MATHEMATICS AND COMPUTATIONAL SCIENCES

Vol 4(1), 2023, pp:9-17

DOI:10.30511/mcs.2023.1987591.1106

ISSN:2717-2708

http://mcs.qut.ac.ir/

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On a fractional differential equation with fractional boundary conditions

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Abstract: In this article, we study a new nonlinear Langevin equation of two fractional orders with fractional boundary value conditions which is a generalization of previous Langevin equations. Based on Banach and Schauder fixed point theorems, the existence and uniqueness of solutions of this equation are investigated. Moreover, our hypotheses are simpler than similar works.

Keywords: Fractional Langevin equation; Fractional boundary condition; Existence solution; Fixed point theorem
2020 Mathematics Subject Classification: 26A33; 34A08; 34A12
Receive: 16 January 2023, **Accepted:** 09 March 2023

1 Introduction

Fractional calculus has infiltrated as a powerful tool in various problems and modern systems. Fractional differential equations, as one of these tools, have high capacity in many engineering sciences and technology and has attracted many researchers. For instance, Jianmin *et al.* [20] applied some fractional derivatives to fractional viscoelastic models and studied the performance of fractional derivatives in describing viscoelastic behaviors. In [22], the generalization of the Lorentz model was obtained by using a fractional differential equation with Caputo fractional derivative. Bas and Ozarslan [5] investigated some modeling problems, logistic equation, population growth, blood alcohol model and Newton,'s law of cooling by applying Atangana-Baleanu fractional derivative. The fractional calculus to model a pn semiconductor diode under sinusoidal operation was considered in [21]. Carmo *et al.* [7] provided, based on the theory of fractional calculus, a new model to predict the relative viscosity of petroleum emulsions. For further reading, see [6, 8, 13, 14, 24, 25, 27].

A type of equations called Langevin equation, with stochastic framework, has been used to describe the dynamical processes in fluctuating environments. Kubo introduced some generalizations of Langevin equation in [16, 17]. Langevin equations and fractional Langevin equations are applicable in many fields. For example, Lim *et al.* [19] introduced, based on both the Weyl and Riemann-Liouville fractional derivatives, a type of fractional Langevin equation of two different orders and obtained its solutions. Eab and Lim [9] proposed the fractional generalized Langevin equation with external force that was used to model single-file diffusion. By using sub diffusive motion of bio molecules observed in living cells, Jeon and Metzler [12] studied a non-Brownian particle and its the stochastic properties which is governed by either fractional

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Brownian motion or the fractional Langevin equation. In [10], four-dimensional Langevin equations were applied to estimate the mean prompt neutron multiplicity, the average pre-scission neutron multiplicity, the fission probability and the mean kinetic energy of fission fragments.

In recent years, on existence and uniqueness of solutions of initial and boundary value problems for differential equations of fractional order by different methods is an important issue that includes fractional Langevin equation. Ahmad *et al.* studied the existence of solutions for several types of nonlinear Langevin equations with boundary conditions in [1, 2, 3]. Also, the existence results for fractional Langevin equations involving two fractional orders was obtained in [4, 11, 18, 26].

In this manuscript, we discuss the existence results for a fractional equation of Langevin type with fractional boundary conditions of the following form:

$$ {}^{c}D_{0+}^{\tau}\left({}^{c}D_{0+}^{\eta}+\lambda\right)u\left(s\right)=f(s,u(s)),\ \ s\in\left(0,1\right),\ 0<\eta\leqslant 1,\ 2<\tau\leqslant 3, \tag{1.1}$$

$$ u\left(0\right)=u\left(1\right)=0,\quad D_{0+}^{\eta}u\left(0\right)=D_{0+}^{\eta}u\left(1\right)=0, \tag{1.2}$$

where ${}^{c}D_{0+}^{\eta}$ is the Caputo fractional derivative of order η, D_{0+}^{η} is the Riemann-Liouville fractional derivative of order η, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function and λ is a real number. By applying some fixed point theorems, the existence results are investigated with simpler conditions than previous papers.

The rest of the work is organized as: In section 2, we propose some preliminaries of fractional integral and derivative. The existence and uniqueness of solutions for BVP (1.1) and (1.2) are studied in section 3. In section 4, we give some concluding remarks.

2 Preliminaries

In this section, some definitions and lemmas necessary are recalled.

