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A new approach to the bipolar Shilkret integral

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Abstract: Capacity, also known as a non-additive measure, is an extension of the Lebesgue measure. In recent years, bi-capacity was presented as a generalization of capacity with several bipolar fuzzy integrals related to bi-capacity, one of them being the bipolar Shilkret integral. In this paper, we propose a new approach to calculating the bipolar Shilkret integral to be suitable for bipolar scales. Then, we give some main properties of this integral related to bi-capacity.

Keywords: Non-additive measure; Shilkret integral; Bipolar scales; Bi-capacity; Bipolar Shilkret integral.
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1 Introduction

There are many applications of capacities (known also under the name non-additive measures) and their integrals in many disciplines of engineering and sciences, particularly in machine learning, data fusion, economics, etc. (see, e.g. [9-11], [13-14], [22-26]). One of the most common nonlinear integrals is the Shilkret integral related to capacity [28], which is defined in a case when its values are in the non-negative interval. Bi-capacities (known also under the name monotone bi-cooperative games) are a generalization of capacities that have been introduced and studied in [15,19,1]. Recently, several bipolar fuzzy integrals related to bi-capacity have been introduced in several works of literature [2-6,16,18,20,24]. One of these bipolar fuzzy integrals is Shilkret integral concerning aggregation in the case when their values are on the bipolar scales has been studied in [18].

In this paper, we propose a new approach to calculating the bipolar Shilkret integral to be suitable for bipolar scales. Then, we give some main properties of this integral related to bi-capacity. This approach is an alternative approach to the bipolar Shilkret integral that was presented in [18] and allows a simple way to propose other bipolar fuzzy integrals concerning the bi-capacities.

The article is organized as follows: Section 2 summarizes the works related to the proposed theme. The proposed approach of bipolar Shilkret integral is explained in Section 3. In Section 4, the main

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properties of our approach are shown and discussed with illustrated examples. Finally, section 5 presents the conclusion and a proposal for future work.

2 Preliminary Notions

Let X be a non empty finite set, the binary operators $\wedge, \vee$ on [0,1] is defined as follows: For any $s, t \in [0,1]$, $s \wedge t := \min\{s, t\}$ and $s \vee t := \max\{s, t\}$. According to [28, 20], the symmetric minimum $\curlywedge$ and the symmetric maximum $\curlyvee$ are operations have been introduced as follows:
For any $s, t \in [-1,1]$, $s \curlywedge t = sign(s + t) \cdot (|s| \wedge |t|)$ and $s \curlyvee t = sign(s \cdot t) \cdot (|s| \vee |t|)$,
where sign(r) = −1 if $r < 0$, = 0 if $r = 0$, and = 1 if $r > 0$.
Moreover,

$$\curlyvee_{s_i \in I} s_i = \bigvee_{s_i \geq 0} s_i \curlyvee \bigwedge_{s_i \leq 0} s_i \tag{1}$$

for any subset I of the interval [−1,1].

Let $(X, §)$ be a measurable space, where X is a non-empty set and § is a σ- algebra of subsets of X. A capacity [12] is a generalization of classical measure using non-additivity property instead of additivity property. The definition of the capacity is as follows.

Definition 1. [14] Let X be a universal set. A capacity is a function $\mu : § \rightarrow [0, 1]$ satisfies:
1. $\mu(\emptyset) = 0,\ \mu(X) = 1$,
2. for all $A, B \in §,\ A \subseteq B,\ \mu(A) \leq \mu(B)$.

In decision analysis, especially in multiple-criteria decision analysis, has introduced several non-additive integrals for capacities in the last 60 years. Let Ж be a class of all finite nonnegative real-valued measurable functions on measurable space $(X, §)$. For any $\boldsymbol{x} = (x_1, \ldots, x_i, \ldots, x_m) \in$ Ж, the Shilkret integral of $\boldsymbol{x}$ with respect to μ is defined as follows.

