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The Approximate Solutions of Fractional Differential Equations with Antagana-Baleanu Fractional Operator

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Abstract: This article presents the iterative method with the natural variational iteration technique (NVIM) to obtain an approximate and numerical solution to the fractional differential equations involving the fractional derivative of Atangana-Baleaneau in the Caputo sense. The algorithm of this method has been analyzed, then the method was applied to two nonlinear fractional differential equations and a linear system. In addition, tables of values and graphs of approximate and accurate solutions were presented. In the end, this method was effective for solving these fractional equations.

Keywords: Antagana-Baleanu Operator; Fractional Variational Iteration Method; Natural Transform.
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1 Introduction

Differential equations are used to represent a variety of events in physics, biology, chemistry, engineering, finance, and other applied disciplines. Because of its significance in many practical fields, such as biological population diffusion, fluid flow, electromagnetic waves, control theory of dynamical systems, and so on, fractional partial differential equations (FPDEs), especially nonlinear ones, have piqued interest in recent years. [2,14,22,36].

The world we live in is a jumble of scientific occurrences. Mathematics is the foundation of all sciences. Many of our everyday concerns and natural events are guided or influenced by mathematical laws. Using calculus methods of integration and derivative, some real-world issues may be represented and examined in depth. Many issues, however, may be investigated more precisely by utilizing fractional calculus approaches. As a result, fractional differential equations (FDEs) have piqued the interest of several well-known scholars and scientists. A fractional derivative of any order can be either a real or a complex number. L. Hopital and Leibnitz are credited as being the first to think of fractional derivatives in 1695 [34].

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Researchers in this field flourished after this initial step because of the various applications in physics, economics, engineering, and biological sciences. To find the solutions to fractional differential equations, several complex and efficient algorithms have been designed and developed, such as the FVIM, FDTM, FSEM, FSVIM, FLTM, FHPM, FSDM, FFSM, FRDTM, FADM, FLDM, FLHPM, and FLVIM [3,5-9,11-13,15-21,23-33,35,37,38]. We aim to present the two methods NVIM and NHPM.

2 Basic Concepts

In this section, we will remember definitions of the Atangana-Baleanu-Caputo operator and natural transform in addition to some of their properties.

Definition 1 [4] Let f $\in H1(\varepsilon 1,\varepsilon 2), \varepsilon 1>\varepsilon 2$, the Atangana-Baleanu operator in Caputo sense for $0<\delta\leq 1$ is

$$ {}^{ABC}D_{\tau}^{\delta}f(\tau)=\frac{B(\delta)}{1-\delta}\int_0^{\tau}f'(x)\,E_{\delta}\left(-\frac{\delta}{1-\delta}(\tau-x)^{\delta}\right)dx, \quad (2.1) $$

where $B(\delta)$ is a normalization function satisfies $B(0)=B(1)=1$.

Definition 2 [10] The natural transform is defined over the set of function

$$ A=\left\{f(\tau)\middle|\exists M,\alpha_1,\alpha_2>0,|f(\tau)|<Me^{\frac{|\tau|}{\alpha_j}},\tau\in(-1)^j\times[0,\infty)\right\}, $$

by the following formula

$$ N\big(f(\tau)\big)=R(u,s)=\int_0^{\infty}e^{-st}f(ut)\,dt. \quad (2.2) $$

The Laplace transform can be obtained by the natural transform through the following relationship [37],

$$ R(u,s)=\frac{1}{u}\int_0^{\infty}e^{-st/u}f(t)\,dt=\frac{1}{u}F\left(\frac{s}{u}\right). \quad (2.3) $$

From [1] and Eq. (2.3), we get this relation

$$ N\left({}^{ABC}D_{\tau}^{\delta}f(\tau)\right)=\frac{B(\delta)}{1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}}\left(R(u,s)-\frac{1}{s}f(0)\right) \quad (2.4) $$

Definition 3 [1] The inverse natural transform of a function is defined by

$$ N^{-1}\big(R(u,s)\big)=f(t)=\frac{1}{2i\pi}\int_{p-i\infty}^{p+i\infty}e^{\frac{st}{u}}R(u,s)dt, \quad (2.5) $$

where s and u are natural transform variables and p is a real constant.

3 Analysis of NVIM

Suppose that fractional partial differential equation with Atangana-Baleanu-Caputo operator

$$ {}^{ABC}D_{\tau}^{\delta}\varphi(\mu,\tau)+L\big(\varphi(\mu,\tau)\big)+M\big(\varphi(\mu,\tau)\big)=f(\mu,\tau), \quad (3.1) $$

with initial condition $\varphi(\mu,0)=\varphi_0(\mu)$, where ${}^{ABC}D_{\tau}^{\delta}$ is the Atangana-Baleanu-Caputo operator, L is the linear operator, M is the nonlinear operator and $f(\mu,\tau)$ is a source term.

