alpha}}{\Gamma_{\alpha}}\right)-q\left(1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right)$$

$$-\,{}^{AB}I^{\alpha}\left[\{[-4p^2\,sin(x+y]\}+\ \left[1-\alpha+\frac{\xi^{\alpha}}{\Gamma(\alpha)}\right]\right]$$

$$= sin(x+y)-2p\,(sin(x+y)\left(1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right)+4p^2\,sin(x+y)\,\{(1-\alpha)^2$$

$$+\,2(1-\alpha)\frac{t^{\alpha}}{\Gamma(\alpha)}+\frac{\alpha^2\,t^2}{\Gamma(2\alpha+1)}\ \}-q\left(1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right)$$

$$\vdots$$

Therefore, the solution of (4.3) is

$$u = -sin(x+y)+2p\,(sin\,x+y)\left(1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right)-4p^2\,sin(x+y)\{\,(1-\alpha)^2+\,2(1-\alpha)\frac{t^{\alpha}}{\Gamma(\alpha)}$$

$$+\frac{\alpha^2\,t^2}{\Gamma(2\alpha+1)}\ \}+q\left(1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right)+\cdots$$

$$v = sin(x+y)-2p\,(sin(x+y)\left(1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right)$$

$$+4p^2\,sin\,(x+y)\,\{(1-\alpha)^2+\,2(1-\alpha)\frac{t^{\alpha}}{\Gamma(\alpha)}+\frac{\alpha^2\,t^2}{\Gamma(2\alpha+1)}\}-q\left(1-\alpha+\frac{t^{\alpha}}{\Gamma(\alpha)}\right)+\cdots$$

If $\alpha = 1$ and $q = 0$, then the closed form solution of (4.3) is
$u(x\,,y\,,t\,) = -\,e^{-\,2p\,t}\,sin(\,x+y\,)$
$v(x\,,y\,,t\,) = \ e^{-\,2p\,t}\,sin(\,x+y\,)$

5 Conclusion

The fractional-order multi-dimensional Navier-Stokes equations were evaluated using FVIM with ABFO in this research. The Navier-Stokes equations' solutions were given using the FVIM. Within the fractal Lagrange multipliers, which can be ideally defined using fractional variational theory, the iteration functions may be easily produced. The FVIM is demonstrated to be a useful and easy-to-use tool for dealing with PDEs using fractional differential operators.

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