Definition 2. [28] Let $(X, §)$ be a measurable space and $\boldsymbol{x} \in$ Ж. The Shilkret integral of $\boldsymbol{x}$ with respect to capacity μ is defined by

$$(BSu) \int (\boldsymbol{x}, \mu) d\mu = \bigvee_{i \in \boldsymbol{x}} \{ x_i . \mu(\{j \in \boldsymbol{x} \,|x_j \geq x_i\})\}\}. \tag{2}$$

Let $Q(X) = 3^X = \{(A_1, A_2) \in P(X) \times P(X)|\ A_1 \cap A_2 = \emptyset\}$ be the set all disjoint pairs of sets and equip it with a binary relation $\sqsubseteq$ for arbitrary $(A_1, A_2), (B_1, B_2) \in Q(X)$ such that:

$$(A_1, A_2) \sqsubseteq (B_1, B_2)\ iff\ A_1 \subseteq B_1\ and\ A_2 \supseteq B_2 \tag{3}$$

There is another order relation on the structure $Q(X)$ has introduced by Bilbao et al. [8]. This order is simply the product order on $P(X) \times P(X)$:

$$(A_1, A_2) \sqsubseteq (B_1, B_2) \; iff \; A_1 \subseteq B_1 and \; A_2 \subseteq B_2 \tag{4}$$

Definition 3. [15] A bi-capacity on X is a set function $\mathcal{v}_b : Q(X) \to [-1, 1]$ if

1- $\mathcal{v}_b(\emptyset, \emptyset) = 0,\; \mathcal{v}_b(N, \emptyset) = 1 \; and \; \mathcal{v}_b(\emptyset, N) = -1$

2- $\forall \; (A_1, A_2), (B_1, B_2) \in \; Q(X),\;\; (A_1, A_2) \sqsubseteq (B_1, B_2) \implies \mathcal{v}_b(A_1, A_2) \leq \; \mathcal{v}_b(B_1, B_2)$.

Given $\boldsymbol{x} = (x_1, \dots, x_i, \dots, x_m) \in [-1,1]^n$, the *bipolar maximum* of $\boldsymbol{x}$, denoted by

$$\bigvee_{i \in \boldsymbol{x}}^{b} \boldsymbol{x} = \left(\bigvee_{i=1}^{m} x_i \right) \curlyvee \left(\bigwedge_{i=1}^{m} x_i \right), \tag{5}$$

where the operator $\curlyvee : [-1,1]^2 \to [-1,1]$ is the symmetric maximum has been introduced by Grabisch [17]. In [18], the bipolar Shilkret integral based on bi-capacities with order $\sqsubseteq$ is defined as follows**.**

Definition 4 [18]: The bipolar Shilkret integral of a vector $\boldsymbol{x} = (x_1, \dots, x_i, \dots, x_m) \in [-1,1]^n$ with respect to the bi-capacity v on $\boldsymbol{x}$ is given by

$$Sh_b \int (\boldsymbol{x}, v)\, dv = \bigvee_{i \in \boldsymbol{x}}^{b} \{ |x_i| .\; v(\{ j \in \boldsymbol{x} : x_j \geq |x_i| \}, \{ j \in \boldsymbol{x} : x_j \leq -|x_i| \}) \}, \tag{6}$$

$$Sh_b \int (\boldsymbol{x}, v)\, dv = \bigvee_{i \in \boldsymbol{x}}^{b} \{ Sh_{b^+} \int (\boldsymbol{x}, v)\, dv, Sh_{b^-} \int (\boldsymbol{x}, v)\, dv \},$$

where, $Sh_{b^+} \int (\boldsymbol{x}, v)\, dv$ is the right bipolar Shilkret integral of a vector $\boldsymbol{x} = (x_1, \dots, x_i, \dots, x_m) \in [-1,1]^n$ with respect to the bi-capacity v on S is given by

