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An extension of Lagrange interpolation formula and its applications

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Abstract: In this work, a new type of interpolation formulas is introduced. These formulas can be an extension of the Lagrange interpolation formula. The error of this new type of interpolation is calculated. In order to display efficiency of the proposed formulas, three numerical examples are presented.

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1 Introduction

Let $y_0, y_1, \dots, y_n$ be $n+1$ known values for an arbitrary function $y : \mathbb{R} \longrightarrow \mathbb{R}$, at $a = x_0 < x_1 < \cdots < x_n = b$. Then the Lagrange interpolation formula is defined as [1, 8]

$$p_n(x) = \sum_{i=0}^{n} L_i(x) y_i, \tag{1.1}$$

where

$$L_i(x) = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{x - x_k}{x_i - x_k}, \ i = 0, 1, \cdots, n. \tag{1.2}$$

Clearly, for $i = 0, 1, \cdots, n$, we have

$$p_n(x_i) = y_i. \tag{1.3}$$

Moreover, for $y \in \mathcal{C}^{n+1}[a, b]$, we have[1, 8]

$$y(x) = p_n(x) + (x - x_0)(x - x_1) \cdots (x - x_n) \frac{y^{(n+1)}(\eta_x)}{(n+1)!}, \quad \eta_x \in [a, b]. \tag{1.4}$$

Above mentioned interpolation formula is a special case of the following interpolation formula [1]

$$\Phi_n(x) = \sum_{i=0}^{n} \phi_i(x) y_i, \tag{1.5}$$

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where ϕ_i for $i = 0, 1, \cdots, n$, are defined as

$$\phi_i(x) = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{\phi(x, x_k)}{\phi(x_i, x_k)}, \tag{1.6}$$

and $\phi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function subject to

$$\phi(x, y) = 0 \Longleftrightarrow x = y. \tag{1.7}$$

Recently, some new type of interpolation formulas are introduced. For further see [2, 3, 4, 5, 6] and references therein. In this paper, a new type of interpolation formulas are introduced.

This paper is orginazed as follows. In the next section, a new type of interpolation function is introduced. The error function of this new type of interpolation formulas are also considered. In section 3, some numerical examples are presented. The results show the accuracy of the proposed new interpolation formula. Finally, conclusions are given in section 4.

2 A new type of interpolation formulas

Let $y_0, y_1, \ldots, y_n$ be $n+1$ known values for arbitrary function $y : \mathbb{R} \longrightarrow \mathbb{R}$, at $a = x_0 < x_1 < \cdots < x_n = b$. Also, let f be an arbitrary function on $[a, b]$ (i.e. $f : [a, b] \longrightarrow \mathbb{R}$) provided that $f(x - y) = 0$ if and only if $x = y$. Then the new type of interpolation formulas are defined as

$$I_n(x; f) = \sum_{i=0}^{n} \ell_i(x; f) y_i, \tag{2.1}$$

where $\ell_i(x; f)$, for $i = 0, 1, \cdots, n$ are defined as

$$\ell_i(x; f) = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{f(x - x_k)}{f(x_i - x_k)}. \tag{2.2}$$

Clearly, we have

$$\ell_i(x_k; f) = \delta_{i,k}. \tag{2.3}$$

Hence, for $i = 0, 1, 2, \cdots, n$, we have

$$I_n(x_i; f) = y_i. \tag{2.4}$$

Remark 1 In the interpolation formula (2.1), if $f(x) = x$, the Lagrange interpolation formula is obtained.

Remark 2 In the interpolation formula (2.1), if $f(x) = \sin\left(\frac{x}{2}\right)$, the trigonometric interpolation formula is obtained [7].

Theorem 2.1. *If $f \in \mathcal{C}^{(n+1)}[a, b]$ then the error function of the interpolation formula (2.1), for $y \in \mathcal{C}^{(n+1)}[a, b]$ is obtained as*

$$E(x) = y(x) - I_n(x; f) = \frac{y^{(n+1)}(\xi_x) - I_n^{(n+1)}(\xi_x; f)}{(n+1)!} \prod_{k=0}^{n} (x - x_k). \tag{2.5}$$

Proof. *Let $\Psi(x)$ is defined as*

$$\Psi(x) = y(x) - I_n(x;f) - \eta(\widehat{x}) \prod_{k=0}^{n} (x - x_k), \tag{2.6}$$

where $\eta(\widehat{x})$ is defined as we have $\Psi(\widehat{x}) = 0$ for $\widehat{x} \notin \{x_k\}_{k=0}^{n}$ and $\widehat{x} \in [a,b]$. Therefore, we have

$$\eta(\widehat{x}) = \frac{y(\widehat{x}) - I_n(\widehat{x};f)}{\prod_{k=0}^{n} (\widehat{x} - x_k)}. \tag{2.7}$$

Hence, $\Psi(x)$ has at least $n+2$ distinct roots at $[a,b]$. Using the Rolle's theorem results in that $\Psi^{(n+1)}(x)$ has at least one root at (a,b). Consequently, there exists $\xi_x \in (a,b)$ since $\Psi^{(n+1)}(\xi_x) = 0$. So we have

$$\eta(\widehat{x}) = \frac{y^{(n+1)}(\xi_x) - I_n^{(n+1)}(\xi_x;f)}{(n+1)!}. \tag{2.8}$$

Finally, by substituting the relation (2.8) in the relation (2.6), the relation (2.5) is obtained.