Definition 2.1. *[15, 23] The Riemann-Liouville fractional integral of order $\rho>0$ of a function $h:(0,\infty)\to\mathbb{R}$ is defined by*

$$I_{0+}^{\rho}h(\nu)=\frac{1}{\Gamma(\rho)}\int_{0}^{\nu}(\nu-s)^{\rho-1}h(s)ds,$$

provided that the integral exists.

Definition 2.2. *[15, 23] The Riemann-Liouville fractional derivative of order $\rho>0$ of a continuous function $h:(0,\infty)\to\mathbb{R}$ is defined by*

$$D_{0+}^{\rho}h(\nu)=\frac{1}{\Gamma(r-\rho)}\left(\frac{d}{dt}\right)^{r}\int_{0}^{\nu}(\nu-s)^{r-\rho-1}h(s)ds,$$

where $r=[\rho]+1$, provided that the integral exists.

Definition 2.3. *[15, 23] For a function h given on the interval $[0,\infty)$, the Caputo fractional derivative of order $\rho>0$ of h is defined by*

$${}^{c}D_{0+}^{\rho}h(\nu)=\frac{1}{\Gamma(r-\rho)}\int_{0}^{\nu}(\nu-s)^{r-\rho-1}h^{(r)}(s)ds,$$

where $r=[\rho]+1$.

Lemma 2.4. *[15, 23] Let $\eta,\tau\geqslant 0$ and $n\in\mathbb{N}$, then the following relations hold:*

1. $D_{a+}^{\eta}I_{a+}^{\eta}h(s)=h(s)$,
2. $D_{a+}^{\tau}I_{a+}^{\eta}h(s)=D_{a+}^{\tau-\eta}h(s)$, *(if $\tau\geqslant\eta$),*

3. $D_{a+}^{\tau} I_{a+}^{\eta} h(s) = I_{a+}^{\eta-\tau} h(s)$, (if $\eta \geqslant \tau$),
4. $D_{a+}^{\eta} (s-a)^{\tau} = \frac{\Gamma(\tau+1)}{\Gamma(\tau-\eta+1)} (s-a)^{\tau-\eta}$.

Lemma 2.5. *[15] Let $\rho > 0$, then the fractional differential equation ${}^{c}D^{\rho}u(s) = 0$, has a general solution as*

$$u(s) = a_0 + a_1 s + a_2 s^2 + \cdots + a_{r-1} s^{r-1},$$

for some $a_k \in \mathbb{R}$, $k = 1, \ldots, r-1$, and $m = [\rho] + 1$.

Lemma 2.6. *[15] Let $\rho > 0$, then we have*

$$I_{0+}^{\rho}\ {}^{c}D_{0+}^{\rho} u(s) = u(s) + a_0 + a_1 s + \cdots + a_{r-1} s^{r-1},$$

where $a_k \in \mathbb{R}$, $k = 1, \ldots, r-1$, and $r = [\rho] + 1$.

Lemma 2.7. *Let $z \in C[0,1]$, then the following boundary value problem for fractional Langevin equation*

$$\begin{aligned} &{}^{c}D_{0+}^{\tau} \left({}^{c}D_{0+}^{\eta} + \lambda \right) u(s) = z(s), \ s \in (0,1), \ 0 < \eta \leqslant 1, \ 2 < \tau \leqslant 3, \\ &u(0) = u(1) = 0, \quad D_{0+}^{\eta} u(0) = D_{0+}^{\eta} u(1) = 0, \end{aligned}$$

has a unique solution as

$$u(s) = \int_0^1 G(s,x) z(x) dx + \int_0^1 H(s,x) z(x) dx,$$

where

$$G(s,x) = \begin{cases} \frac{(s-x)^{\eta+\tau-1}}{\Gamma(\eta+\tau)} + \frac{s^{\eta+1}(1-x)^{\eta+\tau-1}}{\Gamma(\eta+\tau)} - \frac{2s^{\eta+2}(1-x)^{\eta+\tau-1}}{(\eta+2)\Gamma(\eta+\tau)} \\ + \frac{2s^{\eta+2}(1-x)^{\tau-1}}{\Gamma(\eta+3)\Gamma(\tau)} - \frac{2s^{\eta+1}(1-x)^{\tau-1}}{\Gamma(\eta+2)\Gamma(\tau)}, & x \leqslant s, \\ \\ \frac{s^{\eta+1}(1-x)^{\eta+\tau-1}}{\Gamma(\eta+\tau)} - \frac{2s^{\eta+2}(1-x)^{\eta+\tau-1}}{(\eta+2)\Gamma(\eta+\tau)} + \frac{2s^{\eta+2}(1-x)^{\tau-1}}{\Gamma(\eta+3)\Gamma(\tau)} \\ - \frac{2s^{\eta+1}(1-x)^{\tau-1}}{\Gamma(\eta+2)\Gamma(\tau)}, & s \leqslant x, \end{cases}$$

