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Approximation of inverse source problem for time fractional pseudo-parabolic equation in L^p

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Abstract: In this work, we focus on the final value problem of an inverse problem for the pseudo-parabolic equation. This study aims to provide a regularization method for this equation, once the measurement data are obtained at the final time in $L^r(0,\pi)$. We obtain an approximated solution using the Fourier method and the final input data $L^r(0,\pi)$ for $r \neq 2$. Using embedding between $L^r(0,\pi)$ and Hilbert scales $\mathcal{H}^\rho(0,\pi)$, this study is the error between the exact and regularized solutions to be estimated in $L^r(0,\pi)$.

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1 Introduction

In this paper, we consider the fractional diffusion equation as follows

$$\begin{cases} \partial_t^\alpha w(x,t) - \Delta w(x,t) - b\partial_t^\alpha \Delta w(x,t) = \theta(t) f(x), & 0 \le x \le \pi, 0 \le t \le T, \\ w(0,t) = w(\pi,t) = 0 & 0 \le t \le T, \\ \int_0^1 w(x,t)dt = \rho(x), & 0 \le x \le \pi, \end{cases} \tag{1.1}$$

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where $b > 0$. Here $(0, \pi) \in \mathbb{R}$. The symbol $\partial_t^\alpha u(x,t)$ is the Caputo derivative

$$\partial_t^\alpha w(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{w_s(x,s)}{(t-s)^\alpha} d\tau, \quad 0 < \alpha < 1, \tag{1.2}$$

and $\Gamma(.)$ denotes the standard Gamma function. Here, the $\theta(t)$ and $\rho(x)$ are given, and $f(x)$ is unknown. It is known that the inverse source problem mentioned above is ill-posed in general, i.e., a solution does not always exist, and in the case of the existence of a solution, it does not depend continuously on the given data. In fact, $(\rho, \theta) \in (L^r(0,\pi), L^r(0,T))$ represented the exact data and $(\rho_\epsilon, \theta_\epsilon) \in (L^r(0,\pi), L^r(0,T))$ represents the observation data with noise level $\epsilon > 0$ such that

$$\|\rho - \rho_\epsilon\|_{L^r(0,\pi)} + \|\theta - \theta_\epsilon\|_{L^r(0,T)} \le \epsilon. \tag{1.3}$$

In this work, we used the condition $\int\limits_0^1 u(x,t)dt = \rho(x)$, instead of $u(x,T) = \rho(x)$. Some of the applied studies, interested readers can find out in the following references [12, 6, 3, 5, 7, 8, 10, 2, 16].

Regarding the ill-posed problems, we can list some of the research works as follows: In [1], we study the backward problem for the LotkaVolterra system to determine the population density of species at preceding times. The problem is ill-posed in the sense that if the solution exists it does not depend continuously on the given data. We propose two stable regularization methods to regularize the system. In [13], the authors considered an inverse problem to determine a source term in the parabolic equation. In general, this problem is non-well posed. Therefore the Tikhonov regularization method with a priori and a posteriori parameter choice rule strategies is proposed to solve the problem. In [14], The main purpose of this article is to present a new method to regularize the initial inverse heat problem with an inhomogeneous source. There are many regularization methods with error estimators of logarithmic order, but an improved regularization method proposed with the error estimates of the Hlder type is obtained. Regarding the methods of regularizations, In [19, 11], with the Fractional Landweber method, in [4, 15], with the Tikhonov method, and in [17], with the quasi boundary value method, the authors study the problem of finding the source function for diffusion equations with non-integer derivatives, with the estimation between the exact solution and the regularized solution in L^2. However, for evaluation cases in $L^r(0,\pi)$ spaces, the Parseval's equality was not usable; therefore, we applied the embedding between $L^r(0,\pi)$ and Hilbert scales spaces $\mathcal{H}^\rho(0,\pi)$ to overcome this limitation; with ρ is a number, and the Lemma 2.5 will be used throughout this article. The manuscript is proceeding as follows:

- The first part deals introduce the mild solution of Problem (1.1) with the observed data $\rho_\epsilon, \theta_\epsilon \in L^r(0,\pi) \times L^r(0,T)$. Next, applying the Fourier truncation method, the error between the regular and exact solution in the $L^{\frac{2N}{N-4k}}(0,\pi)$, whereby N is the number of dimensions of the space and $k \in \mathbb{R}$.

