MATHEMATICS AND COMPUTATIONAL SCIENCES

Vol 1(1), 2020, pp:9-15

DOI:10.30511/mcs.2020.44462

ISSN:2717-2708

http://mcs.qut.ac.ir/

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Tripled fixed point results via fractional differential equations

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Abstract: This article intends to study a class of mixed monotone operators with convexity on ordered Banach spaces and investigate some new tripled fixed point results, also in this article, we examine the existence and uniqueness of tripled fixed points without assuming the operator to be compact or continuous. As applications, we apply the results obtained in this paper to study the existence and uniqueness of positive solutions for a fractional boundary value problem.

Keywords: Fractional boundary value problem; Tripled fixed point; Positive solution.
2020 Mathematics Subject Classification: 46T99; 47H10; 54H25.
Receive: 9 May 2020, **Accepted:** 5 August 2020

1 Introduction

In 2006, Bhaskar and Lakshmikantham introduced the concept of a coupled fixed point and studied existence and uniqueness theorems in partially ordered metric spaces. They also applied their results to problems of the existence of solution for a periodic boundary value problem [5]. In 2011 Zhai proved some results on a class of mixed monotone operators with perturbations (see [9]).
In 2012, Berinde and Borcut in [3] introduced the concept of tripled fixed point for nonlinear mappings in partially ordered complete metric spaces and obtained its existence.
Following the paper of Zhai, we will study tripled fixed point results for a class of mixed monotone operators with perturbations on ordered Banach spaces and we examine the existence and uniqueness of tripled fixed points without assuming the operator to be compact or continuous. As an application of our results, we study, an application to a fractional boundary value problem.
Suppose $(E, \| \,.\, \|)$ is a Banach space which is partially ordered by a cone $P \subseteq E$, that is, $x \leq y$ if and only if $y - x \in P$. If $x \neq y$, then we denote $x < y$ *or* $x > y$. We denote the zero element of E by θ. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P,\ \lambda \geq 0 \Longrightarrow \lambda x \in P$; (ii) $x \in P,\ -x \in P \Longrightarrow x = \theta$. A cone P is called normal if there exists a constant $N > 0$ such that $\theta \leq x \leq y$ implies $\| x \| \leq N \| y \|$. Also we define the order interval $[x_1, x_2] = \{x \in E | x_1 \leq x \leq x_2\}$ for all $x_1, x_2 \in E$. We say that and operator $A : E \to E$ is increasing whenever $x \leq y$ implies $Ax \leq Ay$.

Definition 1.1. *[6, 7] $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x and decreasing in y, i.e., u_i, v_i $(i = 1, 2) \in P$, $u_1 \leq u_2, v_1 \geq v_2$ imply $A(u_1, v_1) \leq A(u_2, v_2)$. The element $x \in P$ is called a fixed point of A if $A(x, x) = x$.*

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Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. On the product space $X \times X \times X$, consider the following partial order: for $(x, y, z), (u, v, w) \in X \times X \times X$,

$$(u, v, w) \leq (x, y, z) \Leftrightarrow x \geq u, y \leq v, z \geq w. \tag{1.1}$$

Definition 1.2. *[3] Let $(X, \leq)$ be a partially ordered set and $F : X \times X \times X \to X$. We say F has the mixed monotone property if for any $x, y, z \in X$,*

$$x_1, x_2 \in X, x_1 \leq x_2 \quad \textit{implies} \quad F(x_1, y, z) \leq F(x_2, y, z), \tag{1.2}$$
$$y_1, y_2 \in X, y_1 \leq y_2 \quad \textit{implies} \quad F(x, y_1, z) \geq F(x, y_2, z), \tag{1.3}$$
$$\textit{and} \quad z_1, z_2 \in X, z_1 \leq z_2 \quad \textit{implies} \quad F(x, y, z_1) \leq F(x, y, z_2). \tag{1.4}$$