Applying the natural transform to Eq. (3.1) subject to the given initial condition, we get

$$ \frac{B(\delta)}{1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}}\left(N(\varphi)-\frac{1}{s}\varphi(\mu,0)\right)=N[f(\mu,\tau)-L(\varphi)-M(\varphi)], \quad (3.2) $$

by substituting initial condition of Eq. (3.2),

$$\bar{\varphi} = \frac{1}{s}\varphi_0(\mu) - \frac{1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}}{B(\delta)} N[f(\mu,\tau) - L(\varphi) - M(\varphi)], \tag{3.3}$$

applying variation iteration method

$$\bar{\varphi}_{n+1} = \bar{\varphi}_n + \lambda\left(\bar{\varphi}_n - \frac{1}{s}\varphi_0(\mu) + \frac{1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}}{B(\delta)} N[L(\varphi_n) + M(\varphi_n) - f(\mu,\tau)]\right), \tag{3.4}$$

where λ is the Lagrange multiplier, since $0 < \delta < 1$, then λ = −1, after applying the inverse of the natural transform to both sides of the equation, we get

$$\varphi_{n+1} = \varphi_0(\mu) - N^{-1}\left(\frac{1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}}{B(\delta)} N[L(\varphi_n) + M(\varphi_n) - f(\mu,\tau)]\right), \tag{3.5}$$

where is the initial iteration is $\varphi(\mu,0) = \varphi_0(\mu)$, consequently, we have

$$\varphi(\mu,\tau) = \lim_{k\to\infty} \varphi_k(\mu,\tau).$$

4 Application

In this section two nonlinear equations and a linear system will be solved using two methods NVIM, we will suppose that $B(\delta) = 1$.

Example 4.1 Let us consider the following nonlinear equation with the Atangana-Baleanu-Caputo sense

$$^{\mathcal{ABC}}\mathcal{D}_\tau^{\delta}\varphi(\mu,\tau) = -\frac{\partial}{\partial\mu}\left(\frac{12}{\mu}\varphi - \mu\right)\varphi + \frac{\partial^2}{\partial\mu^2}\varphi^2, \qquad 0 < \delta \le 1, \tag{4.1}$$

subject to the initial condition $\varphi(\mu,0) = \mu^2$.
Applying the NVIM to Eq. (4.1), we get

$$\varphi_{n+1} = \mu^2 - N^{-1}\left(\left(1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}\right) N\left(\frac{12}{\mu}\varphi_{\mu n}\varphi_n - \frac{12}{\mu^2}\varphi_n^2 - (\varphi_n^2)_{\mu\mu} + \varphi_n\right)\right), \tag{4.2}$$

now, we find the approximate solutions as,

$$\varphi_0 = \mu^2,$$

$$\varphi_1 = \mu^2 + N^{-1}\left(\left(1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}\right) N(\mu^2)\right) = \mu^2\left((2-\delta) + \delta\frac{\tau^{\delta}}{\Gamma(\delta+1)}\right),$$

$$\varphi_2 = \mu^2 + N^{-1}\left(\left(1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}\right) N\left((2-\delta) + \delta\frac{\tau^{\delta}}{\Gamma(\delta+1)}\right)\right)$$

$$= \mu^2\left(1 + (1-\delta)(2-\delta) + (\delta(1-\delta) + \delta(2-\delta))\frac{\tau^{\delta}}{\Gamma(\delta+1)} + \delta^2\frac{\tau^{2\delta}}{\Gamma(2\delta+1)}\right), \tag{4.3}$$

and so on.
Therefore, the series solution $\varphi(\mu,\tau)$ of Eq.(4.1) is given by

$$\varphi(\mu,\tau) = \mu^2\left((\delta^2 - 3\delta + 3) + (3\delta - 2\delta^2)\frac{\tau^{\delta}}{\Gamma(\delta+1)} + \delta^2\frac{\tau^{2\delta}}{\Gamma(2\delta+1)} + \cdots\right), \tag{4.4}$$

when choosing $\delta = 1$ in Eq.(4.4), it becomes

$$\varphi(\mu,\tau) = \mu^2\left(1 + \tau + \frac{\tau^2}{2!} + \cdots\right), \tag{4.5}$$

Ultimately, the exact solution of Eq. (4.1),

$$\varphi(\mu,\tau) = \mu^2 e^{\tau}. \tag{4.6}$$

Table 1: The values of the approximate and exact solutions of Eq.(4.1) at different values of $\tau, \mu, and\ \delta$.