and

$$H(s,x) = \begin{cases} \frac{2\lambda s^{\eta+2}(1-x)^{\eta-1}}{(\eta+2)\Gamma(\eta)} - \frac{\lambda s^{\eta+1}(1-x)^{\eta-1}}{\Gamma(\eta)} - \frac{\lambda(s-x)^{\eta-1}}{\Gamma(\eta)}, & x \leqslant s, \\ \\ \frac{2\lambda s^{\eta+1}(1-x)^{\eta-1}}{(\eta+2)\Gamma(\eta)} - \frac{\lambda s^{\eta+1}(1-x)^{\eta-1}}{\Gamma(\eta)}, & s \leqslant x. \end{cases}$$

Proof. By Lemma 2.6, for $2 < \tau \leqslant 3$, we have

$${}^{c}D_{0+}^{\eta} u(s) = I_{0+}^{\tau} z(s) - \lambda u(s) + c_0 + c_1 s + c_2 s^2,$$

where $c_0, c_1, c_2 \in \mathbb{R}$. Then for $0 < \eta \leqslant 1$,

$$u(s) = I_{0^+}^{\eta+\tau} z(s) - \lambda I_{0^+}^{\eta} u(s) + c_0 \frac{s^{\eta}}{\Gamma(\eta+1)} + c_1 \frac{s^{\eta+1}}{\Gamma(\eta+2)} + c_2 \frac{2s^{\eta+2}}{\Gamma(\eta+3)} + c_3. \tag{2.1}$$

By the boundary condition $u(0) = 0$, we get $c_3 = 0$. Also, the Riemann-Liouville fractional derivative of order η of $u(s)$ is as

$$D_{0+}^{\eta} u(s) = I_{0+}^{\tau} z(s) - \lambda u(s) + c_0 + c_1 s + c_2 s^2. \tag{2.2}$$

For (2.2) by $D^{\eta} u(0) = 0$, we obtain $c_0 = 0$. The conditions $u(1) = 0$ and $D_{0+}^{\eta} u(1) = 0$, yield

$$\frac{1}{\Gamma(\eta+\tau)} \int_0^1 (1-x)^{\eta+\tau-1} z(x) dx - \frac{\lambda}{\Gamma(\eta)} \int_0^1 (1-x)^{\eta-1} u(x) dx + \frac{c_1}{\Gamma(\eta+2)} + \frac{2c_2}{\Gamma(\eta+3)} = 0, \tag{2.3}$$

and

$$\frac{1}{\Gamma(\tau)}\int_0^1 (1-x)^{\tau-1} z(x)dx + c_1 + c_2 = 0, \tag{2.4}$$

respectively. By solving the system of (2.3) and (2.4), we have

$$\begin{aligned} c_1 &= \frac{\Gamma(\eta+2)}{\Gamma(\eta+\tau)}\int_0^1 (1-x)^{\eta+\tau-1} z(x)dx - \frac{2}{\Gamma(\tau)}\int_0^1 (1-x)^{\tau-1} z(x)dx \\ &\quad -\frac{\lambda\Gamma(\eta+2)}{\Gamma(\eta)}\int_0^1 (1-x)^{\eta-1} u(x)dx, \end{aligned}$$

and

$$\begin{aligned} c_2 &= -\frac{\Gamma(\eta+2)}{\Gamma(\eta+\tau)}\int_0^1 (1-x)^{\eta+\tau-1} z(x)dx + \frac{1}{\Gamma(\tau)}\int_0^1 (1-x)^{\tau-1} z(x)dx \\ &\quad +\frac{\lambda\Gamma(\eta+2)}{\Gamma(\eta)}\int_0^1 (1-x)^{\eta-1} u(x)dx. \end{aligned}$$