Definition 1.3. *[3] An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of a mapping $F : X \times X \times X \to X$ if $F(x, y, z) = x, F(y, x, y) = y$ and $F(z, y, x) = z$.*

The following conditions will be assumed:
(A_1) there exists $h \in P$ with $h \neq \theta$ such that $A(h, h) \in P_h$,
(A_2) for any $u, v \in P$ and $t \in (0, 1)$, there exists $\varphi(t) \in (t, 1]$ such that

$$A(tu, t^{-1}v) \geq \frac{\varphi(t)}{t} A(u, v). \tag{1.5}$$

Lemma 1.4. *[9] Let the conditions (A_1), (A_2) are satisfies, then $A : P_h \times P_h \to P_h$; and there exist $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that*

$$rv_0 \leq u_0 < v_0, u_0 \leq A(u_0, v_0) \leq A(v_0, u_0) \leq v_0.$$

2 Main Results

Now we consider the mixed monotone operator $A : P \times P \times P \to P$. The following conditions will be assumed:
(A_1) there exists $h \in P$ with $h \neq \theta$ such that $A(h, h, h) \in P_h$,
(A_2) for any $u, v, w \in P$ and $t \in (0, 1)$, there exists $\varphi(t) \in (t, 1]$ such that

$$A(tu, t^{-1}v, tw) \geq \frac{\varphi(t)}{t} A(u, v, w). \tag{2.1}$$

Lemma 2.1. *Let the conditions (A_1), (A_2) are satisfies, then $A : P_h \times P_h \times P_h \to P_h$; and there exist $u_0, v_0, w_0 \in P_h$ and $r \in (0, 1)$ such that*

$$rv_0 \leq u_0 \leq w_0 < v_0, u_0 \leq A(u_0, v_0, w_0) \leq A(v_0, u_0, w_0) \leq v_0, A(w_0, u_0, w_0) \geq w_0.$$

Proof. The proof process is similar to the proof process from Lemma 1.4 in [9]. □

Theorem 2.2. *Let P be a normal cone of E, and $(A_1), (A_2)$ hold. Then the operator A defined in Lemma 2.1 has a unique fixed point x in P_h and for $x_0, y_0, z_0 \in P_h$, constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}, z_{n-1}), y_n = A(y_{n-1}, x_{n-1}, z_{n-1}), z_n = A(z_{n-1}, x_{n-1}, z_{n-1})$$
$$n = 1, 2, \ldots,$$

we have $\| x_n - x^* \| \to 0$, $\| y_n - x^* \| \to 0$ and $\| z_n - x^* \| \to 0$ as $n \to \infty$.

Proof. From Lemma (2.1), there exist $u_0, v_0, w_0 \in P_h$ and $r \in (0,1)$ such that

$$rv_0 \le u_0 \le w_0 < v_0, u_0 \le A(u_0, v_0, w_0) \le A(v_0, u_0, w_0) \le v_0, A(w_0, u_0, w_0) \ge w_0.$$

Construct successively the sequences

$$u_n = A(u_{n-1}, v_{n-1}, w_{n-1}), v_n = A(v_{n-1}, u_{n-1}, w_{n-1}), w_n = A(w_{n-1}, u_{n-1}, w_{n-1}),$$

$n = 1, 2, \ldots.$

Evidently $u_1 \le v_1$ and $w_1 \ge w_0$. By the mixed monotone properties of A, we obtain $u_n \le v_n$ and $w_n \ge \ldots \ge w_1 \ge w_0,\ n = 1, 2, \ldots$. It also follows from Lemma 2.1 and the mixed monotone properties of A that

$$u_0 \le u_1 \le \ldots \le u_n \le \ldots \le w_0 \le w_1 \le \ldots \le w_n \le \ldots \le v_n \le \ldots \le v_1 \le v_0, \tag{2.2}$$