$$Sh_{b^+} \int (\boldsymbol{x}, v)\, dv = \bigvee_{i \in \boldsymbol{x}}^{b^+} \{ |x_i| .\; v(\{ j \in \boldsymbol{x} : x_j \geq |x_i| \}, \{ j \in \boldsymbol{x} : x_j \leq -|x_i| \}) \}, \tag{7}$$

and $Su_{b^-} \int (\boldsymbol{x}, v)\, dv$ is the left bipolar Shilkret integral of a vector $\boldsymbol{x} = (x_1, \dots, x_i, \dots, x_m) \in [-1,1]^n$ with respect to the bi-capacity v on S is given by

$$Sh_{b^-} \int (\boldsymbol{x}, v)\, dv = \bigvee_{i \in \boldsymbol{x}}^{b^-} \{ |x_i| .\; v(\{ j \in \boldsymbol{x} : x_j \geq |x_i| \}, \{ j \in \boldsymbol{x} : x_j \leq -|x_i| \}) \}. \tag{8}$$

3 A new approach to the bipolar Shilkret integral

At first, we recall the main notions of the ternary-element set and the equivalent definition of bi-capacities. Then we propose the bipolar Shilkret integral defined on ternary-element sets.
We consider every element $i \in X$ that has either a positive effect (i.e., i is positively important criterion of weighted evaluation not only alone but also is interactive with others), or a negative effect (i.e., i is negatively important criterion), or has no effect (i.e., i is criterion at neutral level). Hence, we represent the element i as i^+whenever i is positively important, as i^- whenever i is negatively important, and as $i^\emptyset$ whenever i is neutral, and we call this element a ternary-element. The ternary-element set (or simply ter-element set) is the set which contains only out of $i^+, i^-, and\ \ i^\emptyset$ for all $i,\ i \in \{1,2 \cdots, n\}$.
Thus, in our framework we consider the set of all possible combinations of ternary elements of n criteria given by $T(X) = \left\{\{\tau_1, \dots, \tau_2\} \mid \forall \tau_i \in \{i^+, i^-, i^\emptyset\},\ i = 1, \dots, n\right\}$ which corresponds to $Q(X)$ in the notation of classical bi-capacities.
We have $T\ (X)$ can be identified with $\{-1,0,1\}^n$, hence $|T\ (X)| = 3^n$. Also, simply remarked that for any ter-element set $A \in T\ (X)$, A is equivalent to a ternary alternative $(\tau_1, \dots, \tau_n)$ with $\tau_i = 1\ if\ i^+ \in A$, $\tau_i = 0\ if\ i^\emptyset \in A, and\ \tau_i = -1\ \ if\ i^- \in A,\ \forall\ \ i = 1,2 \cdots, n$.
We introduce the order relation $\sqsubseteq$ between ter-element sets of $T\ (X)$ as follows.

Definition 5 Let $T\ (X)$ be the set of all ter-element sets and $A, B \in\ T\ (X)$. Then, $A\ \sqsubseteq\ B\ iff\ i\ \in\ X$,

$$``if\ \ i^+ \in\ A\ \ implies\ \ i^+ \in\ B\text{''}, and\ \ ``if\ \ i^\emptyset\ \in\ A\ implies\ \ i^+\ or\ i^\emptyset\ \in B\text{''}. \tag{9}$$

The following definition is equivalent definition of bi-capacities based on notion of ternary-element sets.

Definition 6 Let $T\ (X)$ be the set of all ternary-element sets. A set function $v : T\ (X) \to\ [-1, 1]$, is called bi-capacity based on the ter-element sets if it satisfies the following requirements:
(i) $v(X^-) = v(\{1^-, 2^- \cdots, n^-\}) = -1,\ v(X^\emptyset) = v(\{1^\emptyset, 2^\emptyset \cdots, n^\emptyset\}) = 0,$
$and\ \ v(X^+) = v(\{1^+, 2^+ \cdots, n^+\}) = 1.$
(ii) $\forall\ A, B \in\ T(X)\ ,\ A \sqsubseteq B\ implies\ v(A) \leq v(B).$

Bi-capacities are functions defined on the structure of the underlying partially ordered set. Hence, we can introduce an order on the structure $T\ (X)$ different from the order $\sqsubseteq$ described in definition 5.
We consider the following definition of an order on $T\ (X)$ which is equivalent to Bilbao order on bi-cooperative game [8]. For convenience, we denote by $\subseteq$ the order relation defined on $T\ (X)$ as in the classical order relation, and we will use the order $\subseteq$ on $T\ (X)$ to establish our next results of this research.