- The second part deals the convergent rate between the regularized solution and the exact solution through the estimation of $\left\| f_\epsilon^{\mathcal{A}_\epsilon} - f \right\|_{L^{\frac{2N}{N-4k}}(0,\pi)}$.

This paper is organized as follows. In Section 2, some preliminaries such as definition and Lemmas are given. Section 3 introduces the solution, and in Section 4, we provided the regularization solution of inverse source problem and the error estimation with observed data in $L^q(\Omega)$.

2 Preliminary

Definition 2.1 (see [9]). *For $\alpha > 0$ and an arbitrary constant $\beta \in \mathbb{R}$, the Mittag-Leffler function is defined by*

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad z \in \mathbb{C}, \tag{2.1}$$

where Γ is the usual Gamma function.

Lemma 2.2. *[11] For $0 < \alpha_1 < \alpha_2 < 1$ and $\alpha \in [\alpha_1, \alpha_2]$, there exist positive constants $\mathcal{M}_1$ and $\mathcal{M}_2$, depending only on α such that*

1. $E_{\alpha,1}(-z) > 0, \quad$ *for* $z > 0$.
2. $\dfrac{\mathcal{M}_1}{1+z} \le E_{\alpha,1}(-z) \le \dfrac{\mathcal{M}_2}{1+z}$, *for* $z > 0$.

Lemma 2.3. *[11] For $\alpha > 0$, and $j \in \mathbb{N}^*$, we have:*

$$\frac{d}{dt}\big(tE_{\alpha,2}(-j^2 t^\alpha)\big) = E_{\alpha,1}(-j^2 t^\alpha), \quad \frac{d}{dt}\big(E_{\alpha,1}(-j^2 t^\alpha)\big) = -j^2 t^{\alpha-1} E_{\alpha,\alpha}(-j^2 t^\alpha). \tag{2.2}$$

Lemma 2.4. *For any $\alpha \in (0,1)$, and $b > 0$ then the following estimate hold*

$$\frac{1}{j^2}\Big(1 - E_{\alpha,1}\big(-\frac{1}{1+b}\big)\Big) \le \int_0^1 \frac{(1-s)^{\alpha-1}}{1+bj^2} E_{\alpha,\alpha}\Big(\frac{-j^2(1-s)^\alpha}{1+bj^2}\Big) ds \le \frac{1}{j^2}. \tag{2.3}$$

Proof. See in [11], page 4, the Lemma 2.7. □

Lemma 2.5. *(See [18]) Character $\hookrightarrow$ representing the embedding between two spaces A and B.*
The following statement are true:

$$\left.\begin{aligned} L^p(\mathcal{D}) &\hookrightarrow \mathcal{H}^\mu(\mathcal{D}), \quad \text{if} \quad -\frac{N}{4} < \mu \le 0, \quad p \ge \frac{2N}{N-4\mu}, \\ \mathcal{H}^\mu(\mathcal{D}) &\hookrightarrow L^p(\mathcal{D}), \quad \text{if} \quad 0 \le \mu < \frac{N}{4}, \quad p \le \frac{2N}{N-4\mu}. \end{aligned}\right\} \tag{2.4}$$

3 The solution, the ill-posed and the conditional stability of problem (1.1)

Theorem 3.1. *The solution of problem* (1.1) *represented by the formula* (3.4)*.*

Proof. If the inner product on $L^2(0,\pi)$ is denoted by $\sqrt{\frac{2}{\pi}}\Big(\int\limits_0^\pi \nu(x)\sin(jx)dx\Big)$, then the Fourier series of a function u in $L^2(0,\pi)$ with $e_j = \sqrt{\frac{2}{\pi}}\sin(jx)$, $\lambda_j = j^2$ can be formulated as

$$u(x,t) = \frac{2}{\pi}\sum_{j=1}^{\infty}\Big(\int\limits_0^\pi u(x,t)\sin(jx)dx\Big)\sin(jx).$$