Now, Substituting the values of c_0, c_1, c_2, c_3 in (2.1), we obtain

$$\begin{aligned} u(s) &= \frac{1}{\Gamma(\eta+\tau)}\int_0^s (s-x)^{\eta+\tau-1} z(x)dx - \frac{\lambda}{\Gamma(\eta)}\int_0^s (s-x)^{\eta-1} u(x)dx \\ &+ \left(\frac{s^{\eta+1}}{\Gamma(\eta+\tau)} - \frac{2s^{\eta+2}}{(\eta+2)\Gamma(\eta+\tau)}\right)\int_0^1 (1-x)^{\eta+\tau-1} z(x)dx \\ &+ \left(\frac{2s^{\eta+2}}{\Gamma(\eta+3)\Gamma(\tau)} - \frac{2s^{\eta+1}}{\Gamma(\eta+2)\Gamma(\tau)}\right)\int_0^1 (1-x)^{\tau-1} z(x)dx \\ &+ \left(\frac{2\lambda s^{\eta+2}}{(\eta+2)\Gamma(\eta)} - \frac{\lambda s^{\eta+1}}{\Gamma(\eta)}\right)\int_0^1 (1-x)^{\eta-1} u(x)dx \\ &= \int_0^s \left(\frac{(s-x)^{\eta+\tau-1}}{\Gamma(\eta+\tau)} + \frac{s^{\eta+1}(1-x)^{\eta+\tau-1}}{\Gamma(\eta+\tau)} - \frac{2s^{\eta+2}(1-x)^{\eta+\tau-1}}{(\eta+2)\Gamma(\eta+\tau)}\right. \\ &\left. + \frac{2s^{\eta+2}(1-x)^{\tau-1}}{\Gamma(\eta+3)\Gamma(\tau)} - \frac{2s^{\eta+1}(1-x)^{\tau-1}}{\Gamma(\eta+2)\Gamma(\tau)}\right) z(x)dx \\ &+ \int_s^1 \left(\frac{s^{\eta+1}(1-x)^{\eta+\tau-1}}{\Gamma(\eta+\tau)} - \frac{2s^{\eta+2}(1-x)^{\eta+\tau-1}}{(\eta+2)\Gamma(\eta+\tau)} + \frac{2s^{\eta+2}(1-x)^{\tau-1}}{\Gamma(\eta+3)\Gamma(\tau)} - \frac{2s^{\eta+1}(1-x)^{\tau-1}}{\Gamma(\eta+2)\Gamma(\tau)}\right) z(x)dx \\ &+ \int_0^s \left(\frac{2\lambda s^{\eta+2}(1-x)^{\eta-1}}{(\eta+2)\Gamma(\eta)} - \frac{\lambda s^{\eta+1}(1-x)^{\eta-1}}{\Gamma(\eta)} - \frac{\lambda(s-x)^{\eta-1}}{\Gamma(\eta)}\right) u(x)dx \\ &+ \int_s^1 \left(\frac{2\lambda s^{\eta+2}(1-x)^{\eta-1}}{(\eta+2)\Gamma(\eta)} - \frac{\lambda s^{\eta+1}(1-x)^{\eta-1}}{\Gamma(\eta)}\right) u(x)dx \\ &= \int_0^1 G(s,x)\, z(x)dx + \int_0^1 H(s,x)\, u(x)dx. \quad \Box \end{aligned}$$

3 Existence results

The existence and uniqueness solutions of fractional Langevin differential equation (1.1) and (1.2) is investigate in this section. We consider the Banach space of continuous functions $D = C[0,1]$ equipped with the norm $\|w\| = \max_{s\in[0,1]} |w(s)|$.

The following hypotheses to prove our main results is necessary:

A_1) The function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

A_2) A constant $K > 0$ exists such that

$$|f(s,x) - f(s,z)| \leq K|x - z|,$$

for all $s \in [0,1]$, $x, z \in \mathbb{R}$.

A_3) $(\Upsilon_1 + \Upsilon_2) < 1$, where $\Upsilon_1 = \left\{\frac{2\alpha+6}{\Gamma(\eta+\tau+1)(\eta+2)} + \frac{2\alpha+6}{\Gamma(\tau+1)\Gamma(\eta+3)}\right\}$, $\Upsilon_2 = \left\{\frac{|\lambda|(2\alpha+6)}{\Gamma(\eta+1)(\eta+2)}\right\}$.