Remark 3 To reduce the error formula (2.5), the nodes are given by the zeros of the first Chebyshev polynomial $T_{n+1}(x)$ as [1, 7, 8]

$$x_k = a + \left(\frac{b-a}{2}\right)\left[\cos\left(\frac{(2k-1)\pi}{2(n+1)}\right) + 1\right], k = 1, 2, \cdots, n+1. \tag{2.9}$$

In the next section, some applications of the new interpolation formula, for $f(x) = x$, $\sin(x)$, $\sin\left(\frac{x}{2}\right)$, $\tan(x)$, $\sinh(x)$, $\tanh(x)$ are given.

3 Numerical Experiments

In this section, to show the efficiency of the proposed interpolation formula, introduced in the relation (2.1), some numerical examples are presented.

Example 1 (Closed type quadrature formula) As the first example, consider the following definite integral

$$I = \int_0^1 \exp(x^4)dx \approx 1.271287105. \tag{3.1}$$

To compute I, the interpolation formula (2.1) is applied. Let $x_k = \frac{k}{n}$ for $k = 0, 1, 2, \ldots, n$. Therefore, we have

$$I \approx \int_0^1 I_n(x;f)dx = \sum_{k=0}^{n} w_k y_k, \tag{3.2}$$

where $y_k = \exp\left(x_k^4\right)$, $w_k = \int_0^1 \ell_k(x;f)dx$, and $\ell_k(x;f) = \prod_{\substack{i=0 \\ k \neq i}}^{n} \frac{f(x - x_i)}{f(x_k - x_i)}$. The absolute error of the formula (3.2), for $n = 5, 10, 15$, and $f(x) = x$, $\sin(x)$, $\sin\left(\frac{x}{2}\right)$, $\tan(x)$, $\sinh(x)$, $\tanh(x)$ are given in the table 1.

Table 1. Absolute errors of the quadrature formula (3.2), for $n = 5, 10, 15$, in six different cases.

	n=5	n=10	n=15
$f(x) = x$	2.84(-3)	4.28(-6)	2.70(-8)
$f(x) = \sin(x)$	5.07(-3)	1.50(-5)	3.47(-7)
$f(x) = \sin(\frac{x}{2})$	3.29(-3)	5.97(-6)	5.0(-8)
$f(x) = \tan(x)$	1.11(-4)	8.05(-7)	5.40(-8)
$f(x) = \sinh(x)$	1.53(-3)	9.11(-7)	2.10(-9)
$f(x) = \tanh(x)$	6.47(-3)	2.39(-5)	7.97(-7)

Example 2 (Open type quadrature formula) As the second example, consider the following definite integral

$$J = \int_0^1 \exp(-x^6)dx \approx 0.8882636988. \tag{3.3}$$

Similar to the first example, to compute J, the interpolation formula (2.1) is applied. Let $x_k = \dfrac{k}{n}$ for $k = 1, 2, \ldots, n-1$. Therefore, we have

$$J \approx \int_0^1 I_n(x; f)dx = \sum_{k=1}^{n-1} w_k^{'} y_k, \tag{3.4}$$

where $y_k = \exp\left(x_k^{-6}\right)$, $w_k^{'} = \displaystyle\int_0^1 \ell_k^{'}(x; f)dx$, and $\ell_k^{'}(x; f) = \displaystyle\prod_{\substack{i=1\\k\neq i}}^{n-1} \frac{f(x - x_i)}{f(x_k - x_i)}$. The absolute error of the formula (3.4), for $n = 5, 10, 15$, and $f(x) = x$, $\sin(x)$, $\sin\left(\dfrac{x}{2}\right)$, $\tan(x)$, $\sinh(x)$, $\tanh(x)$ are given in the table 2.

Table 2. Absolute errors of the quadrature formula (3.4), for $n = 5, 10, 15$, in six different cases.

	n=5	n=10	n=15
$f(x) = x$	3.94(-3)	1.97(-4)	4.51(-6)
$f(x) = \sin(x)$	6.83(-3)	1.14(-4)	2.87(-8)
$f(x) = \sin(\frac{x}{2})$	5.02(-3)	1.84(-4)	4.02(-6)
$f(x) = \tan(x)$	1.55(-2)	1.04(-4)	2.90(-5)
$f(x) = \sinh(x)$	2.79(-3)	1.73(-4)	2.12(-6)
$f(x) = \tanh(x)$	8.20(-3)	4.27(-5)	3.57(-6)

Example 3 As the third example, consider the Runge's function [8] defined as

$$G(x) = \frac{1}{1 + 25x^2}, \; x \in [-1, 1]. \tag{3.5}$$

The error of the interpolation formula (2.1), at the zeros of Chebyshev polynomial $T_{N+1}(x)$, for $N = 20, 30$ and $f(x) = x$, $\sin(x)$, $\sin\left(\dfrac{x}{2}\right)$ are ploted in figures (1)-(6).