μ	τ	$\varphi_{\delta=0.8}$	$\varphi_{\delta=0.9}$	$\varphi_{\delta=1}$	φ_{exact}	$\lvert\varphi_e - \varphi_1\rvert$
0.1000	0.1000	0.0143	0.0126	0.0111	0.0111	0.0000
0.2000	0.2000	0.0634	0.0556	0.0488	0.0489	0.0001
0.3000	0.3000	0.1558	0.1376	0.1210	0.1215	0.0004
0.4000	0.4000	0.3000	0.2675	0.2368	0.2387	0.0019
0.5000	0.5000	0.5050	0.4551	0.4062	0.4122	0.0059
0.6000	0.6000	0.7805	0.7112	0.6408	0.6560	0.0152
0.7000	0.7000	1.1365	1.0475	0.9530	0.9867	0.0337
0.8000	0.8000	1.5841	1.4764	1.3568	1.4243	0.0675
0.9000	0.9000	2.1347	2.0117	1.8670	1.9923	0.1252
1.0000	1.0000	2.8006	2.6678	2.5000	2.7183	0.2183

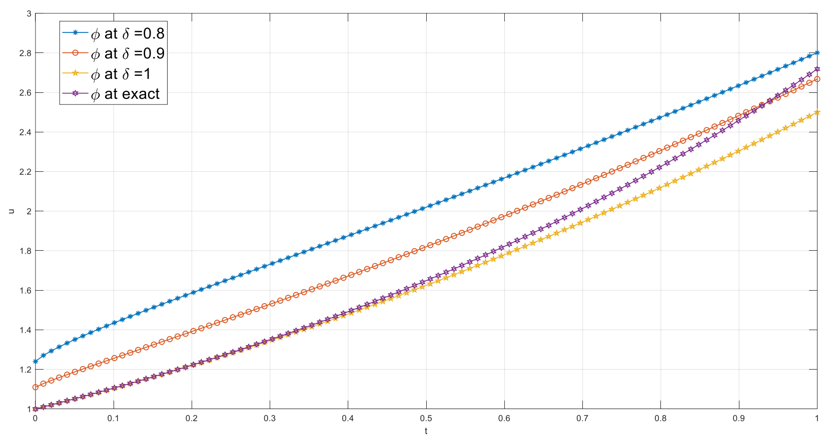

Figure 1: The graphs of the approximate and exact solutions among different values of τ,δ and $\mu = 1$, for Eq.(4.1).

Example 4.2 Let us consider the following nonlinear equation with the Atangana-Baleanu-Caputo sense

$$^{ABC}D_\tau^\delta \varphi(\mu,\tau) + \frac{1}{2}(\varphi^2)_\mu - \varphi + \varphi^2 = 0, \qquad 0 < \delta \leq 1 \tag{4.7}$$

subject to the initial condition $\varphi(\mu, 0) = e^{-\mu}$.
using algorithm of the method, we get

$$\varphi_{n+1} = e^{-\mu} - N^{-1}\left(\left(1-\delta+\delta\left(\frac{u}{s}\right)^\delta\right) N\left(\varphi^2 + \frac{1}{2}(\varphi^2)_\mu - \varphi\right)\right), \tag{4.8}$$

now, we find the approximate solutions as,

$$\varphi_0 = e^{-\mu},$$

$$\varphi_1 = e^{-\mu} - \left(\left(1-\delta+\delta\left(\frac{u}{s}\right)^\delta\right) N(-e^{-\mu})\right) = e^{-\mu}\left(2-\delta+\delta\frac{\tau^\delta}{\Gamma(\delta+1)}\right),$$

$$\varphi_2 = e^{-\mu} - \left(\left(1-\delta+\delta\left(\frac{u}{s}\right)^\delta\right) N\left(-e^{-\mu}\left(2-\delta+\delta\frac{\tau^\delta}{\Gamma(\delta+1)}\right)\right)\right)$$

$$= e^{-\mu}\left((3-3\delta+\delta^2)+(3\delta-2\delta^2)\frac{\tau^\delta}{\Gamma(\delta+1)}+\delta^2\frac{\tau^{2\delta}}{\Gamma(2\delta+1)}\right), \quad (4.9)$$

thus, the approximate solution of Eq. (4.7) can be written,

$$\varphi(\mu,\tau) = e^{-\mu}\left((3-3\delta+\delta^2)+(3\delta-2\delta^2)\frac{\tau^\delta}{\Gamma(\delta+1)}+\delta^2\frac{\tau^{2\delta}}{\Gamma(2\delta+1)}+\cdots\right), \quad (4.10)$$

when choosing $\delta = 1$ in Eq.(4.10), it becomes

$$\varphi(\mu,\tau) = e^{-\mu}\left(1+\tau+\frac{\tau^2}{2!}+\cdots\right), \quad (4.11)$$

Ultimately, the exact solution of Eq. (4.7).

$$\varphi(\mu,\tau) = e^{-\mu+\tau} \quad (4.12)$$

Table 2: The values of the approximate and exact solutions of Eq.(4.7) at different values of $\tau, \mu, and\ \delta$.

