MATHEMATICS AND COMPUTATIONAL SCIENCES

Vol 4(1), 2023, pp:1-8

DOI:10.30511/mcs.2023.1978083.1099

ISSN:2717-2708

http://mcs.qut.ac.ir/

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Numerical solution to Volterra integro-differential equations using collocation approximation

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Abstract: This paper consider collocation method for the numerical solution of Volterra integro-differential equation using polynomial basis functions. We convert modeled equation into a linear algebraic system of equations and matrix inversion is employed to solve the algebraic equation. We substitute the result of algebraic into the approximate solution to obtain the numerical result. Some numerical problems are solved to show the method's efficiency and consistency.

Keywords: Collocation; Volterra; Integro-differential; Basis polynomial; Approximate Solution
2020 Mathematics Subject Classification: 47G20, 81Q05, 45D05
Receive: 4 December 2022, **Accepted:** 09 March 2023

1 Introduction

Volterra integro-differential equations have been the subject of numerous studies due to their frequent appearance in various applications such as fluid mechanics and viscoelasticity. Volterra integral equations, on the other hand, appear in engineering, physics, chemistry, and biological problems such as parabolic boundary value problems, population dynamics, and semi-conductor devices. Many initial and boundary value problems associated with ordinary and partial differential equations can be modeled using the Volterra integral equation types. [20].

A variety of methods have been used to investigate the solution of Volterra integro-differential equations, including the Adomian decompositions method developed by [12, 15], Collocation method by[9, 2, 17, 1], Hybrid linear multistep method[14, 13], Chebyshev-Galerkin method[10], Bernoulli matrix method [5], Differen- tial transform method[6], Pseudospectral Method[7], Bernstein Polynomials Method [11, 18], Mellin transform approach [8], Perturbed Method[19] and Homotopy Perturbation [16]. [18] presented an efficient numerical method for solving Volterra integro-differential equations by using Legendre as a basis function for the solution of the integro-differential equations. Assumed suitable solutions in terms of the Legendre polynomial as the basis function, which was then substituted into the class of integro-differential equations considered. The results obtained for some numerical examples validated the proposed method's efficiency and dependability. [3] considered first order volterra integro-differential equations using standard collocation method. An assumed approximate solution in terms of the constructed polynomial was substituted into the class of integro-differential equation considered. The equation was collocated at appropriate points within the interval of conside- ration [0,1] to obtain a system of algebraic linear equations. Collocation approach for the computa- tional solution of Fredholm-Volterra fractional order of

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integro-differential equations was presented by [4]. They obtained the linear integral form of the problem and transformed it into a system of linear algebraic equations using standard collocation points.

In this research, we consider Volterra Integro-differential equation of the form:

$$y^{(n)}(x) = f(x) + \int_0^x k(x,t)y(t)dt \tag{1.1}$$

subject to initial condition

$$y^{(n)}(0) = a_n,\ n = 0, 1, \ldots, N \tag{1.2}$$

$a \leq x \leq b$, $y^{(n)}(x)$ is the unkown function, $k\,(x,t)$ is the Volterra integral kernel function. $f(x)$ is the known function

2 Basic Definitions

In this section, we provide certain definitions and fundamental ideas for the formulation of the specified problem.

Definition 1: Let $(a_n)\,, n \geq 0$ be a sequence of real numbers. The power series in x with coefficients a_n is an expression.

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots a_Nx^N = \sum_{n=0}^{N} a_nx^n = \ \phi(x)\ \mathbf{A} \tag{2.1}$$

where $\phi(x) = [1 \quad x \quad x^2 \quad \cdots \quad x^N]$, $\mathbf{A} = [a_0 \quad a_1 \quad \cdots \quad a_N]^T$

then $y(x,n) = x^n\mathbf{A},\ n = 0(1)N,\ n \in Z^+$

Definition 2: The desired collocation points within an interval are determined using this method. i.e. [a,b] and is provided by

$$x_i = a + \frac{(b-a)i}{N}, i = 1, 2, 3, ...N \tag{2.2}$$

3 Methodology

In this section, collocation method and approximation are employed for the numerical solution of Volterra integro-differential equations.