Using the Laplace transform method, we can find the formula of the solution to the first equation of (1.1) as follows

$$\begin{aligned} u_j(t) = {} & \sqrt{\frac{2}{\pi}}\sum_{j=1}^{\infty} E_{\alpha,1}\Big(-\frac{j^2t^\alpha}{1+bj^2}\Big)\Big(\int\limits_0^\pi u_0(x)\sin(jx)dx\Big) \\ & + \sqrt{\frac{2}{\pi}}\sum_{j=1}^{\infty}\Big(\int_0^t \frac{(t-s)^{\alpha-1}}{1+bj^2}E_{\alpha,\alpha}\Big(\frac{-j^2(t-s)^\alpha}{1+bj^2}\Big)\theta(s)\Big(\int\limits_0^\pi f(x)\sin(jx)dx\Big)ds\Big). \end{aligned} \tag{3.1}$$

Since the fact that $u(x,0)=0$ and $u(x,1)=\rho(x)$, we follow from (3.1) that

$$\int\limits_0^\pi \rho(x)\sin(jx)dx = \int\limits_0^1 \frac{(1-s)^{\alpha-1}}{1+bj^2}E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2}\Big)\theta(s)\Big(\int\limits_0^\pi f(x)\sin(jx)dx\Big)ds\cdot \tag{3.2}$$

Therefore, the formula of the mild solution to the problem (1.1) can be given by

$$\int_0^\pi f(x)\sin(jx)dx = \Big[\int_0^1 \frac{(1-s)^{\alpha-1}}{1+bj^2} E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2}\Big)\theta(s)ds\Big]^{-1}\Big(\int_0^\pi \rho(x)\sin(jx)dx\Big). \tag{3.3}$$

Therefore, from (3.3), through some basic calculation steps, we have

$$f(x) = \sqrt{\frac{2}{\pi}}\sum_{j=1}^\infty \Big[\int_0^1 \frac{(1-s)^{\alpha-1}}{1+bj^2} E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2}\Big)\theta(s)ds\Big]^{-1}\Big(\int_0^\pi \rho(x)\sin(jx)dx\Big)\sin(jx). \tag{3.4}$$

□

4 Regularization of inverse source problem

In this section, we provide a regularization result concerning on the observed data in L^r spaces. Fourier truncation method is applied to establish approximate solution.

Theorem 4.1. *Let us take $(\theta_\epsilon, \rho_\varepsilon) \in L^r(0,T)\times L^r(0,\pi)$ such that $\theta_\epsilon(t) > \theta_0 > 0$ for any $0 \le t \le 1$ for any $\frac{1}{\alpha} < r < 2$ and*

$$\|\theta_\epsilon - \theta\|_{L^r(0,T)} + \|\rho_\epsilon - \rho\|_{L^r(0,\pi)} \le \epsilon. \tag{4.1}$$

Let us assume that $f \in \mathcal{H}^{n+k}(0,\pi)$ for $k > 0$ and $0 < n < \frac{N}{4}$. We construct a regularized solution as follows

$$f_\epsilon^{\mathcal{A}_\epsilon}(x) = \sqrt{\frac{2}{\pi}}\sum_{j\le\mathcal{A}_\epsilon} \Big[\int_0^1 \frac{(1-s)^{\alpha-1}}{1+bj^2} E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2}\Big)\theta_\epsilon(s)ds\Big]^{-1}\Big(\int_0^\pi \rho_\epsilon(x)\sin(jx)dx\Big)\sin(jx). \tag{4.2}$$

Then the error estimate as follow

$$\begin{aligned}\big\|f_\epsilon^{\mathcal{A}_\epsilon} - f\big\|_{L^{\frac{2N}{N-4k}}(0,\pi)} &\lesssim \sqrt{\frac{2}{\pi}}\mathcal{M}_4(\mathcal{M}_3,b,\theta_1,r,\alpha)\|f\|_{\mathcal{H}^k(0,\pi)}\left|\mathcal{A}_\epsilon\right|^2\epsilon \\ &\quad + \sqrt{\frac{2}{\pi}}\left|\mathcal{A}_\epsilon\right|^{-2n}\big\|f\big\|_{\mathcal{H}^{n+k}(0,\pi)} + \sqrt{\frac{2}{\pi}}\mathcal{M}_5\left|\mathcal{A}_\epsilon\right|^{2k+2+\frac{N}{r}-\frac{N}{2}}\epsilon. \end{aligned}\tag{4.3}$$