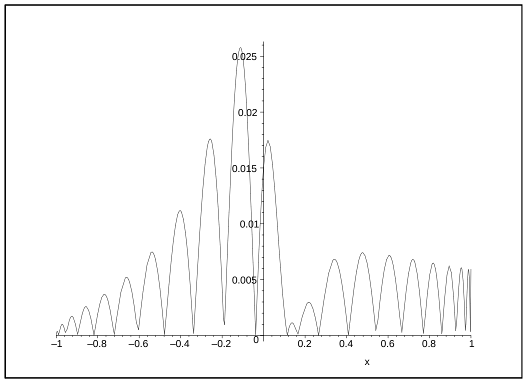

Figure 1: Error function of Runge's function with $N = 20$ and $f(x) = x$.

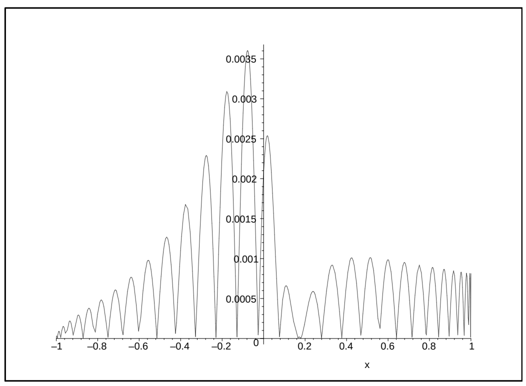

Figure 2: Error function of Runge's function with $N = 30$ and $f(x) = x$.

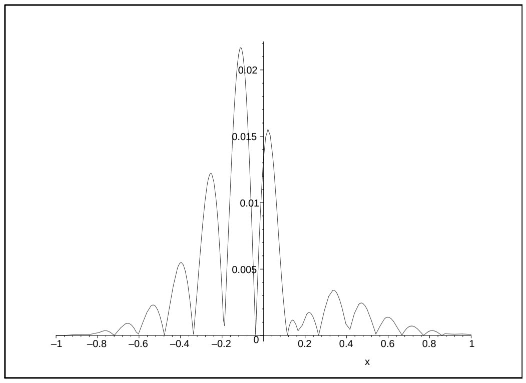

Figure 3: Error function of Runge's function with $N = 20$ and $f(x) = \sin(x)$.

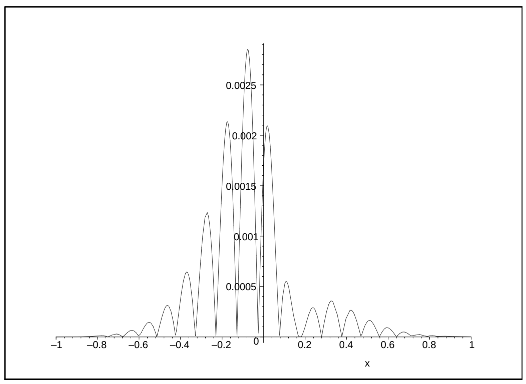

Figure 4: Error function of Runge's function with $N = 30$ and $f(x) = \sin(x)$.

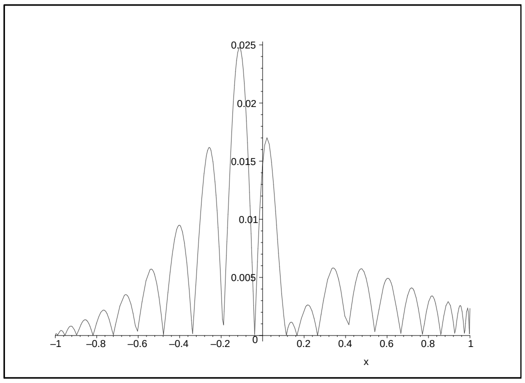

Figure 5: Error function of Runge's function with $N = 20$ and $f(x) = \sin\left(\frac{x}{2}\right)$.

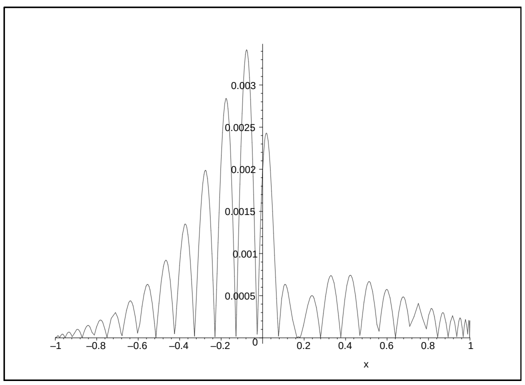

Figure 6: Error function of Runge's function with $N = 30$ and $f(x) = sin\left(\frac{x}{2}\right)$.

4 Conclusions

In this work, a new type of interpolation formulas is presented. This new type of interpolation formulas is extension of Lagrange interpolation formula. The error function for the new type of interpolation formulas is determined. Finally, some applications of this new type of interpolation formulas are given in numerical results section.

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