Let the solution to (1.1) and (2.1) be approximated by

$$y\,(x) = \phi\,(x)\,\mathbf{A} \tag{3.1}$$

$\phi\,(x)$ is an interpolating polynomial and $\mathbf{A}$ are parameters to be determined,

$$\phi\,(x) = \left[\ \phi_0\,(x) \quad \phi_1\,(x) \quad \phi_2\,(x) \quad \cdots \phi_N\,(x)\ \right]$$

$$\mathbf{A} = \left[\ a_0 \quad a_1 \quad a_2 \quad \cdots \quad a_N\ \right]^T$$

substituting (2.1) into (1.1) gives

$$\phi^{(n)}\,(x)\,\mathbf{A} = f(x) + \int_0^x k(x,t)\phi\,(t)\,\mathbf{A}dt \tag{3.2}$$

collecting the like terms

$$\left(\phi^{(n)}\,(x) - \int_0^x k(x,t)\phi\,(t)\,dt\right)\mathbf{A} = f(x) \tag{3.3}$$

Equation (3.3) can be written in this form

$$U(x)\mathbf{A} = f(x) \tag{3.4}$$

where

$$U(x) = \left(\phi^{(n)}(x) - \int_0^x k(x,t)\phi(t)\,dt\right)_{1\times[N+1]}$$

Collocating (3.4) using the standard collocation points

$$x_i = a + \frac{b-a}{N}i$$

$$U(x_i)\mathbf{A} = f(x_i) \tag{3.5}$$

where

$$U(x_i) = \begin{bmatrix} U_0(x_0) & U_1(x_0) & U_2(x_0) & \cdots & U_N(x_0) \\ U_0(x_1) & U_1(x_1) & U_2(x_1) & \cdots & U_N(x_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ U_0(x_N) & U_1(x_N) & U_2(x_N) & \cdots & U_N(x_N) \end{bmatrix}, \; f(x_i) = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix}$$

Using the initial conditions

$$y^{(n)}(a) = 0 \tag{3.6}$$

hence, (3.6) becomes

$$\phi^{(n)}(a) = 0 \tag{3.7}$$

Substituting (3.7) into equation (3.5) gives

$$U^*(x_i)\mathbf{A} = f^*(x_i) \tag{3.8}$$

The unknown values are solved using matrix inversion. Substituting the values of a_i obtained in the approximate solution gives the numerical solution.

$$y(x) = \phi(x_i)U^{*-1}(x_i)f^*(x_i) \tag{3.9}$$

4 Convergence Analysis

We establish the convergence of the method.

$$y_N^{(n)}(x) = f(x) + \int_0^x k(x,t)y(t)dt \tag{4.1}$$

Substract (1.1) from (4.1) gives

$$E_N(x) = y_N^{(n)}(x) - y^{(n)}(x) \tag{4.2}$$

hence

$$|E_N(x)| \leq \left|\int_0^x k(x,t)E_N(t)dt\right|$$

Therefore

$$\frac{\|E_N(x_i)\|_\infty}{\|E_N(t)\|_\infty} \leq \left|\int_0^{x_i} k(x_i,t)E_N(t)dt\right|$$

The method converges

5 Numerical Examples

In this section, two numerical examples with initial conditions are presented to confirm the efficiency and accuracy of the method. Let $y_n(x)$ and $y(x)$ be the approximate and exact solution respectively. $\text{Error}_N = |y(x) - y_n(x)|$

Example 1: [18] Considering Volterra integro-differential equation

$$y^{''}(x) = f(x) + \int_0^x (x-t)\, y(t)dt$$

subject to initial conditions

$$y(0) = 0, y^{'}(0) = 2$$

where

$$f(x) = -x - \frac{x^3}{6}$$

Exact solution $y(x) = x + \sin(x)$

Solution 1

We solve the example 1 at $N = 4, 5$ *and* 7 but we use $N = 4$ for demonstration