μ	τ	$\varphi_{\delta=0.8}$	$\varphi_{\delta=0.9}$	$\varphi_{\delta=1}$	φ_{exact}	$\lvert\varphi_e-\varphi_1\rvert$
0.1000	0.1000	1.2977	1.1362	0.9998	1.0000	0.0002
0.2000	0.2000	1.2986	1.1390	0.9989	1.0000	0.0011
0.3000	0.3000	1.2825	1.1328	0.9964	1.0000	0.0036
0.4000	0.4000	1.2569	1.1207	0.9921	1.0000	0.0079
0.5000	0.5000	1.2253	1.1042	0.9856	1.0000	0.0144
0.6000	0.6000	1.1898	1.0842	0.9769	1.0000	0.0231
0.7000	0.7000	1.1518	1.0615	0.9659	1.0000	0.0341
0.8000	0.8000	1.1121	1.0366	0.9526	1.0000	0.0474
0.9000	0.9000	1.0715	1.0097	0.9371	1.0000	0.0629
1.0000	1.0000	1.0303	0.9814	0.9197	1.0000	0.0803

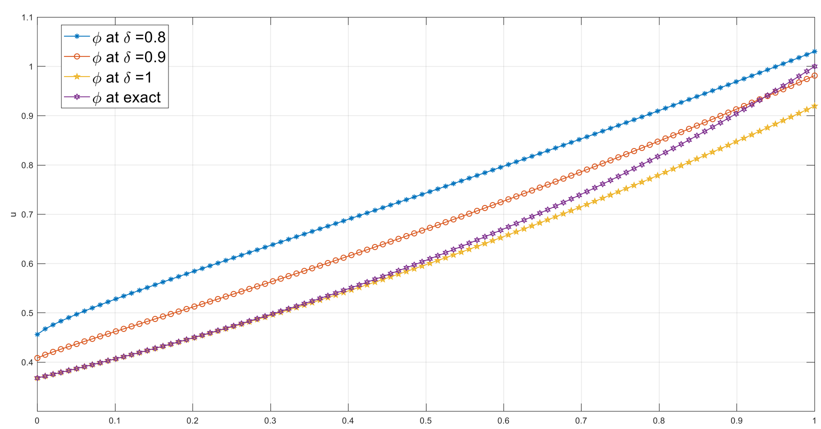

Figure 2: The graphs of the approximate and exact solutions among different values of τ,δ and $\mu = 1$, for Eq.(4.7).

Example 4.3 Consider the nonlinear system of time-fractional differential equation in the Atangana-Baleanu-Caputo operator:

$${}^{ABC}D_\tau^\delta \varphi(\mu,\tau) - \psi_\mu + \psi + \varphi = 0, \ \ 0 < \delta \le 1,$$

$$^{ABC}D_\tau^\lambda \psi(\mu,\tau) - \varphi_\mu + \psi + \varphi = 0,\ \ 0 < \lambda \leq 1, \tag{4.13}$$

where $0 < \delta, \lambda \leq 1$ and the initial conditions are

$$\begin{aligned} \varphi(\mu,0) &= \sinh(\mu), \\ \psi(\mu,0) &= \cosh(\mu). \end{aligned} \tag{4.14}$$

using algorithm of the method, we get

$$\begin{aligned} \varphi_{n+1}(\mu,\tau) &= \varphi(\mu,0) + \mathrm{N}^{-1}\left(\left(1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}\right)\mathcal{N}\{\psi_{\mu n} - \psi_n - \varphi_n\}\right), \\ \psi_{n+1}(\mu,\tau) &= \psi(\mu,0) + \mathrm{N}^{-1}\left(\left(1-\lambda+\lambda\left(\frac{u}{s}\right)^{\lambda}\right)\mathcal{N}\{\varphi_{\mu n} - \psi_n - \varphi_n\}\right), \end{aligned} \tag{4.15}$$