Theorem 3.1. *Suppose that the assumptions (A_1,A_2,A_3) are satisfied. Then the fractional nonlinear Langevin equation (1.1)-(1.2) has a unique solution.*

Proof. The operator $\mathcal{T} : D \to D$ defined as follows:

$$(\mathcal{T}u)(s) = \int_0^1 G(s,x)\, f(x,u(x))\, dx + \int_0^1 H(s,x)\, u(x) dx.$$

By applying the Lemma 2.7, we obtain the fixed points of the operator $\mathcal{T}$ that are the same solutions of the problem (1.1)-(1.2). Since the functions f, $G(s,x)$ and $H(s,x)$ are continuous, then the operator $\mathcal{T}$ is continuous. Let $M = \max_{s\in[0,1]} |f(s,u(s))| + 1$. We consider a ball $B_\delta = \{u \in D, \|u\| \leq \delta, s \in [0,1]\}$ so that $\frac{M\Upsilon_1}{1-K\Upsilon_1-\Upsilon_2} \leq \delta$. First, we show that $\mathcal{T}B_\delta \subset B_\delta$. For any $u \in B_\delta$, then

$$\begin{aligned}
|(\mathcal{T}u)(s)| &\leq \frac{1}{\Gamma(\eta+\tau)} \int_0^s (s-x)^{\eta+\tau-1} |f(x,u(x))|\, dx \\
&+ \left(\frac{s^{\eta+1}}{\Gamma(\eta+\tau)} + \frac{2s^{\eta+2}}{(\eta+2)\Gamma(\eta+\tau)}\right) \int_0^1 (1-x)^{\eta+\tau-1} |f(x,u(x))|\, dx \\
&+ \left(\frac{2s^{\eta+2}}{\Gamma(\eta+3)\Gamma(\tau)} + \frac{2s^{\eta+1}}{\Gamma(\eta+2)\Gamma(\tau)}\right) \int_0^1 (1-x)^{\tau-1} |f(x,u(x))|\, dx \\
&+ \frac{|\lambda|}{\Gamma(\eta)} \int_0^s (s-x)^{\eta-1} |u(x)|\, dx + \left(\frac{2|\lambda|\, s^{\eta+2}}{(\eta+2)\Gamma(\eta)} + \frac{|\lambda|\, s^{\eta+1}}{\Gamma(\eta)}\right) \int_0^1 (1-x)^{\eta-1} |u(x)|\, dx \\
&\leq \frac{1}{\Gamma(\eta+\tau)} \int_0^s (s-x)^{\eta+\tau-1} \left(|f(x,u(x)) - f(x,0)| + |f(x,0)|\right) dx \\
&+ \left(\frac{1}{\Gamma(\eta+\tau)} + \frac{2}{(\eta+2)\Gamma(\eta+\tau)}\right) \int_0^1 (1-x)^{\eta+\tau-1} \left(|f(x,u(x)) - f(x,0)| + |f(x,0)|\right) dx \\
&+ \left(\frac{2}{\Gamma(\eta+3)\Gamma(\tau)} + \frac{2}{\Gamma(\eta+2)\Gamma(\tau)}\right) \int_0^1 (1-x)^{\tau-1} \left(|f(x,u(x)) - f(x,0)| + |f(x,0)|\right) dx \\
&+ \frac{|\lambda|}{\Gamma(\eta)} \int_0^s (s-x)^{\eta-1} |u(x)|\, dx + \left(\frac{2|\lambda|}{(\eta+2)\Gamma(\eta)} + \frac{|\lambda|}{\Gamma(\eta)}\right) \int_0^1 (1-x)^{\eta-1} |u(x)|\, dx \\
&\leq (K\|u\| + M) \left\{\frac{s^{\eta+\tau}}{\Gamma(\eta+\tau+1)} + \frac{\eta+4}{\Gamma(\eta+\tau+1)(\eta+2)} + \frac{2\alpha+6}{\Gamma(\tau+1)\Gamma(\eta+3)}\right\} \\
&+ \|u\| \left\{\frac{|\lambda|\, s^{\eta}}{\Gamma(\eta+1)} + \frac{|\lambda|(\eta+4)}{\Gamma(\eta+1)(\eta+2)}\right\} \\
&\leq (K\delta + M) \left\{\frac{2\alpha+6}{\Gamma(\eta+\tau+1)(\eta+2)} + \frac{2\alpha+6}{\Gamma(\tau+1)\Gamma(\eta+3)}\right\} + \delta \left\{\frac{|\lambda|(2\alpha+6)}{\Gamma(\eta+1)(\eta+2)}\right\} \\
&= K\Upsilon_1\delta + M\Upsilon_1 + \Upsilon_2\delta \\
&\leq \delta.
\end{aligned}$$