Definition 7 Let $T\ (X)$ be the set of all ter-element sets and $A, B\ \in\ T\ (X)$. Then, $A \subseteq\ B\ \ iff\ \ \forall\ i\ \in\ X$,

$$``\ if\ i^+\ \in\ A\ implies\ i^+\ \in\ B\text{''}, \text{and } ``if\ i^- \in\ A\ implies\ \ i^-\ \in\ B\text{''} \tag{10}$$

Now, we propose an alternative bipolar Shilkret integral model with respect to bi-capacity based on the ternary-element set. The basic idea underlying this model is for an input vector $\boldsymbol{x} = (x_1, \dots, x_i, \dots, x_n);\ x_i\ \in R\ with\ i \in \{1,2 \cdots, n\}$. we consider a ternary-element set $X^* = \{\tau_1, \cdots, \tau_n\}$ with $\tau_i =\ \ i^+\ if\ x_i\ >\ 0;\ \tau_i =\ \ i^-\ if\ x_i\ <\ 0;\ and\ \tau_i =\ \ i^\emptyset\ if\ x_i\ =\ 0,\ \forall\ i =\ 1, \dots, n.$

Thus, we define the bipolar Shilkret integral with respect to bi-capacity based on ternary-element set of real input $\boldsymbol{x}$ as follows.

Definition 8 Let $T\ (X)$ be the set of all ternary-element sets and $v :\ T\ (X) \to [-1, 1]$, be a bi-capacity based on ternary-element set. Then, the bipolar Shilkret integral of $\boldsymbol{x}$ with respect to v is given by

$$(BSh) \int (\boldsymbol{x}, v)\ dv = \curlyvee_{i=1}^{n} \ \left[\ |x_{\sigma(\tau_i)}|\ .\ v\left(A_{\sigma(\tau_i)}\right)\right], \tag{11}$$

where $A_{\sigma(\tau_i)} = \left\{\sigma(\tau_1), \dots, \sigma(\tau_i), \sigma\left((i+1)^{\emptyset}\right), \sigma\left((i+2)^{\emptyset}\right), \dots\right\}$ is ternary-element set $\subseteq X^*$, and σ is a permutation on X^* so that $x_{\sigma(\tau_i)}| \geq \cdots \geq |x_{\sigma(\tau_n)}|$.

Example 1: Let us consider $S = \{1,2,3\}$, and we define the bi-capacity values $v : T\ (X) \to [-1, 1]$ as shown in Table 1. The function $\boldsymbol{x}$ on S defined by $\boldsymbol{x} = (0.1, -0.7, 0.4)$. That is, $x_1 = 0.1,\ x_2 = -0.7, x_3 = 0.4$.

Then, the ternary element set corresponding to $\boldsymbol{x}$ is $X^* = \{1^+, 2^-, 3^+\}$. Using definition 8,
$|x_1| = 0.1,\ \ |x_2| = 0.7,\ \ |x_3| = 0.4$,
$(BSh) \int (\boldsymbol{x}, v)\ dv = \curlyvee_{i=1}^{n} \ \left[\ |x_{\sigma(\tau_i)}|\ .\ v\left(A_{\sigma(\tau_i)}\right)\right]$
We obtain,