Noting that $u_0, w_0 \ge rv_0$. We can get $u_n \ge u_0 \ge rv_0 \ge rv_n,\ n = 1, 2, \ldots$. Let

$$t_n = \sup\{t > 0 | u_n \ge tv_n\} \quad n = 1, 2, \ldots.$$

Thus we have $u_n \ge t_n v_n,\ w_n \ge t_n v_n, n = \ldots$, and then

$$u_{n+1} \ge u_n \ge t_n v_n \ge t_n v_{n+1}, n = 1, 2, \ldots.$$

Therefore, $t_{n+1} \ge tn,\ i.e., t_n$ is increasing with $t_n \subset (0, 1]$. Suppose $t_n \to t^*$ as $n \to \infty$, then $t^* = 1$. Otherwise, $0 < t^* < 1$. Then from condition (A_2) and $t_n \le t^*$, we have

$$\begin{aligned} u_{n+1} &= A(u_n, v_n, w_n) \ge A(t_n v_n, \frac{1}{t_n} u_n, t_n v_n) = A(\frac{t_n}{t^*} t^* v_n, \frac{t^*}{t_n} \frac{1}{t^*} u_n, \frac{t_n}{t^*} t^* w_n) \\ &\ge \frac{t_n}{t^*} A(t^* v_n, \frac{1}{t^*} u_n, t^* w_n) \ge \frac{t_n}{t^*} \frac{\varphi(t^*)}{t^*} A(v_n, u_n, w_n) \ge \frac{t_n}{t^*} \varphi(t^*) A(v_n, u_n, w_n) \\ &= \frac{t_n}{t^*} \varphi(t^*) v_{n+1}. \end{aligned}$$

By the definition of $t_n,\ t_{n+1} \ge \frac{t_n}{t^*}.\varphi(t^*)$. Let $n \to \infty$, we get $t^* \ge \varphi(t^*) > t^*$, which is a contradiction. Thus, $\lim_{n\to\infty} t_n = 1$. For any natural number p we have

$$\begin{aligned} &\theta \le u_{n+p} - u_n \le v_n - u_n \le v_n - t_n v_n = (1 - t_n) v_n \le (1 - t_n) v_0, \\ &\theta \le v_n - v_{n+p} \le v_n - u_n \le (1 - t_n) v_0, \\ &\theta \le w_n - w_{n+p} \le v_n - u_n \le v_n - t_n v_n = (1 - t_n) v_n \le (1 - t_n) v_0. \end{aligned}$$

Since the cone P is normal, we have

$$\begin{aligned} &\| u_{n+p} - u_n \| \le N(1 - t_n) \| v_0 \| \to 0, \quad \| v_n - v_{n+p} \| \le N(1 - t_n) \| v_0 \| \to 0, \\ &\| w_n - w_{n+p} \| \le N(1 - t_n) \| v_0 \| \to 0. \quad (n \to \infty), \end{aligned}$$

where N is the normality constant of P . So we can claim that u_n and v_n are Cauchy sequences. Because E is complete, there exist u^*, v^*, w^* such that $u_n \to u^*, v_n \to v^*, w_n \to w^*$ as $n \to \infty$. By (2.2), we know that $u_n \le u^* \le w^* \le v^* \le v_n$ with $u^*, v^*, w^* \in P_h$ and

$$\begin{aligned} &\theta \le v^* - u^* \le v_n - u_n \le (1 - t_n) v_0, \quad \theta \le w^* - v^* \le v_n - u_n \le (1 - t_n) v_0 \\ &\theta \le u^* - w^* \le v_n - u_n \le (1 - t_n) v_0. \end{aligned}$$

Further

$$\| v^* - u^* \| \leq N(1-t_n) \| v_0 \| \to 0 \ \ (n \to \infty),$$
$$\| w^* - v^* \| \leq N(1-t_n) \| v_0 \| \to 0 \ \ (n \to \infty),$$
$$\| u^* - w^* \| \leq N(1-t_n) \| v_0 \| \to 0 \ \ (n \to \infty),$$

and thus $u^* = v^* = w^*$. Let $x^* := u^* = v^* = w^*$ and then we obtain

$$u_{n+1} = A(u_n, v_n, w_n) \leq A(x^*, x^*, x^*) \leq A(v_n, u_n, w_n) = v_{n+1}.$$

