MATHEMATICS AND COMPUTATIONAL SCIENCES

Vol 5(1), 2024, PP:13-19
DOI:10.30511/MCS.2023.543989.1055
ISSN:2717-2708
http://mcs.qut.ac.ir/

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Approximate solutions of Klein-Gordon equation with equal vector and scalar modified Mobius square plus Kratzer potentials with centrifugal term

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Abstract: In this study, we present the analytical solutions of Klein-Gordon equation with modified Mobius square plus Kratzer potential. The energy spectrum and wave functions are obtained using the Nikiforov-Uvarov (NU) method by assuming equal scalar and vector potential. The non relativistic limit is obtained and numerical results are presented. In addition, the energy eigenvalues are obtained for special cases of this potential. Our results show that energy decreases with the screening parameter.

Keywords: Klein-Gordon equation; Nikiforov- Uvarov method; modified Mobius square potential; Kratzer potential

2020 Mathematics Subject Classification: 81Q05
Receive: 30 November 2021, **Accepted:** 09 March 2023

1 Introduction

Over the years, studies have been performed to investigate the various wave equations for known solvable potentials. The reason is that the wave functions provide every useful data to describe a quantum system of importance. For a quantum system in a field of high potential, the relativistic effect is taken into consideration, which in turn leads to corrections to nonrelativistic quantum mechanics [17]. Bearing in mind relativistic effects, a quantum system under study could be investigated within Klein-Gordon and Dirac frameworks, with the former as a tool to study spin 0 particles. Recently, some authors have tried to investigate solutions of the Klein-Gordon equation in different interaction potentials via various methods [1,9,13,11]. However, centrifugal terms appear in some potentials, making it difficult to arrive at exact solutions. Hence, the need for approximation schemes. Recently, interest has been focused on combining different potentials. The reason for combining two or more potentials is to achieve a wide range of applications. For example, Hassanabadi et al. [6] investigated an

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arbitrary state of the Dirac equation with the combination of deformed Hylleraas and modified Eckart potentials. Onate et al. [14] studied solutions of the D -dimensional Schrodinger equation with a combination of inverse Trigonometry scarf and Coulomb potentials. Peyman and Aspoukeh [3] obtained the bound state solution of the Klein-Gordon equation from a combination of Hellmann and modified Kratzer potentials. Ahmadov et al. [1], in their contribution, discussed solutions of the Klein-Gordon equation with a linear combination of Hulten and Yukawa potentials. Some authors have investigated Schrodinger, Klein-Gordon, and Dirac equations with a combination of Hua and modified Eckart potentials [15,5].

The emphasis of this paper is on the combination of modified Mobius square and Kratzer potentials. Already, solutions of the Schrodinger equation with the combination of modified Mobius square and Kratzer potentials have been reported [5]. The modified Mobius square (MMS) potential is the general form of notable potentials, and it is very useful in molecular physics, high energy physics, and chemical physics [5,18,7]. The Kratzer potential is a very useful potential in atomic and molecular physics, it serves as a model to describe vibrations in diatomic molecules [10]. The energy solutions for respective Mobius square and Kratzer potentials have been obtained using various approaches [7,8]. In approach and methodology aspects, various studies on Klein-Gordon equation for central and non central potentials have been reported [19].

The aim of this study is to investigate the relativistic Klein-Gordon equation with the modified Mobius squared plus Kratzer potentials using the NU method. The modified Mobius square plus Kratzer potential considered in the present work is written as [5]

$$V(r) = -V_o\left(\frac{A+Be^{-2\alpha r}}{1-e^{-2\alpha r}}\right)^2 + \frac{V_1 e^{-2\alpha r}}{1-e^{-2\alpha r}} , \quad (1)$$

where V_0, A, B, V_1 and α , represent the potential depth, potential range, bond length, coupling parameter, and screening parameter, respectively.

2 The Nikiforov-Uvarov (NU) Method in the parametric form

The NU method [12] has been successfully applied to solve both relativistic and non-relativistic wave equations of quantum mechanics. This method is easier to use as it is less complicated than most methods. The parametric form of this method is vital tool to solve parametric second order differential equations [16]. Here, we consider second order differential equation of the form [12, 15]

$$\frac{d^2\psi(s)}{ds^2}+\frac{\alpha_1-\alpha_2 s}{s(1-\alpha_3 s)}\frac{d\psi(s)}{ds}+\frac{-\xi_1 s^2+\xi_2 s-\xi_3}{s^2(1-\alpha_3 s)^2}\psi(s)=0 \quad . \tag{2}$$

