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On ∂−locally compact space

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Abstract: The aim of this paper is to introduce and give preliminary investigation of $\partial-$locally compact spaces. Locally compactness and $\partial-$locally compactness are independent of each other. Every locally compact Hausdorff space is $\partial-$locally compact. But the converse is not true even though it be Hausdorff. $\partial-$locally compactness is a topological property. $\partial-$locally compactness is not preserved by the product topology.

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1 Introduction

By a space, we mean a topological space. If $A \subseteq X$, the closure, interior and boundary of A is denoted by clA, $intA$ and ∂A respectively and if Y is a subspace of X and $A \subseteq Y$, the closure, interior and boundary of A in Y is denoted by $cl_Y A$, $int_Y A$ and $\partial_Y A$ respectively. A space X will be called $\partial-$locally compact at $x \in X$ if for every open set U containing x, there exists an open subset V containing x such that ∂V is compact and $V \subseteq U$. A space X is called $\partial-$locally compact if it is $\partial-$locally compact at each of points. Locally compact and $\partial-$locally compact are independent of each other (Example 2.3 and Example 2.4). We show that a locally compact Hausdorff space is $\partial-$locally compact (Lemma 2.5). Open or closed subspace of a $\partial-$locally compact space is $\partial-$locally compact (Lemma 2.7 and Lemma 2.9). We show that $\partial-$locally compactness is a topological property (Theorem 2.10). The product of two $\partial-$locally compact spaces need not to be $\partial-$locally compact (Example 2.11). We give some conditions such that the product of two $\partial-$locally compact spaces is $\partial-$locally compact(Lemma 2.12 and Theorem 2.13).

The space of real numbers with the usual topology is denoted by $\mathbb{R}$, and $\mathbb{Q}$ is the rationals with the subspace topology. Let I be a non-empty index set, and for every $i \in I$, X_i be a space. We denote by $\prod_{i\in I} X_i$, the cartesian product of X_i with the product topology. For each j, $\pi_j : \prod_{i\in I} X_i \to X_j$ is the natural projection map. For more information on topological spaces, see [1].

2 ∂−locally compact space

In this section, we introduce the concept and study some properties of $\partial-$locally compact space.

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Definition 2.1. *A space X is called $\partial-$locally compact at x if for every open set $U \subseteq X$ containing x, there exists an open set $V \subseteq X$ containing x such that ∂V is compact and $V \subseteq U$. The space X is called $\partial-$locally compact if X is $\partial-$locally compact at each of its points.*

Lemma 2.2. *Let β be a basis on X such that ∂B is compact for every $B \in \beta$. Then X is ∂-locally compact.*

Proof. Let $x \in X$ and $U \subseteq X$ be an open set containing x. Then there exists $B \in \beta$ such that $x \in B$ and $B \subseteq U$. By the assumption, ∂B is compact. So X is ∂-locally compact. □

Locally compactness and $\partial-$locally compactness are independent of each other. See the examples 2.3 and 2.4.

Example 2.3. *It is clear that $\{(a,b) \cap \mathbb{Q}; a,b \in \mathbb{R}\}$ is a basis on $\mathbb{Q}$. Also, $\partial((a,b) \cap \mathbb{Q}) = \{a,b\} \cap \mathbb{Q}$ is compact in $\mathbb{Q}$. By Lemma 2.2, $\mathbb{Q}$ is a $\partial-$locally compact space which is not locally compact.*

Example 2.4. *Let X be an infinite set and $p \in X$. We define*

$$\tau = \{\emptyset\} \bigcup \{U \subseteq X; p \in U\}$$

Then τ is a topology on X. It is clear that (X,τ) is locally compact space. Since $\partial\{p\} = X - \{p\}$ is not compact, (X,τ) is not a $\partial-$locally compact space.

Lemma 2.5. *A locally compact Hausdorff space is $\partial-$locally compact.*

Proof. Let $x \in X$ and U be an open set containing x. Since X is locally compact and Hausdorff, there exists an open set $V \subseteq X$ containing x, such that clV is compact and $V \subseteq U$. It follows that ∂V, as a closed subset of compact set clV, is compact. The proof is complete. □

Lemma 2.6. *Every compact space X is $\partial-$locally compact, even though X is not Hausdorff.*

Proof. It is clear. □

Lemma 2.7. *Let X be a $\partial-$locally compact space, and $Y \subseteq X$ an open subspace. Then also Y is $\partial-$locally compact.*

Proof. Let $y \in Y$, and U be an open set in Y containing y. Then U is open in X. Since X is $\partial-$locally compact, it follows that $V \subseteq U$ for some open set V in X containing y such that ∂V is compact. □

Remark 2.8. *Let $Y \subseteq X$ be a closed subspace, and U an open subset in X. Then*

$$\begin{aligned}\partial_Y(U \cap Y) &= cl_Y(U \cap Y) - int_Y(U \cap Y) \\ &= (clU \cap Y) - (U \cap Y) \\ &= \partial U \cap Y\end{aligned}$$

Lemma 2.9. *Let X be a $\partial-$locally compact space, and $Y \subseteq X$ a closed subspace. Then also Y is $\partial-$locally compact.*