Let $n \to \infty$, then we get $x^* = A(x^*, x^*, x^*)$. That is, x^* is a fixed point of A in P_h. In the following, we prove that x^* is the unique fixed point of A in P_h. In fact, suppose $\bar{x}$ is a fixed point of A in P_h.Since $x^*, \bar{x} \in P_h$, there exists positive numbers $\bar{\rho}_1, \bar{\rho}_2, \bar{\lambda}_1, \bar{\lambda}_2 > 0$ such that

$$\bar{\rho}_1 h \leq x^* \leq \bar{\lambda}_1, \quad \bar{\rho}_2 h \leq \bar{x} \leq \bar{\lambda}_2 h.$$

Then we obtain

$$\bar{x} \leq \bar{\lambda}_2 h = \frac{\bar{\lambda}_2}{\bar{\rho}_1}.\bar{\rho}_1 h \leq \frac{\bar{\lambda}_2}{\bar{\rho}_1}.x^*, \quad \bar{x} \geq \bar{\lambda}_2 h = \frac{\bar{\rho}_2}{\bar{\lambda}_1}.\bar{\lambda}_1 h \geq \frac{\bar{\rho}_2}{\bar{\lambda}_1} x^*.$$

Let $e_1 = \sup\{t > 0 | tx^* \leq \bar{x} \leq t^{-1}x^*\}$. Evidently, $0 < e_1 \leq 1, e_1 x^* \leq \bar{x} \leq \frac{1}{e_1} x^*$. Next we prove $e_1 = 1$. If $0 < e_1 < 1$, then

$$\begin{aligned} \bar{x} &= A(\bar{x}, \bar{x}, \bar{x}) \geq A(e_1 x^*, \frac{1}{e_1} x^*, e_1 x^*) \\ &\geq \frac{\varphi(e_1)}{e_1} A(x^*, x^*, x^*) \geq \varphi(e_1) A(x^*, x^*, x^*) \\ &= \varphi(e_1) x^*. \end{aligned}$$

Since $\varphi(e_1) > e_1$, this contradicts the definition of e_1. Hence $e_1 = 1$, and we get $\bar{x} = x^*$. Therefore, A has a unique fixed point x^* in P_h. Note that $[u_0, v_0] \subset P_h$, then we know that x^* is the unique fixed point of A in $[u_0, v_0]$.

Now we construct successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}, z_{n-1}), y_n = A(y_{n-1}, x_{n-1}, z_{n-1}),$$
$$z_n = A(z_{n-1}, x_{n-1}, z_{n-1}), n = 1, 2, \ldots,$$

for any initial points $x_0, y_0, z_0 \in P_h$. Since $x_0, y_0, z_0 \in P_h$ we can choose small numbers $e_2, e_3, e_4 \in (0, 1)$ such that

$$e_2 h \leq x_0 \leq \frac{1}{e_2} h, \quad e_3 h \leq y_0 \leq \frac{1}{e_3} h, \quad e_4 h \leq z_0 \leq \frac{1}{e_4} h.$$

Let $e^* = \min\{e_2, e_3, e_4\}$. Then $e^* \in (0, 1)$ and

$$e^* h \leq x_0, \quad y_0 \leq \frac{1}{e^*} h, \quad e^* h \leq z_0.$$

We can choose a sufficiently large positive integer m such that

$$[\frac{\varphi(e^*)}{e^*}]^m \geq \frac{1}{e^*},$$

and we choose $e_1^* \in (0, 1)$ such that $e^* \leq e_1^* \leq \varphi(e^*) \leq 1$.