In line with parametric NU method, we express the energy equation and eigenfunction as

$$(\alpha_2-\alpha_3)n+\alpha_3 n^2-(2n+1)\alpha_5+(2n+1)\left(\sqrt{\alpha_9}+\sqrt{\alpha_8}\right)+\alpha_7+2\alpha_3\alpha_2+2\sqrt{\alpha_8\alpha_9}=0 \tag{3}$$

and

$$\psi(s)=N_{nl}s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}P_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}(1-2\alpha_3 s). \tag{4}$$

where,

$$\left.\begin{aligned}
&\alpha_4=\frac{1}{2}(1-\alpha_1),\alpha_5=\frac{1}{2}(\alpha_2-2\alpha_3),\alpha_6=\alpha_5^2+\xi_1,\alpha_7=2\alpha_4\alpha_5-\xi_2,\\
&\alpha_8=\alpha_4^2+\xi_3,\alpha_9=\alpha_3\alpha_7+\alpha_3^2\alpha_8+\alpha_6,\alpha_{10}=\alpha_1+2\alpha_4+2\sqrt{\alpha_8},\\
&\alpha_{11}=\alpha_2-2\alpha_5+2\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right),\alpha_{12}=\alpha_4+\sqrt{\alpha_8},\alpha_{13}=\alpha_5-(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8})
\end{aligned}\right\}, \tag{5}$$

are the parametric constants and P_n is the orthogonal Jacobi polynomial.

3 Radial part of Klein-Gordon equation

The radial Klein-Gordon equation with equal scalar and vector potentials in three dimensions is given by

$$\frac{d^2R(r)}{dr^2}+\left[\left(E^2-m^2\right)-2(m+E)V(r)-\frac{l(l+1)}{r^2}\right]. \tag{6}$$

However, following Alhaidari et al's scheme [2], the potential in the above equation is considered to be in non relativistic limit, i. e., $V\to\frac{V}{2}$, the Eq. (6) becomes

$$\frac{d^2R(r)}{dr^2}+\left[\left(E^2-m^2\right)-(m+E)V(r)-\frac{l(l+1)}{r^2}\right]R(r)=0. \tag{7}$$

The radial solutions can be obtained by substituting Eq. (1) into Eq. (7) which gives

$$\frac{d^2R(r)}{dr^2}+\left[\left(E^2-m^2\right)-(m+E)\left\{-V_0\left(\frac{A+Be^{-2\alpha r}}{1-e^{-2\alpha r}}\right)^2+\frac{V_1e^{-2\alpha r}}{1-e^{-2\alpha r}}\right\}-\frac{l(l+1)}{r^2}\right]R(r)=0. \tag{8}$$

A good approximation for centrifugal is given as [4] (see Fig. 1).

$$\frac{1}{r^2}\approx\frac{4\alpha^2e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}. \tag{9}$$

The above approximation is valid for $\alpha << 1$. By substituting Eq. (9) into Eq. (8) we obtain

$$\frac{d^2R(r)}{dr^2}+\left[\left(E^2-m^2\right)-(m+E)\left\{-V_0\left(\frac{A+Be^{-2\alpha r}}{1-e^{-2\alpha r}}\right)^2+\frac{V_1e^{-2\alpha r}}{1-e^{-2\alpha r}}\right\}-\frac{4\alpha^2e^{-2\alpha r}l(l+1)}{(1-e^{-2\alpha r})^2}\right]R(r)=0. \quad (10)$$

With change of variable $S=e^{-2\alpha r}$, Eq. (10) is written as

$$\frac{d^2R(s)}{ds^2}+\frac{1-s}{s(1-s)}\frac{dR(s)}{ds}+\frac{-\varepsilon^2s^2+\varphi s-\phi}{s^2(1-2)^2}R(s)=0, \quad (11)$$

where,

$$\varepsilon^2=\frac{-\left(m+E_{nl}\right)\left(E_{nl}-m+V_1+V_0B^2\right)}{4\alpha^2}, \quad (12)$$

$$\varphi=\frac{\left(m+E_{nl}\right)\left(2V_0AB-2\left(E_{nl}-m\right)-V_1\right)}{4\alpha^2}-l(l+1), \quad (13)$$

$$\phi=\frac{-\left(m+E_{nl}\right)\left(E_{nl}-m+V_0A^2\right)}{4\alpha^2}. \quad (14)$$

Comparing Eq. (11) with Eq. (2) gives the parameters.

$$\left.\begin{array}{l}\alpha_1=\alpha_2=\alpha_3=1,\alpha_4=0,\alpha_5=-\frac{1}{2},\ \alpha_6=\frac{1}{4}+\varepsilon^2,\ \alpha_7=-\varphi,\alpha_8=\phi,\alpha_9=-\varphi+\phi+\frac{1}{4},\\ \alpha_{10}=1+2\sqrt{\phi},\alpha_{11}=2\left(1+\sqrt{\phi}\right)+\sqrt{(2l+1)^2-\frac{V_0(m+E_{nl})(A+B)^2}{\alpha^2}},\ \alpha_{12}=\sqrt{\phi},\\ \alpha_{13}=-\frac{1}{2}\left(1+\sqrt{(2l+1)^2-\frac{V_0(m+E_{nl})(A+B)^2}{\alpha^2}}\right)-\sqrt{\phi}\end{array}\right\}. \quad (15)$$