Using approximate solution (2.1) on example 1 gives

$$\phi^{''}(x)\,\mathbf{A} = f(x) + \int_0^x (x-t)\;\; \phi(t)\,\mathbf{A}dt \tag{5.1}$$

$$\left[\phi^{''}(x) - \int_0^x (x-t)\;\; \phi(t)\,dt\right]\mathbf{A} = f(x)$$

Equation (5.1) gives

$$U(x)\mathbf{A} = f(x)$$

where

$$U(x) = \phi^{''}(x) - \int_0^x (x-t)\;\; \phi(t)\,dt$$

collocating at $x_4 = \begin{bmatrix} 0 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 \end{bmatrix}$ and substituting the initial conditions gives

$$U(x_i)^*\mathbf{A} = f(x_i)^* \tag{5.2}$$

where

$$U(x_i)^* = \begin{bmatrix} 0.000000000 & 0.000000000 & 2.000000000 & 0.000000000 & 0.000000000 \\ -0.31250000e-1 & -0.2604167e-2 & 1.999674479 & 1.499951172 & 0.749991862 \\ -0.125000000 & -0.20833330e-1 & 1.994791667 & 2.998437500 & 2.999479167 \\ -0.281250000 & -0.70312500e-1 & 1.9736328120 & 4.488134766 & 6.744067383 \\ -0.500000000 & -0.166666700 & 1.916666667 & 5.950000000 & 11.966666670 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$f(x_i)^* = \begin{bmatrix} 0.000000000 & -0.252604166700 & -0.520833300 & -0.820312500 & -1.1666666670 & 0 & 2 \end{bmatrix}^T$$

We now solve for the unknown values $\mathbf{A}$ (5.2) making use of matrix inversion results in;

$$y_4 = \begin{pmatrix} -0.2397100e-5 + 1.999915481800x + 0.1355244200e-2x^2 \\ -0.180169550500x^3 + 0.19503866100e-1x^4 \end{pmatrix}$$

Table 1: Exact and approximate values for example 1

x	Exact	$N=4$	$N=5$	$N=7$
0.2	0.398669330800	0.398624758900	0.398667390500	0.398669337200
0.4	0.789418342300	0.789149082600	0.789409656400	0.789418354300
0.6	1.164642473000	1.164045858000	1.164630051000	1.164642493000
0.8	1.517356091000	1.516539319000	1.517343505000	1.517356116000
1.0	1.841470985000	1.840602644000	1.841452966000	1.841471017000

Applying the same procedure for $N = 5$ *and* 7 gives

$$y_5 = \begin{pmatrix} 2.386504745000 \times 10^{-8} + 2.000000004700x + 0.26518200e - 4x^2 \\ -0.167307258000x^3 + 0.1537709100e - 2x^4 + 0.7195968200e - 2x^5 \end{pmatrix}$$

$$y_7 = \begin{pmatrix} -5.2153246999 \times 10^{-13} + 2.0000000000x - 1.5123070796 \times 10^{-8}x^2 \\ -0.1666633875x^3 - 0.194459e - 4x^4 + 0.0083804079x^5 \\ -0.541076e - 4x^6 - 0.1724342e - 3x^7 \end{pmatrix}$$

Table 2: Absolute Error for example 1

x	error$_4$	[18]$_4$	error$_5$	[18]$_5$	error$_7$	[18]$_7$
0.2	4.4571900e-5	4.4645e-2	1.940300e-6	2.2430e-2	6.400e-9	1.0211e-5
0.4	2.69259700e-4	1.7431e-1	8.685900e-6	8.5189e-2	1.200e-8	3.3753e-4
0.6	5.96615000e-4	3.8369e-1	1.2422000e-5	1.8286e-1	2.000e-8	2.5434e-3
0.8	8.16772000e-4	6.6918e-1	1.2586000e-5	3.1171e-1	2.500e-8	1.0586e-2
1.0	8.68341000e-4	1.0291e+00	1.8019000e-5	4.6949e-1	3.200e-8	3.1841e-2

Example 2 [18] Considering Volterra integro-differential equation

$$y'''(x) = f(x) + \int_0^x (x-t)\, y(t)dt$$

subject to initial conditions

$$y(0) = 1, y'(0) = 0, y''(0) = 1$$

where

$$f(x) = 1 + x + \frac{x^3}{6}$$

Exact solution $y(x) = e^x - x$

Solution 2

Using approximate solution (2.1) on example 2 gives

$$\phi'''(x)\,\mathbf{A} = f(x) + \int_0^x (x-t)\,\phi(t)\,\mathbf{A}dt \tag{5.3}$$

$$\left[\phi'''(x) - \int_0^x (x-t)\,\phi(\mathbf{t})dt\right]\mathbf{A} = f(x)$$

