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Qom University of Technology

Some new kinds of interpolation formulas and its applications

M. A. Jafari [1], A. Aminataei [2]

Abstract: In this work, using the determination function, some new kinds of interpolation formulas are presented. These novel formulas are extensions of Lagrange interpolation. Error formula for these new kind of interpolation formulas are obtained. Finally, using these formulas, some new numerical integration formulas in closed-type and open-type are presented. Absolute error of quadrature formulas show the efficiency of the proposed formulas.

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1 Introduction

Interpolation formulas have many applications in numerical analysis. For example, Newton-Cotes quadrature formulas, Adams-Bashforth formulas, and Adams-Moulton formulas are obtained from Lagrange interpolation formula. Recently, different kinds of interpolation formulas have been introduced (for further see [1, 2, 3, 4, 5] and references therein). Let $y_0,y_1,\cdots,y_n$ be $n+1$ known values for an arbitrary function $y:[a,b]\to\mathbb{R}$ at $a=x_0<x_1<\cdots<x_n=b$. Then the Lagrange interpolation formula is defined as [6, 7]

$$L_n(x;y)=\sum_{i=0}^{n} I_i(x)y_i, \tag{1.1}$$

where

$$I_i(x)=\prod_{\substack{k=0\\k\neq i}}^{n}\frac{x-x_k}{x_i-x_k},\ i=0,1,\cdots,n. \tag{1.2}$$

Clearly, for $i=0,1,\cdots,n$, we have

$$L_n(x_i;y)=y_i. \tag{1.3}$$

In addition, for $y\in\mathcal{C}^{n+1}[a,b]$, we have [6, 7]

$$y(x)=L_n(x;y)+(x-x_0)(x-x_1)\cdots(x-x_n)\frac{y^{(n+1)}(\eta_x)}{(n+1)!},\quad \eta_x\in[a,b]. \tag{1.4}$$

[1]Corresponding Author: Department of Financial Sciences, Kharazmi University, Tehran, Iran, Email: m.a.jafari.math@gmail.com, m.a.jafari@khu.ac.ir
[2]Faculty of Mathematics, K. N. Toosi University of Technology, ataei@kntu.ac.ir

Using properties of the Vandermonde matrix [6, 7], the $I_i(x)$ (for $i = 0, 1, \cdots, n$) defined in the relation (1.2), can be defined as

$$
\begin{aligned}
I_0(x) &= \frac{det\begin{pmatrix} 1 & x & \cdots & x^n \\ 1 & x_1 & \cdots & x_1^n \\ 1 & x_2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^n \end{pmatrix}}{det\begin{pmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ 1 & x_2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^n \end{pmatrix}}, \\
I_1(x) &= \frac{det\begin{pmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x & \cdots & x^n \\ 1 & x_2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^n \end{pmatrix}}{det\begin{pmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ 1 & x_2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^n \end{pmatrix}}, \\
&\vdots \\
I_n(x) &= \frac{det\begin{pmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ 1 & x_2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x & \cdots & x^n \end{pmatrix}}{det\begin{pmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ 1 & x_2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^n \end{pmatrix}}.
\end{aligned}
\tag{1.5}
$$

In the next section, using an extension of the above mentioned structure, some new kinds of interpolation formulas are presented.

This paper is arranged as follows. In section 2, a new type of interpolation formula is presented. In section 3, some numerical examples are presented. Finally, a short conclusion is given in section 4.

2 A new type of interpolation formulas

Theorem 2.1. *Let f_0, f_1,$\cdots$,f_n be $n+1$ independent functions on $[a, b]$. Again, let y_0,y_1,$\cdots$,y_n be $n+1$ known values for an arbitrary function $y : [a, b] \to \mathbb{R}$ at $a = x_0 < x_1 < \cdots < x_n = b$. Then the new* *interpolation formula is defined as*

$$\mathfrak{L}_{\mathfrak{n}}(x;y)=\sum_{i=0}^{n}\mathfrak{I}_i(x)y_i, \tag{2.1}$$