now, we find the approximate solutions as,

$\varphi_0 = \sinh(\mu)\,, \psi_0 = \cosh(\mu)$,

$$\varphi_1 = \sinh(\mu) - \cosh(\mu)\left(1-\delta+\delta\frac{\tau^\delta}{\Gamma(\delta+1)}\right),$$

$$\psi_1 = \cosh(\mu) - \sinh(\mu)\left(1-\lambda+\lambda\frac{\tau^\lambda}{\Gamma(\lambda+1)}\right),$$

$$\varphi_2 = \sinh(\mu) + \begin{bmatrix} (1-\delta)(1-\lambda) + \lambda(1-\delta)\,\dfrac{\tau^\lambda}{\Gamma(\lambda+1)} \\ +\delta(1-\lambda)\dfrac{\tau^\delta}{\Gamma(\delta+1)} + \delta\lambda\dfrac{\tau^{\delta+\lambda}}{\Gamma(\delta+\lambda+1)} \end{bmatrix}(\sinh(\mu)-\cosh(\mu))$$
$$+\begin{bmatrix} -\delta(1-\delta) + \delta(1-\delta)\dfrac{\tau^\delta}{\Gamma(\delta+1)} \\ -\delta^2\dfrac{\tau^\delta}{\Gamma(\delta+1)} + \delta^2\dfrac{\tau^{2\delta}}{\Gamma(2\delta+1)} \end{bmatrix}\cosh(\mu),$$

$$\psi_2 = \cosh(\mu) + \begin{bmatrix} (1-\delta)(1-\lambda) + \delta(1-\lambda)\dfrac{\tau^\delta}{\Gamma(\delta+1)} \\ +\lambda(1-\delta)\,\dfrac{\tau^\lambda}{\Gamma(\lambda+1)} + \delta\lambda\dfrac{\tau^{\delta+\lambda}}{\Gamma(\delta+\lambda+1)} \end{bmatrix}(\cosh(\mu)-\sinh(\mu))$$
$$+\begin{bmatrix} -\lambda(1-\lambda) + \lambda(1-\lambda)\dfrac{\tau^\lambda}{\Gamma(\lambda+1)} \\ -\lambda^2\dfrac{\tau^\lambda}{\Gamma(\lambda+1)} + \lambda^2\dfrac{\tau^{2\lambda}}{\Gamma(2\lambda+1)} \end{bmatrix}\sinh(\mu).$$

Therefore, the approximate solution of Eq. (4.13) is given by

$$\varphi = \sinh(\mu) + \begin{bmatrix} (1-\delta)(1-\lambda) + \lambda(1-\delta)\,\dfrac{\tau^\lambda}{\Gamma(\lambda+1)} \\ +\delta(1-\lambda)\dfrac{\tau^\delta}{\Gamma(\delta+1)} + \delta\lambda\dfrac{\tau^{\delta+\lambda}}{\Gamma(\delta+\lambda+1)} \end{bmatrix}(\sinh(\mu)-\cosh(\mu))$$
$$+\begin{bmatrix} -\delta(1-\delta) + \delta(1-\delta)\dfrac{\tau^\delta}{\Gamma(\delta+1)} \\ -\delta^2\dfrac{\tau^\delta}{\Gamma(\delta+1)} + \delta^2\dfrac{\tau^{2\delta}}{\Gamma(2\delta+1)} \end{bmatrix}\cosh(\mu) - \cdots,$$

$$\psi = \cosh(\mu) + \begin{bmatrix} (1-\delta)(1-\lambda) + \delta(1-\lambda)\dfrac{\tau^{\delta}}{\Gamma(\delta+1)} \\ +\lambda(1-\delta)\,\dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)} + \delta\lambda\dfrac{\tau^{\delta+\lambda}}{\Gamma(\delta+\lambda+1)} \end{bmatrix}(\cosh(\mu)-\sinh(\mu))$$
$$+ \begin{bmatrix} -\lambda(1-\lambda) + \lambda(1-\lambda)\dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)} \\ -\lambda^2\dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)} + \lambda^2\dfrac{\tau^{2\lambda}}{\Gamma(2\lambda+1)} \end{bmatrix}\sinh(\mu) - \cdots. \quad (4.16)$$

If we put $\delta \to 1$ and $\lambda \to 1$ in Eq.(4.16), we reproduce the solution of the problem as follows

$$\varphi(\mu,\tau) = \sinh(\mu)\left(1 + \frac{\tau^2}{2!} + \cdots\right) - \cosh(\mu)\left(\tau + \frac{\tau^3}{3!} + \cdots\right),$$
$$\psi(\mu,\tau) = \cosh(\mu)\left(\tau + \frac{\tau^3}{3!} + \cdots\right) - \sinh(\mu)\left(1 + \frac{\tau^2}{2!} + \cdots\right). \quad (4.17)$$

This solution is equivalent to the exact solution in closed form:

$$\varphi(\mu,\tau) = \sinh(\mu)\cosh(\tau) - \cosh(\mu)\sinh(\tau),$$
$$\psi(\mu,\tau) = \cosh(\mu)\sinh(\tau) - \sinh(\mu)\cosh(\tau). \quad (4.18)$$

Table 2: The values of the approximate and exact solutions of φ for Eq.(4.13) at different values of $\tau, \mu, and\ \delta$.