Proof. Let $y \in Y$, and $U \subseteq Y$ be an open set containing y. There exists an open set $W \subseteq X$ such that $U = W \cap Y$. Since X is $\partial-$locally compact, there exists an open set $V \subseteq X$ containing y such that ∂V is compact and $V \subseteq W$. By Remark 2.8, $\partial_Y(V \cap Y) = \partial V \cap Y$ which is compact. So Y is $\partial-$locally compact. □

Theorem 2.10. *$\partial-$locally compactness is a topological property.*

Proof. Let X and Y be two spaces, and $f : X \to Y$ is a homeomorphism such that X is $\partial-$locally compact. Let $y \in Y$, and $V \subseteq Y$ be an open set containing y. Since f is a homeomorphism, $f(x) = y$ for some $x \in X$ and $f^{-1}(V) \subseteq X$ is an open set containing x. On the other hand, X is $\partial-$locally compact. So there exists an open set $U \subseteq X$ containing x such that $U \subseteq f^{-1}(V)$ and ∂U is compact. Since f is open, $f(U)$ is open in Y and is contained in V. Also

$$\partial f(U) = clf(U) - f(U) = f(clU) - f(U) = f(\partial U)$$

Hence $\partial f(U)$ is compact and the proof is complete. □

If X and Y be two $\partial-$locally compact spaces, $X \times Y$ need not to be $\partial-$locally compact. See the Example 2.11.

Example 2.11. *Let $N = (0,1)^2 \cap (\mathbb{Q} \times \mathbb{R})$ ($(0,1)^2 = (0,1) \times (0,1)$). We claim that the boundary of every nonempty open subset of N is not compact. Let $U \subseteq N$ be an open set. First, we show that $\pi_1(U) \subseteq \pi_1(\partial U)$. Let $x \in \pi_1(U)$. Assume to contrary, $x \notin \pi_1(\partial U)$. Then $\{x\} \times \mathbb{R} \subseteq U \subseteq N$. So $\pi_2(\{x\} \times \mathbb{R}) \subseteq \pi_2(N) = (0,1)$ which is a contradiction. Now, if ∂U is compact, $cl\pi_1(U)$ is compact in $\mathbb{Q}$ which is a contradiction (since $int\pi_1(U) \neq \emptyset$). Hence $\mathbb{Q} \times \mathbb{R}$ is not $\partial-$locally compact.*

Lemma 2.12. *Let X be a discrete space and Y, a $\partial-$locally compact space. Then $X \times Y$ is $\partial-$locally compact.*

Proof. Let $N \subseteq X \times Y$ be an open set containing (x, y). Then $\pi_2(N)$ is an open set in Y containing y. Hence, there exists an open set $V \subseteq Y$ containing y such that ∂V is compact. It is clear that $(\{x\} \times V) \cap N$ is an open subset of $X \times Y$ containing (x, y). Also

$$\partial((\{x\} \times V) \cap N) \subseteq (\{x\} \times \partial V) \cap \partial N$$

So $\partial((\{x\} \times V) \cap N)$ is compact. Hence $X \times Y$ is $\partial-$locally compact. □

Theorem 2.13. *Let Y be a compact space. Then $X \times Y$ is $\partial-$locally compact if and only if X is $\partial-$locally compact.*

Proof. First, let $X \times Y$ be $\partial-$locally compact and, $U \subseteq X$ an open set containing x. Then $U \times Y \subseteq X \times Y$ is an open set containing (x, y) for some $y \in Y$. It follows that there exists an open subset N of $X \times Y$ containing (x, y) such that ∂N is compact and $N \subseteq X \times Y$. Since Y is compact, π_1 is a closed map. Hence $\partial\pi_1(N) \subseteq \pi_1(\partial N)$. So $\partial\pi_1(N)$ is compact and $\pi_1(N) \subseteq U$. This shows that X is $\partial-$locally compact. Conversely, let $N \subseteq X \times Y$ be an open set containing (x, y). Then $\pi_1(N)$ is an open set in X containing x. Since X is $\partial-$locally compact, there exists an open set $U \subseteq X$ containing x such that ∂U is compact. Clearly, $(x, y) \in (U \times Y) \cap N \neq \emptyset$. Since $\partial((U \times Y) \cap N) \subseteq (\partial U \times Y) \cap \partial N$, $(U \times Y) \cap N$ is an open set containing (x, y) such that $\partial((U \times Y) \cap N)$ is compact. It shows that $X \times Y$ is $\partial-$locally compact. □

Corollary 2.14. *Let X and Y be two Hausdorff spaces. If $X \times Y$ is $\partial-$locally compact, then X and Y are $\partial-$locally compact.*

Proof. Let $y \in Y$. By Lemma 2.9, $X \times \{y\}$ is $\partial-$locally compact. Hence by Theorem 2.13, X is $\partial-$locally compact. Similarly, Y is $\partial-$locally compact. □

Corollary 2.15. *Let $\{X_i; i \in I\}$ be an arbitrary family of Hausdorff spaces. If $\prod_i X_i$ is a $\partial-$locally compact space, then each X_i is $\partial-$locally compact.*

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