Put $\bar{u}_0 = {e^*}^m h, \bar{v}_0 = \frac{1}{{e^*}^m} h, \bar{w}_0 = {e_1^*}^m h$. It easy to see that $\bar{u}_0, \bar{v}_0, \bar{w}_0 \in P_h$ and $\bar{u}_0 < x_0, y_0 < \bar{v}_0, w_0 < \bar{z}_0$. Let

$$\bar{u}_n = A(\bar{u}_{n-1}, \bar{v}_{n-1}, \bar{w}_{n-1}), \bar{v}_n = A(\bar{v}_{n-1}, \bar{u}_{n-1}, \bar{w}_{n-1}),$$
$$\bar{w}_n = A(\bar{w}_{n-1}, \bar{u}_{n-1}, \bar{w}_{n-1}), n = 1, 2, \ldots.$$

Similarly, it follows that there exists $y^* \in P_h$ such that $A(y^*, y^*, y^*) = y^*, \lim_{n\to\infty} \bar{u}_n = \lim_{n\to\infty} \bar{v}_n = \lim_{n\to\infty} \bar{w}_n = y^*$. By the uniqueness of fixed point of operator A in P_h. We get $x^* = y^* = z^*$ and by induction $\bar{u}_n \leq x_n, y_n \leq \bar{v}_n, \bar{w}_n \leq z_n,\ n = 1, 2, \ldots$. Since cone P is normal we have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = x^*$. □

3 Application

Lemma 3.1. *[4, 1] If $h \in C(I \times X, \mathbb{R})$, $X = C([0,1] \times [0,1] \times [0,1], \mathbb{R})$ and $h(t, u(r,s,t)) \leq 0$, then the problem*

$$-\frac{D^\alpha}{Dt} u(r,s,t) = h(t, u(r,s,t)), \quad (0 < t < 1,\ 3 < \alpha \leq 4), \tag{3.1}$$

$$u(r,s,0) = \frac{\partial u}{\partial t} u(r,s,0) = \frac{\partial^2 u}{\partial t^2}(r,s,0) = \frac{\partial^2 u}{\partial t^2}(r,s,1) = 0.$$

(where D^α is the Riemann-Liouville derivative and $h : I \times X \to \mathbb{R}$ is continuous.)
has a unique positive solution

$$u(r,s,t) = \int_0^1 G(t,\zeta) h(\zeta, u(r,s,\zeta)) d\zeta,$$

that $G(t,\zeta)$ given by

$$G(t,\zeta) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-\zeta)^{\alpha-3} - (t-\zeta)^{\alpha-1}, & 0 \leq \zeta \leq t \leq 1, \\ t^{\alpha-1}(1-\zeta)^{\alpha-3}, & 0 \leq t \leq \zeta \leq 1. \end{cases} \tag{3.2}$$

Lemma 3.2. *[10] $G(t,\zeta)$ in Lemma 3.1 has the following property*

$$\frac{1}{\Gamma(\alpha)} \zeta(2-\zeta)(1-\zeta)^{\alpha-3} t^{\alpha-1} \leq G(t,\zeta) \leq \frac{1}{\Gamma(\alpha)} (1-\zeta)^{\alpha-3} t^{\alpha-1},$$

where, $t, \zeta \in I$, $3 < \alpha \leq 4$. and

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx. \tag{3.3}$$

Let

$$E = C([0,1] \times [0,1] \times [0,1], \mathbb{R}).$$

Consider the Banach space of continuous functions on $[0,1] \times [0,1] \times [0,1]$ with sup norm and set

$$P = \{y \in C([0,1] \times [0,1] \times [0,1], \mathbb{R}) : \min_{(s,r,t) \in [0,1]\times[0,1]\times[0,1]} y(s,r,t) \geq 0\}.$$

Then P is a normal cone.