For any $u, v \in D$, $s \in [0,1]$, we have

$$\begin{aligned}
&|(\mathcal{T}u)(s) - (\mathcal{T}v)(s)| \leq \frac{1}{\Gamma(\eta+\tau)} \int_0^s (s-x)^{\eta+\tau-1} |f(x,u(x)) - f(x,v(x))|\, dx \\
&+ \left(\frac{s^{\eta+1}}{\Gamma(\eta+\tau)} + \frac{2s^{\eta+2}}{(\eta+2)\Gamma(\eta+\tau)} \right) \int_0^1 (1-x)^{\eta+\tau-1} |f(x,u(x)) - f(x,v(x))|\, dx \\
&+ \left(\frac{2s^{\eta+2}}{\Gamma(\eta+3)\Gamma(\tau)} + \frac{2s^{\eta+1}}{\Gamma(\eta+2)\Gamma(\tau)} \right) \int_0^1 (1-x)^{\tau-1} |f(x,u(x)) - f(x,v(x))|\, dx \\
&+ \frac{|\lambda|}{\Gamma(\eta)} \int_0^s (s-x)^{\eta-1} |u(x) - v(x)|\, dx \\
&+ \left(\frac{2|\lambda| s^{\eta+2}}{(\eta+2)\Gamma(\eta)} + \frac{|\lambda| s^{\eta+1}}{\Gamma(\eta)} \right) \int_0^1 (1-x)^{\eta-1} |u(x) - v(x)|\, dx \\
&\leq \frac{\|u-v\|}{\Gamma(\eta+\tau)} \int_0^s (s-x)^{\eta+\tau-1}\, dx \\
&+ \|u-v\| \left(\frac{1}{\Gamma(\eta+\tau)} + \frac{2}{(\eta+2)\Gamma(\eta+\tau)} \right) \int_0^1 (1-x)^{\eta+\tau-1}\, dx \\
&+ \|u-v\| \left(\frac{2}{\Gamma(\eta+3)\Gamma(\tau)} + \frac{2}{\Gamma(\eta+2)\Gamma(\tau)} \right) \int_0^1 (1-x)^{\tau-1}\, dx \\
&+ \frac{|\lambda| \|u-v\|}{\Gamma(\eta)} \int_0^s (s-x)^{\eta-1}\, dx + \|u-v\| \left(\frac{2|\lambda|}{(\eta+2)\Gamma(\eta)} + \frac{|\lambda|}{\Gamma(\eta)} \right) \int_0^1 (1-x)^{\eta-1}\, dx \\
&= \|u-v\| \left\{ \frac{s^{\eta+\tau}}{\Gamma(\eta+\tau+1)} + \frac{\eta+4}{\Gamma(\eta+\tau+1)(\eta+2)} + \frac{2\alpha+6}{\Gamma(\tau+1)\Gamma(\eta+3)} \right\} \\
&+ \|u-v\| \left\{ \frac{|\lambda| s^{\eta}}{\Gamma(\eta+1)} + \frac{|\lambda|(\eta+4)}{\Gamma(\eta+1)(\eta+2)} \right\} \\
&\leq (\Upsilon_1 + \Upsilon_2) \|u-v\|.
\end{aligned}$$

By condition (A_3), we have $(\Upsilon_1 + \Upsilon_2) < 1$. Hence the operator $\mathcal{T}$ is a contraction operator. Therefore the Banach fixed point theorem implies that the boundary value problem (1.1) and (1.2) has a unique solution. □

In following theorem, we show that the problem has at least a solution.

Theorem 3.2. *Let the assumption* (A_1) *holds. Then the fractional boundary value problem (1.1) and (1.2) has a solution.*

Proof. We shall prove that the operator $\mathcal{T}$ is completely continuous. For any $u \in B_\delta$, $s \in [0,1]$, we have