By choosing $\mathcal{A}_\epsilon$ satisfies that

$$\lim_{\epsilon\to0}\left|\mathcal{A}_\epsilon\right|^2\epsilon = \lim_{\epsilon\to0}\Big(\big(\mathcal{A}_\epsilon\big)^{2k+2+\frac{N}{r}-\frac{N}{2}}\epsilon\Big) = 0, \quad \lim_{\epsilon\to0}\mathcal{A}_\epsilon = +\infty. \tag{4.4}$$

Remark 4.2. $\mathcal{A}_\epsilon$ *is chosen as follows:*

$$\mathcal{A}_\epsilon = \epsilon^{\overline{\frac{h-1}{2k+2+\frac{N}{r}-\frac{N}{2}}}}, \quad 0 < h < 1.$$

Proof. In view of triangle inequality, we find that

$$\left\| f_\epsilon^{\mathcal{A}_\epsilon} - f \right\|_{\mathcal{H}^k(0,\pi)} \leq \left\| \mathcal{F}_2^{\mathcal{A}_\epsilon} - f \right\|_{\mathcal{H}^k(0,\pi)} + \left\| \mathcal{F}_2^{\mathcal{A}_\epsilon} - \mathcal{F}_1^{\mathcal{A}_\epsilon} \right\|_{\mathcal{H}^k(0,\pi)} + \left\| \mathcal{F}_1^{\mathcal{A}_\epsilon} - f_\epsilon^{\mathcal{A}_\epsilon} \right\|_{\mathcal{H}^k(0,\pi)}, \tag{4.5}$$

where we denote some following functions

$$\mathcal{F}_1^{\mathcal{A}_\epsilon}(x) = \sqrt{\frac{2}{\pi}} \sum_{j \leq \mathcal{A}_\epsilon} \Big[\int_0^1 \frac{(1-s)^{\alpha-1}}{1+bj^2} E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2} \Big) \theta_\epsilon(s) ds \Big]^{-1} \times \Big(\int_0^\pi \rho(x) \sin(jx) dx \Big) \sin(jx), \tag{4.6}$$

and

$$\mathcal{F}_2^{\mathcal{A}_\epsilon}(x) = \sqrt{\frac{2}{\pi}} \sum_{j \leq \mathcal{A}_\epsilon} \Big[\int_0^1 \frac{(1-s)^{\alpha-1}}{1+bj^2} E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2} \Big) \theta(s) ds \Big]^{-1} \times \Big(\int_0^\pi \rho(x) \sin(jx) dx \Big) \sin(jx). \tag{4.7}$$

Next, we considered the upper bound of the expressions on the right of (4.5). For convenience, we consider the following step.
<u>Step 1.</u> *Estimate of* $\left\| \mathcal{F}_2^{\mathcal{A}_\epsilon} - f \right\|_{\mathcal{H}^k(0,\pi)}$. We know that

$$\begin{aligned} \mathcal{F}_2^{\mathcal{A}_\epsilon} - f &= \sqrt{\frac{2}{\pi}} \sum_{j > \mathcal{A}_\epsilon} \Big[\int_0^1 \frac{(1-s)^{\alpha-1}}{1+bj^2} E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2} \Big) \theta(s) ds \Big]^{-1} \\ &\quad \times \Big(\int_0^\pi \rho(x) \sin(jx) dx \Big) \sin(jx) \\ &= \sqrt{\frac{2}{\pi}} \sum_{j > \mathcal{A}_\epsilon} \Big(\int_0^\pi f(x) \sin(jx) dx \Big) \sin(jx). \end{aligned} \tag{4.8}$$

Throught the Parseval equality, the norm on $\mathcal{H}^k(0,\pi)$ is calculated as follow

$$\begin{aligned}\Big\|\mathcal{F}_2^{\mathcal{A}_\epsilon}-f\Big\|^2_{\mathcal{H}^k(0,\pi)} &= \frac{2}{\pi}\sum_{j>\mathcal{A}_\epsilon} j^{4k}\Big(\int\limits_0^\pi f(x)\sin(jx)dx\Big)^2 \\ &= \frac{2}{\pi}\sum_{j>\mathcal{A}_\epsilon} j^{-4n}j^{4n+4k}\Big(\int\limits_0^\pi f(x)\sin(jx)dx\Big)^2.\end{aligned}$$