μ	τ	$\varphi_{\delta=0.8}$	$\varphi_{\delta=0.9}$	$\varphi_{\delta=1}$	φ_{exact}	$\lvert\varphi_e - \varphi_1\rvert$
0.1000	0.1000	-0.1916	-0.1375	0.0002	0.0000	0.0002
0.2000	0.2000	-0.1687	-0.1392	0.0013	0.0000	0.0013
0.3000	0.3000	-0.1308	-0.1272	0.0046	0.0000	0.0046
0.4000	0.4000	-0.0801	-0.1028	0.0112	0.0000	0.0112
0.5000	0.5000	-0.0165	-0.0662	0.0224	0.0000	0.0224
0.6000	0.6000	0.0608	-0.0166	0.0400	0.0000	0.0400
0.7000	0.7000	0.1534	0.0475	0.0658	0.0000	0.0658
0.8000	0.8000	0.2631	0.1277	0.1024	0.0000	0.1024
0.9000	0.9000	0.3925	0.2267	0.1525	0.0000	0.1525
1.0000	1.0000	0.5446	0.3474	0.2197	0.0000	0.2197

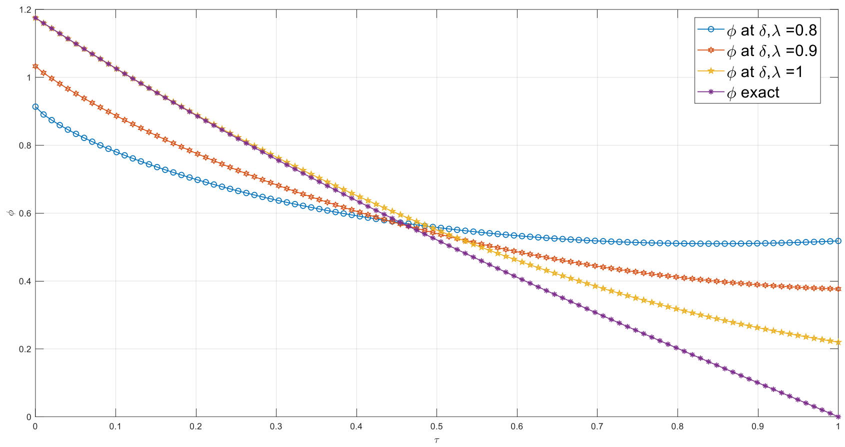

Figure 3: The graphs of the approximate and exact solutions of φ among different values of τ,δ and $\mu = 1$, for Eq.(4.13).

Table 2: The values of the approximate and exact solutions of ψ for Eq.(4.13) at different values of $\tau, \mu, and\ \delta$.

μ	τ	$\psi_{\delta=0.8}$	$\psi_{\delta=0.9}$	$\psi_{\delta=1}$	ψ_{exact}	$\lvert\psi_e - \psi_1\rvert$
0.1000	0.1000	1.0776	1.0246	1.0000	1.0000	0.0000
0.2000	0.2000	1.1044	1.0379	1.0002	1.0000	0.0002
0.3000	0.3000	1.1317	1.0529	1.0010	1.0000	0.0010
0.4000	0.4000	1.1628	1.0714	1.0033	1.0000	0.0033
0.5000	0.5000	1.2005	1.0950	1.0080	1.0000	0.0080
0.6000	0.6000	1.2472	1.1259	1.0169	1.0000	0.0169
0.7000	0.7000	1.3056	1.1664	1.0317	1.0000	0.0317
0.8000	0.8000	1.3785	1.2193	1.0549	1.0000	0.0549
0.9000	0.9000	1.4693	1.2878	1.0896	1.0000	0.0896
1.0000	1.0000	1.5813	1.3756	1.1394	1.0000	0.1394