Equation (5.3) gives

$$U(x)\mathbf{A} = f(x) \tag{5.4}$$

Table 3: Exact and approximate values for example 2

x	Exact	$N=5$	$N=7$
0.2	1.021402758000	1.021362485000	1.021402731000
0.4	1.091824698000	1.091713226000	1.091824494000
0.6	1.222118800000	1.221845998000	1.222118351000
0.8	1.425540928000	1.425013441000	1.425540168000
1.0	1.718281828000	1.717465701000	1.718280607000

where

$$U(x) = \phi^{'''}(x) - \int_0^x (x-t)\,\phi(\mathbf{t})dt$$

collocating at $x_5 = \begin{bmatrix} 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \end{bmatrix}$ and substituting the initial conditions gives

$$U(x_i)^* \mathbf{A} = f(x_i)^* \tag{5.5}$$

where

$$U(x_i)^* = \begin{bmatrix} 0.000000 & 0.000000 & 0.0000000 & 6.000000 & 0.000000 & 0.000000 \\ -0.20000e-1 & -0.1333e-2 & -0.1333e-3 & 5.999984 & 4.7997867 & 2.399695 \\ -0.80000e-1 & -0.10670e-1 & -0.2133e-2 & 5.999488 & 9.599863467 & 9.59996099 \\ -0.180000 & -0.36000e-1 & -0.10800e-1 & 5.996112 & 14.398444800 & 21.59933349 \\ -0.320000 & -0.85340e-1 & -0.34130e-1 & 5.983616 & 19.191261870 & 38.39500678 \\ -0.500000 & -0.166700 & -0.83330e-1 & 5.950000 & 23.9666670 & 59.97619048 \\ 1 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\ 1 & 0 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0 & 0 & 2 & 0.000000 & 0.000000 & 0.000000 \end{bmatrix}$$

$$f(x_i)^* = \begin{bmatrix} 1.000000 & 1.201333 & 1.410667 & 1.636000 & 1.8853333 & 2.166667 & 1 & 0 & 1 \end{bmatrix}^T$$

We now solve for the unknown values $\mathbf{A}$ (5.5) making use of matrix inversion results in;

$$y_5 = \begin{pmatrix} 0.999999073026 - 0.192975973e - 3x + 0.499951473481x^2 \\ +0.167750035773x^3 + 0.35907218999e - 1x^4 + 0.14050876190e - 1x^5 \end{pmatrix}$$

Applying the same procedure for $N = 7$ gives

$$y_7 = \begin{pmatrix} 0.999999999753 - 4.713789797000 \times 10^{-11}x + 0.99999999996x^2 \\ +0.166668024732x^3 + 0.41617891735e - 1x^4 + 0.8495611806e \\ -2x^5 + 0.1167858707e - 2x^6 + 0.331219349e - 3x^7 \end{pmatrix}$$

6 Conclusion

This paper considered collocation method for the solution of Volterra integro-differential equations. The method is consistent, efficient and easy to compute. The results of example 1 as shown in the table 1 and 2 shows that the approximate solutions converges to the exact solution when the values of N increases. In example 2, The approximate solution at $N=5$ gives

$$y_5(x) = 0.999999073026 - 0.192975973e - 3x + 0.499951473481x^2 + 0.167750035773x^3$$

Table 4: Absolute error for example 2

x	$error_5$	error $_7$
0.2	4.0273000e-5	2.700000000000e-8
0.4	1.11472000e-4	2.040000000000e-7
0.6	2.72802000e-4	4.490000000000e-7
0.8	5.27487000e-4	7.600000000000e-7
1.0	8.16127000e-4	1.221000e-6

$$+0.35907218999e - 1x^4 + 0.14050876190e - 1x^5$$

and solving for N=7, we obtained table 3 which shown the results obtained at $x = 0.2\ to\ 1.0$ for various values of N and the exact solution. Error of example 2 as shown in table 4 indicates that as the values of N increases, the error becomes smaller. The results obtained by our method converges faster than the results obtained by Olayiwola *et al* (2020) at all values N. All computations are done with the aid of Maple 18.

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