It is obvious to see that $j^{-4n}\le|\mathcal{A}_\epsilon|^{-4n}$ if $j>\mathcal{A}_\epsilon$ and $n>0$. Therefore, we get that

$$\begin{aligned}\Big\|\mathcal{F}_2^{\mathcal{A}_\epsilon}-f\Big\|^2_{\mathcal{H}^k(0,\pi)} &\le \frac{2}{\pi}\big|\mathcal{A}_\epsilon\big|^{-4n}\sum_{j>\mathcal{A}_\epsilon} j^{4n+4k}\Big(\int\limits_0^\pi f(x)\sin(jx)(x)dx\Big)^2 \\ &= \frac{2}{\pi}\big|\mathcal{A}_\epsilon\big|^{-4n}\big\|f\big\|^2_{\mathcal{H}^{n+k}(0,\pi)},\end{aligned}\tag{4.9}$$

it gives that

$$\big\|\mathcal{F}_2^{\mathcal{A}_\epsilon}-f\big\|_{\mathcal{H}^{n+k}(0,pi)}\le\sqrt{\frac{2}{\pi}}\;|\mathcal{A}_\epsilon|^{-2n}\big\|f\big\|_{\mathcal{H}^{n+k}(0,\pi)}.\tag{4.10}$$

<u>Step 2.</u> *Estimate of* $\Big\|\mathcal{F}_1^{\mathcal{A}_\epsilon}-\mathcal{F}_2^{\mathcal{A}_\epsilon}\Big\|_{\mathcal{H}^k(0,\pi)}$.

Based on two formulas (4.6) and (4.7), we have the difference of two functions $\mathcal{F}_1^{\mathcal{A}_\epsilon}$ and $\mathcal{F}_2^{\mathcal{A}_\epsilon}$ as follows

$$\begin{aligned}&\mathcal{F}_1^{\mathcal{A}_\epsilon}(x)-\mathcal{F}_2^{\mathcal{A}_\epsilon}(x)\\ &=\sqrt{\frac{2}{\pi}}\sum_{j\le\mathcal{A}_\epsilon}\frac{\displaystyle\int_0^1\frac{(1-s)^{\alpha-1}}{1+bj^2}E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2}\Big)\Big(\theta_\epsilon(s)-\theta(s)\Big)ds}{\displaystyle\int_0^1\frac{(1-s)^{\alpha-1}}{1+bj^2}E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2}\Big)\theta_\epsilon(s)ds}\\ &\qquad\times\Big[\int_0^1\frac{(1-s)^{\alpha-1}}{1+bj^2}E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2}\Big)\theta(s)ds\Big]^{-1}\Big(\int\limits_0^1\rho(x)\sin(jx)dx\Big)\;\sin(jx).\end{aligned}\tag{4.11}$$

We follows from (4.11) that

$$\mathcal{F}_1^{\mathcal{A}_\epsilon}(x) - \mathcal{F}_2^{\mathcal{A}_\epsilon}(x) = \sqrt{\frac{2}{\pi}} \sum_{j \le \mathcal{A}_\epsilon} \frac{\int_0^1 \frac{(1-s)^{\alpha-1}}{1+bj^2} E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2}\Big)\Big(\theta_\epsilon(s)-\theta(s)\Big)ds}{\int_0^1 \frac{(1-s)^{\alpha-1}}{1+bj^2} E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2}\Big)\theta_\epsilon(s)ds} \times \Big(\int_0^\pi f(x)\sin(jx)dx\Big)\sin(jx). \tag{4.12}$$

From (4.12), we provide that

$$\Big\|\mathcal{F}_1^{\mathcal{A}_\epsilon} - \mathcal{F}_2^{\mathcal{A}_\epsilon}\Big\|^2_{\mathcal{H}^k(0,\pi)} = \frac{2}{\pi} \sum_{j \le \mathcal{A}_\epsilon} \left[\frac{\int_0^1 \frac{(1-s)^{\alpha-1}}{1+bj^2} E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2}\Big)\Big(\theta_\epsilon(s)-\theta(s)\Big)ds}{\int_0^1 \frac{(1-s)^{\alpha-1}}{1+bj^2} E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2}\Big)\theta_\epsilon(s)ds}\right]^2 \times j^{4k}\Big(\int_0^\pi \rho(x)\sin(jx)dx\Big)^2. \tag{4.13}$$