$$\begin{aligned}
&|(\mathcal{T}u)(s)| \leq \frac{1}{\Gamma(\eta+\tau)} \int_0^s (s-x)^{\eta+\tau-1} |f(x,u(x))|\, dx \\
&+ \left(\frac{s^{\eta+1}}{\Gamma(\eta+\tau)} + \frac{2s^{\eta+2}}{(\eta+2)\Gamma(\eta+\tau)} \right) \int_0^1 (1-x)^{\eta+\tau-1} |f(x,u(x))|\, dx \\
&+ \left(\frac{2s^{\eta+2}}{\Gamma(\eta+3)\Gamma(\tau)} + \frac{2s^{\eta+1}}{\Gamma(\eta+2)\Gamma(\tau)} \right) \int_0^1 (1-x)^{\tau-1} |f(x,u(x))|\, dx
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|\lambda|}{\Gamma(\eta)}\int_0^s (s-x)^{\eta-1}|u(x)|\,dx+\left(\frac{2|\lambda|\,s^{\eta+2}}{(\eta+2)\Gamma(\eta)}+\frac{|\lambda|\,s^{\eta+1}}{\Gamma(\eta)}\right)\int_0^1(1-x)^{\eta-1}|u(x)|\,dx\\
&\le \frac{M}{\Gamma(\eta+\tau)}\int_0^s(s-x)^{\eta+\tau-1}\,dx+M\left(\frac{1}{\Gamma(\eta+\tau)}+\frac{2}{(\eta+2)\Gamma(\eta+\tau)}\right)\int_0^1(1-x)^{\eta+\tau-1}\,dx\\
&+M\left(\frac{2}{\Gamma(\eta+3)\Gamma(\tau)}+\frac{2}{\Gamma(\eta+2)\Gamma(\tau)}\right)\int_0^1(1-x)^{\tau-1}\,dx\\
&+\frac{|\lambda|\,\|u\|}{\Gamma(\eta)}\int_0^s(s-x)^{\eta-1}\,dx+\|u\|\left(\frac{2|\lambda|}{(\eta+2)\Gamma(\eta)}+\frac{|\lambda|}{\Gamma(\eta)}\right)\int_0^1(1-x)^{\eta-1}\,dx\\
&\le M\left\{\frac{s^{\eta+\tau}}{\Gamma(\eta+\tau+1)}+\frac{\eta+4}{\Gamma(\eta+\tau+1)(\eta+2)}+\frac{2\alpha+6}{\Gamma(\tau+1)\Gamma(\eta+3)}\right\}\\
&+\delta\left\{\frac{|\lambda|\,s^{\eta}}{\Gamma(\eta+1)}+\frac{|\lambda|(\eta+4)}{\Gamma(\eta+1)(\eta+2)}\right\}\\
&\le M\Upsilon_1+\delta\Upsilon_2,
\end{aligned}
$$

where $M=\max_{s\in[0,1]}|f(s,u(s))|+1$. Thus, $\mathcal{T}$ is uniformly bounded on B_δ. Moreover, for $u\in B_\delta$ and $s_1,s_2\in[0,1]$ such that $s_1<s_2$, we have

$$
\begin{aligned}
&|(\mathcal{T}u)(s_2)-(\mathcal{T}u)(s_1)|\le\left|\frac{1}{\Gamma(\eta+\tau)}\int_0^{s_2}(s_2-x)^{\eta+\tau-1}f(x,u(x))\,dx\right.\\
&\left.-\frac{1}{\Gamma(\eta+\tau)}\int_0^{s_1}(s_1-x)^{\eta+\tau-1}f(x,u(x))\,dx\right|\\
&+\left|\left(\frac{s_2^{\eta+1}-s_1^{\eta+1}}{\Gamma(\eta+\tau)}+\frac{2s_2^{\eta+2}-2s_1^{\eta+2}}{(\eta+2)\Gamma(\eta+\tau)}\right)\int_0^1(1-x)^{\eta+\tau-1}f(x,u(x))\,dx\right|\\
&+\left|\left(\frac{2s_2^{\eta+2}-2s_1^{\eta+2}}{\Gamma(\eta+3)\Gamma(\tau)}+\frac{2s_2^{\eta+1}-2s_1^{\eta+1}}{\Gamma(\eta+2)\Gamma(\tau)}\right)\int_0^1(1-x)^{\tau-1}f(x,u(x))\,dx\right|\\
&+\left|\frac{\lambda}{\Gamma(\eta)}\int_0^{s_2}(s_2-x)^{\eta-1}u(x)dx-\frac{\lambda}{\Gamma(\eta)}\int_0^{s_1}(s_1-x)^{\eta-1}u(x)dx\right|\\
&+\left|\left(\frac{2\lambda\left(s_2^{\eta+2}-s_1^{\eta+2}\right)}{(\eta+2)\Gamma(\eta)}+\frac{\lambda\left(s_2^{\eta+1}-s_1^{\eta+1}\right)}{\Gamma(\eta)}\right)\int_0^1(1-x)^{\eta-1}u(x)dx\right|\\
&\le\frac{2M}{\Gamma(\eta+\tau+1)}(s_2-s_1)^{\eta+\tau}+\frac{M}{\Gamma(\eta+\tau+1)}\left(s_2^{\eta+\tau}-s_1^{\eta+\tau}\right)\\
&+\frac{M}{\Gamma(\eta+\tau+1)}\left(s_2^{\eta+1}-s_1^{\eta+1}+\frac{2\left(s_2^{\eta+2}-s_1^{\eta+2}\right)}{(\eta+2)}\right)+\frac{2M}{\Gamma(\tau+1)}\left(\frac{s_2^{\eta+2}-s_1^{\eta+2}}{\Gamma(\eta+3)}+\frac{s_2^{\eta+1}-s_1^{\eta+1}}{\Gamma(\eta+2)}\right)\\
&+\frac{2\delta|\lambda|}{\Gamma(\eta+1)}(s_2-s_1)^{\eta}+\frac{\delta|\lambda|}{\Gamma(\eta+1)}\left(s_2^{\eta}-s_1^{\eta}\right)+\frac{\delta|\lambda|}{\Gamma(\eta+1)}\left(\frac{2\left(s_2^{\eta+2}-s_1^{\eta+2}\right)}{(\eta+2)}+s_2^{\eta+1}-s_1^{\eta+1}\right).
\end{aligned}
$$

