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Novel existence results for sequential Caputo FDE with antiperiodic and integral boundary conditions

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Abstract: In this paper, by assuming certain assumptions, we study a novel class of sequential Caputo fractional differential equations (FDE) Consist of antiperiodic and Riemann-Liouville (R-L) fractional integral boundary conditions. The Existence and uniqueness of the solution to the proposed class of problem utilizing the fixed point theory and some new equalities of norm form are investigated. At the end of the paper, two specific examples of the study results are offered to demonstrate its performance and effectiveness.

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1 Introduction

There is a long history behind fractional calculus going back to the advent of classical calculus. In the past, some researchers studied this field; however, researchers show more interest to the new calculus and dynamic equation. Riemann-Liouville and Caputo fractional derivatives are among the most common definitions offered in several classes of fractional derivatives. Riemann-Liouville comes with a mathematical abstraction, while mostly, the engineers use the Caputo fractional derivative[3, 10, 19, 25, 29].
There seems to be substantial growth in the field of FDE demonstrating the position and status of fractional calculus in science and engineering. It is worth mentioning that in natural phenomena like chemical physics, fluid flows, electrical neetworks, visoelasticity, and porous media, the factional calculus is broadly applied; therefore, scientists highlight this field[14, 15, 17].
Recently, scientists have followed the solubility of linear initial FDE regarding specific functions in different problems in which the presence of solutions (or positive solutions) is proposed via the fixed point theorem, Leray-Schoder theory [1, 9, 13, 20].
An interesting branch of this type of equations are FDE sequential type. For example, in [4, 5, 8, 26] some of the investigations carried out on these equations can be seen.
The HIV infection model motivated the authors to investigate the existence and uniqueness of the below

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sequential system in [18]

$$\begin{cases} \left(\mathfrak{D}^{\alpha_1} + k_1\mathfrak{D}^{\alpha_1 - 1}\right)\omega_1(\tau) = F_1\left(\tau, \omega_1(\tau), \omega_1(\tau)\right) & \tau \in (0,1), \\ \left(\mathfrak{D}^{\alpha_2} + k_2\mathfrak{D}^{\alpha_2 - 1}\right)\omega_2(\tau) = F_2\left(\tau, \omega_1(\tau), \omega_1(\tau)\right) & \tau \in (0,1), \\ \omega_1(0) = \omega_1'(0) = 0, \quad \omega_1(1) = a\omega_2(\xi), \quad \xi \in (0,1) \\ \omega_2(0) = \omega_2'(0) = 0, \quad \omega_2(1) = b\omega_1(\eta), \quad \eta \in (0,1) \end{cases}$$

where $k_1, k_2 \in \mathbb{R}^+$, $a, b \in \mathbb{R}$, $2 < \alpha_1, \alpha_2 \leq 3$, $\mathfrak{D}^{\alpha_1}$, and $\mathfrak{D}^{\alpha_2}$ are the Caputo sense of fractional derivatives, and terms $F_1, F_2 : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are the given continuous function. There, the axial tools for obtaining the results were two basic theorems in fixed point theory; Leray-Schauders alternative and Banachs contraction principle.

Another category of problems that has attracted a lot of attention is the antiperiodic boundary problems that occur in the mathematical modeling of some special physical problems and events. Especially the fractional type of this problems has been studied by many researchers [2, 6, 22, 31]. For example, see the below nonlinear antiperiodic BVPs discussed by the authors in [7]

$$\begin{cases} \left(\mathfrak{D}^{q} + k\mathfrak{D}^{q-1}\right)\omega(\tau) = F_1\left(\tau, \omega(\tau)\right) \quad q \in (2,3], \quad \tau \in (0,T), \\ \epsilon_1\omega(0) + \varepsilon_1\omega(T) = \zeta_1, \quad \epsilon_2\omega'(0) + \varepsilon_2\omega'(T) = \zeta_2, \quad \epsilon_3\omega''(0) + \varepsilon_3\omega''(T) = \zeta_3, \end{cases}$$

where $\mathfrak{D}^q$ is the Caputo fractional derivatives of order q, $\epsilon_i, \varepsilon_i, \zeta_i \in \mathbb{R}$ $(i = 1, 2, 3)$, $k > 0$, F is a continuous function.