Using Hölder inequality, we have that the upper bound

$$\begin{aligned} &\Big|\int_0^1 \frac{(1-s)^{\alpha-1}}{1+bj^2} E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2}\Big)\big(\theta_\epsilon(s)-\theta(s)\big)ds\Big| \\ &\le \Big(\int_0^1 \Big|\theta_\epsilon(s)-\theta(s)\Big|^r ds\Big)^{\frac{1}{r}} \left(\int_0^1 \frac{(1-s)^{r^*(\alpha-1)}}{(1+bj^2)^{r^*}}\Big|E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2}\Big)\Big|^{r^*} ds\right)^{\frac{1}{r^*}}, \end{aligned} \tag{4.14}$$

where $r^* = 1 + \frac{1}{r-1}$. It is obvious to provide the following statement

$$\Big(\int_0^1 \big|\theta_\epsilon(s)-\theta(s)\big|^r dr\Big)^{\frac{1}{r}} = \big\|\theta_\epsilon - \theta\big\|_{L^r(0,1)}, \tag{4.15}$$

and using the Lemma 3.3, one has

$$\begin{aligned} \int_0^1 \frac{(1-s)^{r^*(\alpha-1)}}{(1+bj^2)^{r^*}}\Big|E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2}\Big)\Big|^{r^*} ds &\le \mathcal{M}_3^{r^*}\int_0^1 \frac{(1-s)^{r^*(\alpha-1)}}{\big(1+bj^2\big)^{r^*}}ds \\ &\le \frac{\mathcal{M}_3^{r^*}}{s^*(\alpha-1)+1} = \mathcal{M}_3^{r^*}\frac{r-1}{\alpha-1}\cdot \end{aligned} \tag{4.16}$$

where we note that $r > \frac{1}{\alpha}$, combining three evaluations (4.14), (4.15) and (4.16), we derive that the following estimate

$$\Big| \int_0^1 \frac{(1-s)^{\alpha-1}}{1+bj^2} E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2}\Big)\big(\theta_\epsilon(s)-\theta(s)\big)ds\Big| \le \mathcal{M}_3\Big(\frac{r-1}{r\alpha-1}\Big)^{\frac{r-1}{r}} \big\|\theta_\epsilon-\theta\big\|_{L^r(0,1)}. \tag{4.17}$$

Next, let assume that θ_ϵ by a positive constant θ_1, we have immediately

$$\begin{aligned}\int_0^1 \frac{(1-s)^{\alpha-1}}{1+bj^2} E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2}\Big)\theta_\epsilon(s)ds &\ge \theta_1 \int_0^1 \frac{(1-s)^{\alpha-1}}{1+bj^2} E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2}\Big)ds \\ &\le \frac{\theta_1}{j^2}\Big(1-E_{\alpha,1}\big(-\frac{1}{1+b}\big)\Big).\end{aligned} \tag{4.18}$$

From the two closest observations, we assert that

$$\begin{aligned}&\frac{\displaystyle\int_0^1 \frac{(1-s)^{\alpha-1}}{1+bj^2} E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2}\Big)\Big(\theta_\epsilon(s)-\theta(s)\Big)ds}{\displaystyle\int_0^1 \frac{(1-s)^{\alpha-1}}{1+bj^2} E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2}\Big)\theta_\epsilon(s)ds} \\ &\le \big\|\theta_\epsilon-\theta\big\|_{L^r(0,1)} j^2 \ \mathcal{M}_3\Big(\frac{r-1}{r\alpha-1}\Big)^{\frac{r-1}{r}}\bigg(\theta_1\Big(1-E_{\alpha,1}(-\frac{1}{1+b})\Big)\bigg)^{-1},\end{aligned} \tag{4.19}$$

where we denote

$$\mathcal{M}_4(\mathcal{M}_3, b, \theta_1, r, \alpha) = \ \mathcal{M}_3\Big(\frac{r-1}{r\alpha-1}\Big)^{\frac{r-1}{r}} \Big[\theta_1\Big(1-E_{\alpha,1}(-\frac{1}{1+b})\Big)\Big]^{-1}.$$

Combining (4.13) and (4.19), we find that

$$\Big\|\mathcal{F}_1^{\mathcal{A}_\epsilon}-\mathcal{F}_2^{\mathcal{A}_\epsilon}\Big\|^2_{\mathcal{H}^k(0,\pi)} \le \frac{2}{\pi}\mathcal{M}_4^2(\mathcal{M}_3, b, \theta_1, r, \alpha)\big\|\theta_\epsilon-\theta\big\|^2_{L^r(0,1)} \sum_{j\le\mathcal{A}_\epsilon} j^{4k+4}\Big(\int\limits_0^\pi f(x)\sin(jx)dx\Big)^2. \tag{4.20}$$