Since the functions $(s_2-s_1)^{\eta+\tau}$, $s_2^{\eta+\tau}-s_1^{\eta+\tau}$, $s_2^{\eta+2}-s_1^{\eta+2}$, $s_2^{\eta+1}-s_1^{\eta+1}$, $(s_2-s_1)^{\eta}$ and $s_2^{\eta}-s_1^{\eta}$ are uniformly continuous on $[0,1]$, then $\mathcal{T}(B_\delta)$ is equicontinuous. By the Arzela–Ascoli theorem, we have $\mathcal{T}(B_\delta)$ relativity compact. Hence, the operator $\mathcal{T}:B_\delta\to B_\delta$ is completely continuous. Therefore, By the Schauder fixed-point theorem, the BVP problem (1.1) and (1.2) has a solution. □

For a fractional Langevin equation, we show the existence of solutions in the following example .

Example 3.3. *Consider the following boundary value problem of fractional orders*

$$\begin{cases} {}^{c}D_{0+}^{2.8}\left({}^{c}D_{0+}^{0.8}+0.1\right)u\left(s\right)=\frac{s-\tan^{-1}u(s)}{4(s+2)^{2}}, \ s\in(0,1) \\ u\left(0\right)=u\left(1\right)=0, \quad D_{0+}^{0.8}u\left(0\right)=D_{0+}^{0.8}u\left(1\right)=0, \end{cases} \tag{3.1}$$

Here, $\tau = 2.8$, $\eta = 0.8$, $\lambda = 0.1$ *and* $f(s,u(s)) = \frac{s-\tan^{-1}u(s)}{4(s+1)^2}$. *Clearly that the function* f *is continuous and*

$$|f(s,u)-f(s,v)| \leq \frac{1}{4\left(s+1\right)^{2}}\left|\tan^{-1}u-\tan^{-1}v\right| \leq \frac{1}{16}\left|u-v\right|.$$

Also, $\Upsilon_1 = \frac{2\alpha+6}{\Gamma(\eta+\tau+1)(\eta+2)} + \frac{2\alpha+6}{\Gamma(\tau+1)\Gamma(\eta+3)} = \frac{2(0.8)+6}{\Gamma(0.8+2.8+1)(0.8+2)} + \frac{2(0.8)+6}{\Gamma(2.8+1)\Gamma(0.8+3)} \approx 0.5477$ *and* $\Upsilon_2 = \frac{|\lambda|(2\alpha+6)}{\Gamma(\eta+1)(\eta+2)} = \frac{0.1(2(0.8)+6)}{\Gamma(0.8+1)(0.8+2)} \approx 0.2914$, *then* $\Upsilon_1 + \Upsilon_2 \approx 0.8391 < 1$. *Hence the assumptions* (A_1-A_3) *are satisfied. Thus by applying Theorem 3.1, the BVP (3.1) has a unique solution.*

4 Conclusion

This paper is investigated a new Langevin equation of fractional orders with boundary conditions. We have proved the uniqueness solution in Theorem 3.1. Then, we have showed the existence of solution of the problem only by one condition. Moreover, an example have proposed to illustrate our results.

Acknowledgment. The financial support of this study was provided by Sari Agricultural Sciences and Natural Resources University in Iran in the form of research project (Project No. 03-1397-11).

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