Substituting $\alpha_i\,(i=1,2....9)$ in Eq. (15) into Eq. (3) gives the relativistic energy eigenvalues of Klein-Gordon equation for modified Mobius square plus Kratzer potential as

$$\left(E_{nl}-m\right)=$$
$$-V_oA^2-\frac{4\alpha^2}{\left(m+E_{nl}\right)}\left[\frac{\frac{\left(m+E_{nl}\right)\left(V_1-2V_0A(B+A)\right)}{4\alpha^2}+\frac{1}{2}+n(n+1)+l(l+1)+\left(n+\frac{1}{2}\right)\sqrt{(2l+1)^2-\frac{V_o(m+E_{nl})(A+B)^2}{\alpha^2}}}{1+2n+\sqrt{(2l+1)^2-\frac{V_o(m+E_{nl})(A+B)^2}{\alpha^2}}}\right]^2. \quad (16)$$

With Eq. (4), we express the radial wave function as

$$R(s)=N_{nl}s^{\sqrt{\phi}}(1-s)^{\delta}P_n^{\left(2\sqrt{\phi},2\delta-1\right)}\left(1-2s\right). \quad (17)$$

Where,

$$\delta = \frac{1}{2} + \frac{1}{2}\sqrt{(2l+1)^2 - \frac{V_o(m+E_{nl})(A+B)^2}{\alpha^2}} \ . \quad (18)$$

4 Discussions

In Table 1, we have computed numerical results showing energy variation for various values. We also plot the graph of non relativistic energies with respect to the screening parameters for modified Mobius square plus Kratzer potential, Kratzer potential, and modified Mobius square potential, as shown in Figs 2- 4, respectively. It is observed that the energies of the modified Mobius squared plus Kratzer and modified Mobius squared potentials tend to lower values as the screening parameter increases showing an inversely proportional relationship between the energies of both potentials and the screening parameter. On the other hand, the energy of the Kratzer potential increases as the screening parameter increases. Thus, the energy of the Kratzer potential is directly proportional to the screening parameter. A clear observation shows that the ground state of the modified Mobius squared potential is considerably higher than that of the modified Mobius squared plus Kratzer and Kratzer potentials. This implies that particles will be seen to vibrate more in the modified Mobius square potential.

4. 1 Non-relativistic limit for modified Mobius square plus Kratzer potential

Here, the non relativistic energy limit of the energy is considered. With the transformation $E_{nl} + M = \frac{2\mu}{\hbar^2}$,

and $E_{nl} - M = E_{nl}$, the relativistic energy of Klein-Gordon equation reduces to

$$E_{nl} = -V_o A^2 - \frac{2\alpha^2\hbar^2}{\mu}\left[\frac{\frac{\mu(V_1 - 2V_o A(B+A))}{2\hbar^2\alpha^2} + \frac{1}{2} + n(n+1) + l(l+1) + \left(n+\frac{1}{2}\right)\sqrt{(2l+1)^2 - \frac{2\mu V_o(A+B)^2}{\hbar^2\alpha^2}}}{1 + 2n + \sqrt{(2l+1)^2 - \frac{2\mu V_o(A+B)^2}{\hbar^2\alpha^2}}}\right]^2 . \quad (19)$$

4.2. Modified Mobius square potential

By setting $V_1 = 0$, Eq. (1) reduces to the modified Mobius square potential

$$V(r) = -V_O\left(\frac{A + Be^{-2\alpha r}}{1 - e^{-2\alpha r}}\right)^2 . \quad (20)$$

With corresponding eigenvalue as

$$E_{nl} = -V_o A^2 - \frac{2\alpha^2\hbar^2}{\mu}\left[\frac{-\frac{\mu V_o A(B+A)}{\hbar^2\alpha^2} + \frac{1}{2} + n(n+1) + l(l+1) + \left(n+\frac{1}{2}\right)\sqrt{(2l+1)^2 - \frac{2\mu V_o (A+B)^2}{\hbar^2\alpha^2}}}{1+2n+\sqrt{(2l+1)^2 - \frac{2\mu V_o (A+B)^2}{\hbar^2\alpha^2}}}\right]^2 \quad (21)$$

5.3 Kratzer potential

If $V_0 = 0$, Eq. (1) reduces to the Kratzer potential as :

$$V(r) = \frac{V_1 e^{-2\alpha r}}{1-e^{-2\alpha r}}. \quad (22)$$

And the corresponding energy expression becomes

$$E_{nl} = -\frac{2\alpha^2\hbar^2}{\mu}\left[\frac{\frac{\mu V_1}{2\hbar^2\alpha^2} + (n+l+1)^2}{2(n+l+1)}\right]^2. \quad (23)$$

The above result is consistent with Eq. (19) in Refs. [5].

5 Conclusions

In this paper, we have obtained the approximate solution of Klein-Gordon equation with modified Mobius square plus Kratzer potential using the NU method. With a good approximation to the centrifugal term, the energy eigenvalues and the corresponding wave functions are obtained. We have also investigated the non-relativistic limit and special cases of modified Mobius square plus Kratzer potential. Numerical results of the energy spectrum are reported. Our results show that the energy decreases with the screening parameter. This study could find applications in atomic and molecular physics and also in chemical physics.

Declarations

The authors declare that there is no conflict of interest.

Acknowledgement

The authors sincerely appreciate the reviewer for all the useful comments which have greatly improved the manuscript.

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