Noting that the finite sum $\sum_{j\le\mathcal{A}_\epsilon} j^{4k+4}\Big(\int\limits_{\mathcal{D}} f(x)\sin(jx)dx\Big)^2$ is bounded by

$$|\mathcal{A}_\epsilon|^4 \sum_{j\le\mathcal{A}_\epsilon} j^{4k}\Big(\int\limits_{\mathcal{D}} f(x)\sin(jx)dx\Big)^2 \le |\mathcal{A}_\epsilon|^4\big\|f\big\|^2_{\mathcal{H}^k(0,\pi)}.$$

Therefore, we follows from (4.20) that

$$\begin{aligned}\Big\|\mathcal{F}_1^{\mathcal{A}_\epsilon}-\mathcal{F}_2^{\mathcal{A}_\epsilon}\Big\|_{\mathcal{H}^k(0,\pi)} &\le \sqrt{\frac{2}{\pi}}\mathcal{M}_4(\mathcal{M}_3,b,\theta_1,r,\alpha)\big\|\theta_\epsilon-\theta\big\|_{L^r(0,1)}\big|\mathcal{A}_\epsilon\big|^2\|f\|_{\mathcal{H}^k(0,\pi)}\\ &\le \sqrt{\frac{2}{\pi}}\mathcal{M}_4(\mathcal{M}_3,b,\theta_1,r,\alpha)\|f\|_{\mathcal{H}^k(0,\pi)}\big|\mathcal{A}_\epsilon\big|^2\epsilon. \end{aligned}\tag{4.21}$$

Here we have used (4.1).
Step 3. Estimate of $\Big\|\mathcal{F}_1^{\mathcal{A}_\epsilon}-f_\epsilon^{\mathcal{A}_\epsilon}\Big\|_{\mathcal{H}^k(0,\pi)}$.
From (4.2) and (4.6), we receive

$$\begin{aligned}\mathcal{F}_1^{\mathcal{A}_\epsilon}(x)-f_\epsilon^{\mathcal{A}_\epsilon}(x) &= \sqrt{\frac{2}{\pi}}\sum_{j\le\mathcal{A}_\epsilon}\left[\int_0^1\frac{(1-s)^{\alpha-1}}{1+bj^2}E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2}\Big)\theta_\epsilon(s)ds\right]^{-1}\\ &\quad\times\Big(\int_0^\pi\big(\rho_\epsilon(x)-\rho(x)\big)\sin(jx)dx\Big)\sin(jx). \end{aligned}\tag{4.22}$$

From (4.22), by taking the norm of both sides of the above expression in space $\mathcal{H}^k(0,\pi)$ and using Parseval' s equality, we obtain that

$$\begin{aligned}\Big\|\mathcal{F}_1^{\mathcal{A}_\epsilon}-f_\epsilon^{\mathcal{A}_\epsilon}\Big\|^2_{\mathcal{H}^k(0,\pi)} &= \frac{2}{\pi}\sum_{j\le\mathcal{A}_\epsilon}\Big[\int_0^1\frac{(1-s)^{\alpha-1}}{1+bj^2}E_{\alpha,\alpha}\Big(-\frac{j^2(1-s)^\alpha}{1+bj^2}\Big)\theta_\epsilon(s)ds\Big]^{-2}\\ &\quad\times j^{4k}\Big(\int_0^\pi\big(\rho_\epsilon(x)-\rho(x)\big)\sin(jx)dx\Big)^2. \end{aligned}\tag{4.23}$$

Thank to the inequality (4.18), we get

$$\begin{aligned}\Big\|\mathcal{F}_1^{\mathcal{A}_\epsilon}-f_\epsilon^{\mathcal{A}_\epsilon}\Big\|^2_{\mathcal{H}^k(0,\pi)} &= \frac{2}{\pi}\Big[\theta_1\Big(1-E_{\alpha,1}(-\frac{1}{1+b})\Big)\Big]^{-2}\\ &\quad\times\sum_{j\le\mathcal{A}_\epsilon}j^{4k+4}\Big(\int_0^\pi\big(\rho_\epsilon(x)-\rho(x)\big)\sin(jx)dx\Big)^2. \end{aligned}\tag{4.24}$$