$(BSh) \int (\boldsymbol{x}, v)\ dv = [0.7\ .\ \ v\left(\{1^{\emptyset}, 2^-, 3^{\emptyset}\}\right)] \curlyvee [0.4\ .\ \ v\left(\{1^{\emptyset}, 2^-, 3^+\}\right)] \ \curlyvee\ [0.1\ .\ \ v(\{1^+, 2^-, 3^+\})]$
$(BSh) \int (\boldsymbol{x}, v)\ dv = [0.7\ .\ \ -0.6] \curlyvee [0.4\ .\ \ 0.5] \ \curlyvee\ [0.1\ .\ \ 0.8]$
$(BSh) \int (\boldsymbol{x}, v)\ dv = [-0.42] \curlyvee [0.20\ ] \ \curlyvee\ [0.08]$
$(BSh) \int (\boldsymbol{x}, v)\ dv = -0.42.$

4 The main properties of bipolar Shilkret integral

Proposition 1: For any bi-capacity based on the ternary-element sets (v) on $T\ (X)$,

$$(BSh) \int \left((1_A\ , -1_A, 0_A), v\right)\ dv\ =\ v(A), \qquad \forall\ A\ \in\ T\ (X).$$

Proof: For ternary vector $(1_A\ , -1_A, 0_A)$, $\ |x_{\sigma(\tau_i)}|\ = 1\ or\ \ |x_{\sigma(\tau_i)}|\ = 0,\ \forall\ \tau_i \in \{i^+, i^-, i^{\emptyset}\}$
and $v\left(\{A_{\sigma(\tau_i)}\}\right)\ =\ v(1_A\ , -1_A, 0_A) =\ v(A)$. Therefore, from the bipolar Shilkret integral with respect to bi-capacity based on the ternary-element sets (Formula (11)), we have

$$(BSh) \int \left((1_A\ , -1_A, 0_A), v\right)\ dv = \curlyvee_{i=1}^{n} \left[|x_{\sigma(\tau_i)}|\ .\ v\left(A_{\sigma(\tau_i)}\right)\right]$$

$$(BSh) \int \left((1_A\ , -1_A, 0_A), v\right)\ dv\ =\ v(A),\ \ \forall\ A\ \in\ T\ (X) \qquad \blacksquare$$

Table 2: Values of bi-capacities

Sets	$\{1^{\emptyset}, 2^{\emptyset}, 3^{\emptyset}\}$	$\{1^{+}, 2^{\emptyset}, 3^{\emptyset}\}$	$\{1^{\emptyset}, 2^{+}, 3^{\emptyset}\}$
Values	**0**	**0.7**	**0.5**
Sets	$\{1^{\emptyset}, 2^{-}, 3^{-}\}$	$\{1^{-}, 2^{-}, 3^{\emptyset}\}$	$\{1^{-}, 2^{\emptyset}, 3^{-}\}$
Values	**-0.6**	**-0.6**	**-0.7**
Sets	$\{1^{\emptyset}, 2^{\emptyset}, 3^{+}\}$	$\{1^{+}, 2^{+}, 3^{\emptyset}\}$	$\{1^{+}, 2^{\emptyset}, 3^{+}\}$
Values	**0.6**	**0.7**	**0.8**
Sets	$\{1^{\emptyset}, 2^{+}, 3^{+}\}$	$\{1^{-}, 2^{\emptyset}, 3^{\emptyset}\}$	$\{1^{\emptyset}, 2^{-}, 3^{\emptyset}\}$
Values	**0.7**	**-0.6**	**-0.6**
Sets	$\{1^{\emptyset}, 2^{\emptyset}, 3^{-}\}$	$\{1^{+}, 2^{-}, 3^{\emptyset}\}$	$\{1^{-}, 2^{+}, 3^{\emptyset}\}$
Values	**-0.4**	**0.2**	**0.3**
Sets	$\{1^{\emptyset}, 2^{-}, 3^{+}\}$	$\{1^{\emptyset}, 2^{+}, 3^{-}\}$	$\{1^{+}, 2^{\emptyset}, 3^{-}\}$
Values	**0.5**	**-0.2**	**-0.3**
Sets	$\{1^{-}, 2^{\emptyset}, 3^{+}\}$	$\{1^{+}, 2^{+}, 3^{-}\}$	$\{1^{+}, 2^{-}, 3^{+}\}$
Values	**0.3**	**0.8**	**0.8**
Sets	$\{1^{-}, 2^{+}, 3^{-}\}$	$\{1^{-}, 2^{+}, 3^{+}\}$	$\{1^{-}, 2^{-}, 3^{+}\}$
Values	**-0.5**	**0.5**	**-0.5**
Sets	$\{1^{+}, 2^{-}, 3^{-}\}$	$\{1^{-}, 2^{-}, 3^{-}\}$	$\{1^{+}, 2^{+}, 3^{+}\}$
Values	**-0.6**	**-1**	**1**