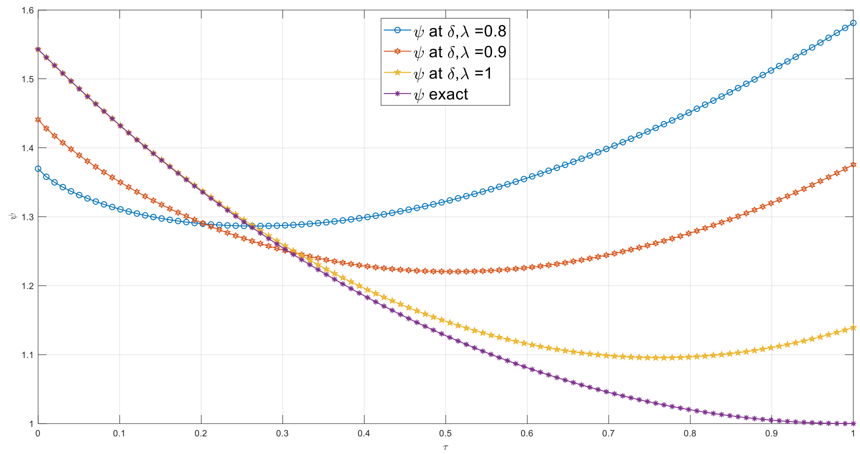

Figure 4: The graphs of the approximate and exact solutions of ψ among different values of τ,δ and $\mu = 1$, for Eq.(4.13).

5 Conclusion

Using the Atangana-Baleaneu fractional operator in the Caputo sense, two nonlinear equations and a linear system of partial differential equations have successfully been solved approximatively using the NVIM in this study. The examples demonstrate the great agreement between the NVIM findings. The technique is particularly effective in solving a variety of classes of linear and non-linear fractional differential equations analytically as well as numerically.

Reference

[1] A.S. Abdel-Rady, et al, Natural transform for solving fractional models, Journal of Applied Mathematics and Physics 3(12) 2015, 1633.

[2] M. Alquran, I. Jaradat, A novel scheme for solving Caputo time-fractional nonlinear equations: Theory and application. Nonlinear Dyn., 91(4) 2018, 2389-2395.

[3] L.K. Alzaki, H.K. Jassim, The approximate analytical solutions of nonlinear fractional ordinary differential equations, International Journal of Nonlinear Analysis and Applications, 12(2) 2021, 527-535.

[4] A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model, arXiv preprint arXiv:1602.03408, 2016.

[5] D. Baleanu, H.K. Jassim, A Modification Fractional Homotopy Perturbation Method for Solving Helmholtz and Coupled Helmholtz Equations on Cantor Sets, Fractal and Fractional, 3(30) 2019, 1-8.

[6] D. Baleanu, H.K. Jassim, Approximate Solutions of the Damped Wave Equation and Dissipative Wave Equation in Fractal Strings, Fractal and Fractional, 3(26) 2019, 1-12.

[7] D. Baleanu, H.K. Jassim, Exact Solution of Two-dimensional Fractional Partial Differential Equations, Fractal Fractional, 4(21) 2020, 1-9.