where $\mathfrak{I}_i$*, for* $i=0,1,\cdots,n$*,*

$$\mathfrak{I}_0(x)=\frac{det\begin{pmatrix} f_0(x) & f_1(x) & \cdots & f_n(x)\\ f_0(x_1) & f_1(x_1) & \cdots & f_n(x_1)\\ f_0(x_2) & f_1(x_2) & \cdots & f_n(x_2)\\ \vdots & \vdots & \vdots & \vdots\\ f_0(x_n) & f_1(x_n) & \cdots & f_n(x_n)\end{pmatrix}}{det\begin{pmatrix} f_0(x_0) & f_1(x_0) & \cdots & f_n(x_0)\\ f_0(x_1) & f_1(x_1) & \cdots & f_n(x_1)\\ f_0(x_2) & f_1(x_2) & \cdots & f_n(x_2)\\ \vdots & \vdots & \vdots & \vdots\\ f_0(x_n) & f_1(x_n) & \cdots & f_n(x_n)\end{pmatrix}},$$

$$\mathfrak{I}_1(x)=\frac{det\begin{pmatrix} f_0(x_0) & f_1(x_0) & \cdots & f_n(x_0)\\ f_0(x) & f_1(x) & \cdots & f_n(x)\\ f_0(x_2) & f_1(x_2) & \cdots & f_n(x_2)\\ \vdots & \vdots & \vdots & \vdots\\ f_0(x_n) & f_1(x_n) & \cdots & f_n(x_n)\end{pmatrix}}{det\begin{pmatrix} f_0(x_0) & f_1(x_0) & \cdots & f_n(x_0)\\ f_0(x_1) & f_1(x_1) & \cdots & f_n(x_1)\\ f_0(x_2) & f_1(x_2) & \cdots & f_n(x_2)\\ \vdots & \vdots & \vdots & \vdots\\ f_0(x_n) & f_1(x_n) & \cdots & f_n(x_n)\end{pmatrix}},$$

$$\vdots \tag{2.2}$$

$$\mathfrak{I}_n(x)=\frac{det\begin{pmatrix} f_0(x_0) & f_1(x_0) & \cdots & f_n(x_0)\\ f_0(x_1) & f_1(x_1) & \cdots & f_n(x_1)\\ f_0(x_2) & f_1(x_2) & \cdots & f_n(x_2)\\ \vdots & \vdots & \vdots & \vdots\\ f_0(x) & f_1(x) & \cdots & f_n(x)\end{pmatrix}}{det\begin{pmatrix} f_0(x_0) & f_1(x_0) & \cdots & f_n(x_0)\\ f_0(x_1) & f_1(x_1) & \cdots & f_n(x_1)\\ f_0(x_2) & f_1(x_2) & \cdots & f_n(x_2)\\ \vdots & \vdots & \vdots & \vdots\\ f_0(x_n) & f_1(x_n) & \cdots & f_n(x_n)\end{pmatrix}}.$$

Proof *Clearly, for* $i,j=0,1,\cdots,n$*, we have*

$$\mathfrak{I}_i(x_j)=\begin{cases} 1 & i=j,\\ \\ 0 & i\neq j.\end{cases} \tag{2.3}$$

Therefore $\mathfrak{L}_n(x)$ *defined in the relation (2.1) is interpolation function.*

Remark 1 In the relation (2.1), if $f_0(x) = 1,\ f_1(x) = x,\ f_2(x) = x^2,\ \cdots$, and $f_n(x) = x^n$ then the Lagrange interpolation formula (i.e., the relation (1.1)) is obtained.

Theorem 2.2. *If $y \in \mathcal{C}^{(n+1)}[a,b]$ then the error function of the interpolation formula (2.1), for $x \in [a,b]$ is obtained as*

$$y(x) - \mathfrak{L}_n(x;y) = \frac{y^{(n+1)}(\xi_x) - \mathfrak{L}_n^{(n+1)}(\xi_x)}{(n+1)!} \prod_{k=0}^{n} (x - x_k), \tag{2.4}$$

where $\xi_x \in (a,b)$.