Example 2: Let us consider $S = \{1,2,3\}$, and we define the bi-capacity values $v : T\,(X) \to [-1, 1]$ as shown in Table 1. The function $\boldsymbol{x}$ on S defined by $\boldsymbol{x} = (1,1,1)$.

Then, the ternary element set corresponding to $\boldsymbol{x}$ is $X^{*} = \{1^{+}, 2^{+}, 3^{+}\}$, and the subset of X^{*} are $\{1^{+}, 2^{\emptyset}, 3^{\emptyset}\}$, $\{1^{+}, 2^{+}, 3^{\emptyset}\}$, $\{1^{+}, 2^{+}, 3^{+}\}$

According to the definition 8, we have

$(BSh) \int (\boldsymbol{x}, v)\, dv = \curlyvee_{i=1}^{n} \left[\, |x_{\sigma(\tau_i)}| \ .\ v\big(A_{\sigma(\tau_i)}\big)\right] = v(\{1^{+}, 2^{+}, 3^{+}\})$=1.

The next property shows that the bipolar Shilkret integral with respect to the bi-capacity satisfy the monotonicity.

Proposition 2: For any bi-capacity based on the ter-element sets v on $T(X)$, $\forall\, \boldsymbol{x}, \boldsymbol{x'} \in R$, $if\ x_{\tau_i} \geq x'_{\tau_i}\ \forall\, \tau_i \in \{i^+, i^-, i^\emptyset\}$, then
$(BSh)\int(\boldsymbol{x}, v)dv \geq (BSh)\int(\boldsymbol{x'}, v)dv$.

Proof: First, we assume that for any $\tau_i \in \{i^+, i^-, i^\emptyset\}$, $x_{\tau_i} > x'_{\tau_i}$ and $\forall\, k \in \{1, \ldots, i-1, i+1 \ldots, n\}$, $x_{\tau_k} = x'_{\tau_k}$.
Also, we assume that for all elements of X, the order of each element is the same, i.e., $|x_{\sigma(\tau_i)}| \geq \cdots \geq |x_{\sigma(\tau_n)}|$ and $|x'_{\sigma(\tau_i)}| \geq \cdots \geq |x'_{\sigma(\tau_n)}|$.
Firstly, we prove the monotonicity of this case.
Using the expression of bipolar Shilkret integral with respect to bi-capacity based on the ternary-element sets (Formula (11)), we have

$$(BSh)\int(\boldsymbol{x}, v)dv = \curlyvee_{i=1}^{n}\left[|x_{\sigma(\tau_i)}|\ .\ v\big(A_{\sigma(\tau_i)}\big)\right]. \tag{12}$$

Also,

$$(BSh)\int(\boldsymbol{x'}, v)dv = \curlyvee_{i=1}^{n}\left[|x'_{\sigma(\tau_i)}|\ .\ v\big(A_{\sigma(\tau_i)}\big)\right] \tag{13}$$