[8] D. Baleanu, H.K. Jassim, H. Khan, A Modification Fractional Variational Iteration Method for solving Nonlinear Gas Dynamic and Coupled KdV Equations Involving Local Fractional Operators, Thermal Science, 22(1) 2018, 165-175.
[9] D. Baleanu, H.K. Jassim, M. Al Qurashi, Solving Helmholtz Equation with Local Fractional Derivative Operators, Fractal and Fractional, 3(43) 2019, 1-13.
[10] F.B.M. Belgacem, R. Silambarasan, Theory of natural transform, Math. Engg. Sci. Aeros, 3 2012, 99-124.
[11] A. Bokhari, Application of Shehu transform to Atangana-Baleanu derivatives, J. Math. Computer Sci, 20 2019, 101-107.
[12] Z.P. Fan, H.K. Jassim, R.K. Rainna, X.J. Yang, Adomian Decomposition Method for Three-Dimensional Diffusion Model in Fractal Heat Transfer Involving Local Fractional Derivatives, Thermal Science, 19(1) 2015, 137-141.
[13] J.F. Gmez-Aguilar, et al., Analytical Solutions of the Electrical RLC Circuit via LiouvilleCaputo Operators with Local and Non-Local Kernels, Entropy, 18(8) 2016, 402.
[14] R. Hilfer, Applications of Fractional Calculus in Physics; World Scientific: Singapore, 2000.
[15] M.A. Hussein, H.K. Jassim, New approximate analytical technique for the solution of two dimensional fractional differential equations, NeuroQuantology 20 2022, 3690-3705.
[16] M.A. Hussein, A Review on Algorithms of Sumudu Adomian Decomposition Method for FPDEs, Journal of Research in Applied Mathematics 8 (8) 2022, 36-43.
[17] M.A. Hussein, A Review on Integral Transforms of Fractional Integral and Derivative."International Academic Journal of Science and Engineering 9 2022, 52-56.
[18] M.A. Hussein, A review on integral transforms of the fractional derivatives of CaputoFabrizio and Atangana-Baleanu, Eurasian Journal of Media and Communications, 7 2022, 17-23.
[19] H. Jafari, H.K. Jassim, D. Baleanu, Y.M. Chu, On the approximate solutions for a system of coupled Korteweg-de Vries equations with local fractional derivative, Fractals, 29(5) 2021, 1-7.
[20] H. Jafari, H.K. Jassim, J. Vahidi, Reduced Differential Transform and Variational Iteration Methods for 3D Diffusion Model in Fractal Heat Transfer within Local Fractional Operators, Thermal Science, 22(1) 2018, 301-307.
[21] H. Jafari, H.K. Jassim, On the Existence and Uniqueness of Solutions for Local differential equations, Entropy, 18 2016, 1-9.
[22] A. Jaradat, M.S.M. Noorani, M. Alquran, H. M. Jaradat, A novel method for solving Caputo-time-fractional dispersive long wave Wu-Zhang system. Nonlinear Dyn. Syst. Theory, 18(2) 2018, 182-190.
[23] H.K. Jassim, a new approach to find approximate solutions of Burgers and coupled Burgers equations of fractional order, TWMS Journal of Applied and Engineering Mathematics, 11(2) 2021, 415-423.
[24] H.K. Jassim, Analytical Approximate Solutions for Local Fractional Wave Equations, Mathematical Methods in the Applied Sciences, 43(2) 2020, 939-947.
[25] H.K. Jassim, D. Baleanu, a novel approach for Korteweg-de Vries equation of fractional order, Journal of Applied Computational Mechanics, 5(2) 2019, 192-198.
[26] H.K. Jassim, H.A. Kadhim, Fractional Sumudu decomposition method for solving PDEs of fractional order, Journal of Applied and Computational Mechanics, 7(1) 2021, 302-311.
[27] H.K. Jassim, H. Ahmad, C. Cesarano, An efficient hybrid technique for the solution of fractional-order partial differential equations, Carpathian Mathematical Publications, 13(3) 2021, 790-804.
[28] H.K. Jassim, M.A. Shareef, On approximate solutions for fractional system of differential equations with Caputo-Fabrizio fractional operator, Journal of Mathematics and Computer science, 23 2021, 58-66.
[29] H.K. Jassim, J. Vahidi, A New Technique of Reduce Differential Transform Method to Solve Local Fractional PDEs in Mathematical Physics, International Journal of Nonlinear Analysis and Applications, 12(1) 2021, 37-44.
[30] H.K. Jassim, M.A. Shareef., On approximate solutions for fractional system of differential equations with Caputo-Fabrizio fractional operator, Journal of Mathematics and Computer science, 23 2021, 58-66.
[31] H.K. Jassim, S.A. Khafif, SVIM for solving Burgers and coupled Burgers equations of fractional order, Progress in Fractional Differentiation and Applications, 7(1) 2021, 1-6.
[32] H.K. Jassim, The Approximate Solutions of Three-Dimensional Diffusion and Wave Equations within Local Fractional Derivative Operator, Abstract and Applied Analysis, 2016 2016, 1-5.

[33] H.K. Jassim, W.A. Shahab, Fractional variational iteration method to solve one dimensional second order hyperbolic telegraph equations, Journal of Physics: Conference Series, 1032(1) 2018, 1-9.
[34] I. Podlubny, Fractional Differential Equations. Academic Press, San Diego, CA, 1999.
[35] J. Singh, H.K. Jassim, D. Kumar, An efficient computational technique for local fractional Fokker-Planck equation, Physica A: Statistical Mechanics and its Applications, 555(124525) 2020, 1-8.
[36] M.I, Syam, H.M. Jaradat, M. Alquran, S. Al-Shara, An accurate method for solving a singular second-order fractional Emden- Fowler problem. Adv. Differ. Equ. 2018(1) 2018, 1-6.
[37] H.G. Taher, H. Ahmad, J. Singh, D. Kumar, H.K. Jassim, solving fractional PDEs by using Daftardar-Jafari method, AIP Conference Proceedings, 2386(060002) 2022, 1-10.
[38] S.P. Yan, H. Jafari, H.K. Jassim, Local Fractional Adomian Decomposition and Function Decomposition Methods for Solving Laplace Equation within Local Fractional Operators, Advances in Mathematical Physics, 2014 2014, 1-7.