Proof. *Let*

$$\Gamma(x) = y(x) - \mathfrak{L}_n(x) - \eta(\widehat{x}) \prod_{k=0}^{n} (x - x_k), \tag{2.5}$$

where $\eta(\widehat{x})$ is defined as we have $\Gamma(\widehat{x}) = 0$ for $\widehat{x} \neq x_k$ $(k = 0, 1, \cdots, n)$ and $\widehat{x} \in [a,b]$. Therefore, we have

$$\eta(\widehat{x}) = \frac{y(\widehat{x}) - \mathfrak{L}_n(\widehat{x})}{\prod_{k=0}^{n} (x - x_k)}. \tag{2.6}$$

Hence, $\Gamma(x)$ has at least $n+2$ distinct roots at $[a,b]$. Using the Rolle's theorem successively $n+1$ times, results in that $\Gamma^{(n+1)}(x)$ has at least one root at (a,b). Consequently, there exists $\xi_x \in (a,b)$ since $\Gamma^{(n+1)}(\xi_x) = 0$, So we have

$$\eta(\widehat{x}) = \frac{y^{(n+1)}(\xi_x) - \mathfrak{L}_n^{(n+1)}(\xi_x)}{(n+1)!}. \tag{2.7}$$

Finally, by substituting the relation (2.7) in the relation (2.5) results in

$$\Gamma(x) = y(x) - \mathfrak{L}_n(x) - \frac{y^{(n+1)}(\xi_x) - \mathfrak{L}_n^{(n+1)}(\xi_x)}{(n+1)!} \prod_{k=0}^{n} (x - x_k). \tag{2.8}$$

We know that $\Gamma(\widehat{x}) = 0$, therefore

$$y(\widehat{x}) - \mathfrak{L}_n(\widehat{x}) = \frac{y^{(n+1)}(\xi_x) - \mathfrak{L}_n^{(n+1)}(\xi_x)}{(n+1)!} \prod_{k=0}^{n} (\widehat{x} - x_k), \tag{2.9}$$

which also obtains for any $\widehat{x} = x$.

In the next section, some applications of the formula (2.1) are presented.

3 Applications and Numerical Examples

In this section some applications and numerical examples in numerical integration are presented. Numerical results show the accuracy of the proposed new method.

3.1 Closed-type quadrature formula

Using the new interpolation formula results in the following closed-type quadrature formula

$$\int_a^b y(x)dx \simeq Q_c(y;a,b) = \sum_{i=0}^{n} w_i y_i, \tag{3.1}$$

where w_i for $i = 0, 1, \cdots, n$, are defined as

$$w_i = \int_a^b \Im_i(x)dx. \tag{3.2}$$

Example 1 As the first example, let $f_0(x) = \sin(x)$, and $f_1(x) = \cos(x)$ be two independent functions on the $[a, b]$. Therefore, using the relations (3.1) and (3.2), the following quadrature formula is obtained

$$\int_a^b y(x)dx \simeq \left(\frac{\sin(b) - sin(a)}{\cos(a) + \cos(b)}\right) \Big[y(a) + y(b)\Big]. \tag{3.3}$$

Example 2 As the second example, let $f_0(x) = 1$, $f_1(x) = \sin(x)$, and $f_2(x) = \cos(x)$. These functions are independent on the $[0, 1]$. Therefore, using the relations (3.1) and (3.2), the following quadrature formula is obtained

$$\int_0^1 y(x)dx \simeq \left(0.1680680606y(0) + 0.6638638787y\left(\frac{1}{2}\right) + 0.1680680606y(1)\right). \tag{3.4}$$

Remark 2 There are generally two main schemes to raise the accuracy of a numerical integration formula. In the first approach accuracy raise by increasing the order of the interpolation function while in the second one increases the accuracy by subdividing the interval into smaller subintervals and apply a quadrature rule on each of the subintervals. The second approach leading to composite quadrature rules. In the next example, the composite quadrature rule is applied.

Example 3 To compute I defined as

$$I = \int_0^1 \exp(x^2)dx \simeq 1.462651746, \tag{3.5}$$

the following two cases are considered:

1 $f_0(x) = 1,\ f_1(x) = x^2$, and $f_2(x) = x^4$;

2 $f_0(x) = 1,\ f_1(x) = \sin(x),\ f_2(x) = \cos(x)$.

Subdividing the interval $[0, 1]$ into $n = 8, 16, 32$ smaller intervals, and applying the composite quadrature rules for cases 1, 2, and Simpson's rule result in Table 1.