$A_{\sigma(\tau_i)}$ *and* $A_{\sigma(\tau_{i-1})}$ are the ternary-element sets with $A_{\sigma(\tau_{i-1})} \subseteq A_{\sigma(\tau_i)}$.
Hence, $v\left(A_{\sigma(\tau_i)}) - v(A_{\sigma(\tau_{i-1})}\right) \geq 0$.
Now, Since $x_{\tau_i} \geq x'_{\tau_i}\ \forall\, \tau_i \in \{i^+, i^-, i^\emptyset\}$, it is clear that $(BSh)\int(\boldsymbol{x}, v)\, dv \geq (BSh)\int(\boldsymbol{x'}, v)dv$.
Therefore, if $x_{\tau_i} > x'_{\tau_i}$ then $(BSh)\int(\boldsymbol{x}, v)dv \geq (BSh)\int(\boldsymbol{x'}, v)\, dv$ is proved within the range that the order of values of each element of $\boldsymbol{x}$ and $\boldsymbol{x'}$ does not change. Thus, by repeating the above procedures two times at the point of the change of the order, if $x_{\tau_i} > x'_{\tau_i}$ then $(BSh)\int(\boldsymbol{x}, v)dv \geq (BSh)\int(\boldsymbol{x'}, v)dv$ can be proved even in the range with the change of the order.
By applying this procedure successively for each i, the proposition can be proved. ■

Example 3: Let us consider $S = \{1,2,3\}$, and we define the bi-capacity values $v: T(X) \to [-1, 1]$ as shown in Table 1. If the functions $\boldsymbol{x}$ and $\boldsymbol{x'}$ on S defined by $\boldsymbol{x} = (0.8, 0.6, -0.3)$ and $\boldsymbol{x}' = (-0.6, 0.5, -0.4)$.
Hence according to the definition 8, for $\boldsymbol{x} = (0.8, 0.6, -0.3)$

$(BSh)\int(\boldsymbol{x}, v)\, dv = [0.8\ .\ v(\{1^+, 2^\emptyset, 3^\emptyset\})] \curlyvee [0.6\ .\ v(\{1^+, 2^+, 3^\emptyset\})] \curlyvee [0.3\ .\ v(\{1^+, 2^+, 3^-\})]$
$(BSh)\int(\boldsymbol{x}, v)\, dv = [0.8\ .\ 0.7] \curlyvee [0.6\ .\ 0.7] \curlyvee [0.3\ .\ 0.8]$
$(BSh)\int(\boldsymbol{x}, v)\, dv = [0.56] \curlyvee [0.42] \curlyvee [0.24]$
$(BSh)\int(\boldsymbol{x}, v)\, dv = 0.56.$

Similarly, for $\boldsymbol{x}' = (-0.6, 0.5, -0.4)$ we obtain

$(BSh)\int(\boldsymbol{x}', v)\, dv = [0.6\ .\ v(\{1^-, 2^\emptyset, 3^\emptyset\})] \curlyvee [0.5\ .\ v(\{1^-, 2^+, 3^\emptyset\})] \curlyvee [0.4\ .\ v(\{1^-, 2^+, 3^-\})]$
$(BSh)\int(\boldsymbol{x}', v)\, dv = [0.6\ .\ -0.6] \curlyvee [0.5\ .\ 0.3] \curlyvee [0.4\ .\ -0.5]$
$(BSh)\int(\boldsymbol{x}', v)\, dv = [-0.36] \curlyvee [0.15] \curlyvee [-0.20]$

$(BSh)\int(\boldsymbol{x}', v)\, dv = -0.36.$

That is, $(BSh)\int(\boldsymbol{x}, v)\, dv \ \geq\ (BSh)\int(\boldsymbol{x'}, v) dv.$

5 Conclusion

This paper presented a new approach of bipolar Shilkret integrals related to bi-capacities. Then, some main properties of this integral ``the bipolar Shilkret integral '' had discussed with illustrated examples. This approach is defined on the notion of ternary-element sets and allows a simple way to propose other bipolar fuzzy integrals concerning the bi-capacities. As future works, we intend to use this approach for studying discrete bipolar universal integrals [21] and discrete pseudo integrals [29].

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