Theorem 3.3. *Let*
$f(s,r,t,u(s,r,t), v(s,r,t), \eta(s,r,t)) \in C([0,1] \times [0,1] \times [0,1] \times [0,\infty) \times [0,\infty) \times [0,\infty), [0,\infty))$ and $c \in (0,1)$, there exists $\varphi(t) \in (t,1]$ such that
$f(s,r,t,cu(s,r,t), c^{-1}v(s,r,t), c\eta(s,r,t)) \geq \frac{\varphi(t)}{t} f(s,r,t,u(s,r,t), v(s,r,t), \eta(s,r,t))$. Also assume that there exist $M_1, M_2 > 0$ and $\theta \neq h \in P$ such that

$$M_1 h \leq \int_0^1 G(t,\xi) f(s,r,\xi, h(s,r,\xi), h(s,r,\xi), h(s,r,\xi)) d\xi \leq M_2 h,$$

where $G(t,\xi)$ is the green function defined in lemma (3.1). Then the problem (3.1) has a unique solution in P_h. Moreover, for any initial $u_0, v_0, \eta_0 \in P_h$, constructing successively the sequences

$$u_{n+1} = \int_0^1 G(t,\xi) f(s,r,\xi,u_n(s,r,\xi),v_n(s,r,\xi),\eta_n(s,r,\xi))d\xi,$$
$$v_{n+1} = \int_0^1 G(t,\xi) f(s,r,\xi,v_n(s,r,\xi),u_n(s,r,\xi),\eta_n(s,r,\xi))d\xi,$$
$$\eta_{n+1} = \int_0^1 G(t,\xi) f(s,r,\xi,\eta_n(s,r,\xi),u_n(s,r,\xi),\eta_n(s,r,\xi))d\xi,$$

we have $\| u_n - u^* \| \to 0, \| v_n - u^* \| \to 0, \| \eta_n - u^* \| \to 0$.

Proof. By using Lemma (3.1), the problem is equivalent to the integral equation

$$u(r,s,t) = \int_0^1 G(t,\zeta) h(\zeta, u(r,s,\zeta))d\zeta,$$

where

$$G(t,\zeta) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-\zeta)^{\alpha-3} - (t-\zeta)^{\alpha-1}, & 0 \le \zeta \le t \le 1, \\ t^{\alpha-1}(1-\zeta)^{\alpha-3}, & 0 \le t \le \zeta \le 1. \end{cases}$$

Define the operator $A : P \times P \times P \to P$ by the following,
$A(u(s,r,t), v(s,r,t), \eta(s,r,t)) = \int_0^1 G(t,\zeta) h(s,r,\zeta,u(s,r,t),v(s,r,t),\eta(s,r,t))d\zeta$.
Then u is solution for the problem if and only if $u = A(u,u,u)$.
For $c \in (0,1), s,r,t \in P$, there exists $\varphi(t) \in (t,1]$ such that

$$\begin{aligned} & A(cu(s,r,t), c^{-1}v(s,r,t), c\eta(s,r,t)) \\ &= \int_0^1 G(t,\xi) f(,s,r,\xi,cu(s,r,\xi),c^{-1}v(s,r,\xi),c\eta(s,r,\xi))d\xi \\ &\ge \frac{\varphi(t)}{t} \int_0^1 G(t,\xi) f(s,r,\xi,u(s,r,\xi),,v(s,r,\xi),\eta(s,r,\xi))d\xi \\ &= \frac{\varphi(t)}{t} A(u(s,r,t),v(s,r,t),\eta(s,r,t)). \end{aligned}$$

Since

$$M_1 h \le A(h,h,h) = \int_0^1 G(t,\xi) f(s,r,\xi,h(s,r,\xi),h(s,r,\xi),h(s,r,\xi))d\xi \le M_2 h,$$

we get $A(h,h,h) \in P_h$. Therefore A satisfies all conditions of Theorem (2.2) and so, the operator A has a unique positive solution (u^*, u^*, u^*) such that $A(u^*, u^*, u^*) = u^*$. This completes the proof. □

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