Table 1. The absolute errors of the quadrature formulas for $n = 8, 16, 32$.

	n=8	n=16	n=32
Case 1	4.78(-5)	3.09(-6)	1.94(-7)
Case 2	7.79(-5)	9.17(-6)	2.41(-4)
Simpson's rule	7.17(-5)	4.57(-6)	2.87(-6)

3.2 Open-type quadrature formula

Let $f_1, f_2, \cdots, f_{n-1}$ be $n-1$ linear independent functions on $[a,b]$. Similarly, using the new interpolation formula (introduced in the relation (2.1)) at x_i for $i = 1, 2, \cdots, n-1$, the following open-type quadrature formula is obtained

$$\int_a^b y(x)dx \simeq Q_o(y; a, b) = \sum_{i=1}^{n-1} w_i^{'} y_i, \tag{3.6}$$

where $w_i^{'}$ for $i = 1, 2, \cdots, n-1$, are defined as

$$w_i^{'} = \int_a^b \Im_i^{'}(x)dx. \tag{3.7}$$

Also, $\Im_i^{'}$ for $i = 1, 2, \cdots, n-1$ are defined as

$$\Im_1^{'}(x) = \frac{det\begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_{n-1}(x) \\ f_1(x_2) & f_2(x_2) & \cdots & f_{n-1}(x_2) \\ f_1(x_3) & f_2(x_3) & \cdots & f_{n-1}(x_3) \\ \vdots & \vdots & \vdots & \vdots \\ f_1(x_{n-1}) & f_2(x_{n-1}) & \cdots & f_{n-1}(x_{n-1}) \end{pmatrix}}{det\begin{pmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_{n-1}(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_{n-1}(x_2) \\ f_1(x_3) & f_2(x_3) & \cdots & f_{n-1}(x_3) \\ \vdots & \vdots & \vdots & \vdots \\ f_1(x_{n-1}) & f_2(x_{n-1}) & \cdots & f_{n-1}(x_{n-1}) \end{pmatrix}},$$

$$\Im_2^{'}(x) = \frac{det\begin{pmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_{n-1}(x_1) \\ f_1(x) & f_2(x) & \cdots & f_{n-1}(x) \\ f_1(x_3) & f_2(x_3) & \cdots & f_{n-1}(x_3) \\ \vdots & \vdots & \vdots & \vdots \\ f_1(x_{n-1}) & f_2(x_{n-1}) & \cdots & f_{n-1}(x_{n-1}) \end{pmatrix}}{det\begin{pmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_{n-1}(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_{n-1}(x_2) \\ f_1(x_3) & f_2(x_3) & \cdots & f_{n-1}(x_3) \\ \vdots & \vdots & \vdots & \vdots \\ f_1(x_{n-1}) & f_2(x_{n-1}) & \cdots & f_{n-1}(x_{n-1}) \end{pmatrix}},$$

$$\vdots \tag{3.8}$$

$$\Im'_{n-1}(x) = \frac{det\begin{pmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_{n-1}(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_{n-1}(x_2) \\ f_1(x_3) & f_2(x_3) & \cdots & f_{n-1}(x_3) \\ \vdots & \vdots & \vdots & \vdots \\ f_1(x) & f_2(x) & \cdots & f_{n-1}(x) \end{pmatrix}}{det\begin{pmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_{n-1}(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_{n-1}(x_2) \\ f_1(x_3) & f_2(x_3) & \cdots & f_{n-1}(x_3) \\ \vdots & \vdots & \vdots & \vdots \\ f_1(x_{n-1}) & f_2(x_{n-1}) & \cdots & f_{n-1}(x_{n-1}) \end{pmatrix}}.$$

Example 4 As the fourth example, let $f_0(x) = \sin(x)$, and $f_1(x) = \cos(x)$. These functions are independent on the $[0, 1]$. Therefore, using the relations (3.6) and (3.7), the following quadrature formula is obtained

$$\int_0^1 y(x)dx \simeq 0.4861621756 \left(y\left(\frac{1}{3}\right) + y\left(\frac{2}{3}\right) \right). \tag{3.9}$$

4 Conclusions

In this paper, the new kinds of interpolation formulas are introduced. The error of these interpolation formulas are computed in the formula (2.4). Using these new classes of interpolation formulas, some new type of quadrature formulas are obtained in the numerical section.

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