We continue to deal with the finite series on the right above as follows

$$\sum_{j\le\mathcal{A}_\epsilon} j^{4k+4}\Big(\int_0^\pi \big(\rho_\epsilon(x)-\rho(x)\big)\sin(jx)(x)dx\Big)^2$$
$$= \sum_{j\le\mathcal{A}_\epsilon} j^{4k+4+\frac{2N}{r}-N} j^{\frac{Nr-2N}{r}}\Big(\int_0^\pi \big(\rho_\epsilon(x)-\rho(x)\big)\sin(jx)dx\Big)^2$$
$$\le (\mathcal{A}_\epsilon)^{4k+4+\frac{2N}{r}-N}\sum_{j\le\mathcal{A}_\epsilon} j^{\frac{Nr-2N}{s}}\Big(\int_{\mathcal{D}} \big(\rho_\epsilon(x)-\rho(x)\big)\sin(jx)dx\Big)^2$$
$$= (\mathcal{A}_\epsilon)^{4k+4+\frac{2N}{r}-N}\left\|\rho_\epsilon-\rho\right\|^2_{\mathcal{H}^{\frac{Nr-2N}{2r}}(0,\pi)}. \tag{4.25}$$

Since $1<r<2$, we know that $L^r(0,\pi)\hookrightarrow\mathcal{H}^{\frac{Nr-2N}{2r}}(0,\pi)$. Therefore, we get

$$\left\|\rho_\epsilon-\rho\right\|_{\mathcal{H}^{\frac{Nr-2N}{2r}}(0,\pi)} \le C(N,r)\left\|\rho_\epsilon-\rho\right\|_{L^r(0,\pi)} \le C(N,r)\epsilon. \tag{4.26}$$

By summarizing all three evaluations (4.24), (4.25) and (4.26), we derive that

$$\left\|\mathcal{F}_1^{\mathcal{A}_\epsilon}-f_\epsilon^{\mathcal{A}_\epsilon}\right\|_{\mathcal{H}^k(0,\pi)} \le \sqrt{\frac{2}{\pi}}\mathcal{M}_5\,(\mathcal{A}_\epsilon)^{2k+2+\frac{N}{r}-\frac{N}{2}}\,\epsilon, \tag{4.27}$$

whereby

$$\mathcal{M}_5=\bigg(\theta_1\Big(1-E_{\alpha,1}(-\frac{1}{1+b})\Big)\bigg)C(N,r).$$

From three steps, we can conclude that

$$\begin{aligned}\left\|f_\epsilon^{\mathcal{A}_\epsilon}-f\right\|_{\mathcal{H}^k(0,\pi)} &\le \left\|\mathcal{F}_2^{\mathcal{A}_\epsilon}-f\right\|_{\mathcal{H}^k(0,\pi)}+\left\|\mathcal{F}_2^{\mathcal{A}_\epsilon}-\mathcal{F}_1^{\mathcal{A}_\epsilon}\right\|_{\mathcal{H}^k(0,\pi)}+\left\|\mathcal{F}_1^{\mathcal{A}_\epsilon}-f_\epsilon^{\mathcal{A}_\epsilon}\right\|_{\mathcal{H}^k(0,\pi)}\\ &\le \sqrt{\frac{2}{\pi}}\mathcal{M}_4(\mathcal{M}_3,b,\theta_1,r,\alpha)\|f\|_{\mathcal{H}^k(0,\pi)}\left|\mathcal{A}_\epsilon\right|^2\epsilon\\ &\qquad +\sqrt{\frac{2}{\pi}}\,\left|\mathcal{A}_\epsilon\right|^{-2n}\|f\|_{\mathcal{H}^{n+k}(0,\pi)}+\sqrt{\frac{2}{\pi}}\mathcal{M}_5\left|\mathcal{A}_\epsilon\right|^{2k+2+\frac{N}{r}-\frac{N}{2}}\epsilon.\end{aligned} \tag{4.28}$$

By using Lemma 2.5, since $0<k<\frac{N}{4}$, we know that $\mathcal{H}^k(0,\pi)\hookrightarrow L^{\frac{2N}{N-4k}}(0,\pi)$, which yields to the desired result (4.3). □

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