Many researchers in this field believe that integral boundary conditions are more sensible than local boundary conditions. Among the models that are described with integral boundary conditions, we can refer to population dynamics, modeling of blood flow, heat transmission,cellular systems. Some results about FDE and PDE with integral boundary conditions can be found in references [21, 23, 24, 27, 28, 30].

According to the above literature and in response to the question of whether antiperiodic and integral conditions can be unified in system, in this article, anti-periodic and integral conditions are used for the sequential FDE with the Caputo-type derivative and of the order $2 < \alpha \leq 3$, and new existence results that have not been given so far, are presented.

Let us consider the below problem:

$$\begin{cases} \mathfrak{D}^{\alpha}\omega(\tau) + k\mathfrak{D}^{\alpha-1}\omega(\tau) = F(\tau, \omega(\tau)) \quad \tau \in [0,1], \quad k > 0 \\ a_1\omega(0) + b_1\omega(1) + \gamma_1\mathfrak{I}^r\omega(\varsigma) = \epsilon_1, \quad r > 0, \quad 0 < \varsigma < 1 \\ a_2\omega'(0) + b_2\omega'(1) + \gamma_2\mathfrak{I}^r\omega'(\varsigma) = \epsilon_2, \\ a_3\omega''(0) + b_3\omega''(1) + \gamma_3\mathfrak{I}^r\omega''(\varsigma) = \epsilon_3, \end{cases} \tag{1.1}$$

where $\alpha \in (2,3]$ is an actual number, $k, r > 0$, $a_i, b_i, \gamma_i, \epsilon_i \in \mathbb{R}, i = 1, 2, 3$, $\mathfrak{D}^\alpha$ is considered the Caputo fractional derivative, and the boundary conditions consist of cases of antiperiodic and R-L fractional integral boundary value. The term F is a nonlinear term containing the unknown function. Antiperiodic and the cases of R-L fractional integral boundary value are considered as the linear amalgamation of the values of the unfamiliar function and its first derivatives at the endpoints of the interval, the R-L fractional integral value of the unfamiliar function and its first and second derivatives at an interior point of the interval, are all related to the new boundary conditions.

2 Preliminaries

In this section, some essential definitions and lemmas of the fractional calculus that will be used in the following, are presented [10, 19, 25].

Definition 2.1. *Suppose that $F : [0,\infty) \to \mathbb{R}$. Then, the R-L fractional integral of order $\nu > 0$ foe F is written as*

$$\mathfrak{I}^{\nu} F(\tau) = \frac{1}{\Gamma(\nu)} \int_0^{\tau} (\tau - s)^{\nu-1} F(s) ds, \quad \tau > 0 \quad n-1 < \nu < n.$$

Definition 2.2. *Suppose that $F : [0,\infty) \to \mathbb{R}$ is $n-$times continuously differentiable function. Then, the Caputo fractional derivative of order $\nu > 0$ for F is given by*

$$\mathfrak{D}^{\nu} F(\tau) = \frac{1}{\Gamma(n-\nu)} \int_0^{\tau} (\tau - s)^{n-\nu-1} F^{(n)}(s) ds, \quad \tau > 0 \quad n-1 < \nu < n.$$

Lemma 2.3. *The Caputo FDE $\mathfrak{D}^{\nu} F(\tau) = 0$, for $\nu > 0$, has the generic solution*

$$F(\tau) = \mathfrak{c}_1 + \mathfrak{c}_2 \tau + ... + \mathfrak{c}_n \tau^{n-1},$$

where $\mathfrak{c}_i \in \mathbb{R}$, $i = 1, 2, ..., n$, and $n-1 < \nu < n$.

Lemma 2.4. *[11] Let $n \in \mathbb{N}$, $\nu \in (n-1, n]$. If $u \in C^{n-1}[0, \mathfrak{b})$ and $\mathfrak{D}^{\nu} F \in C[0, \mathfrak{b})$, then*

$$\mathfrak{I}^{\nu} \mathfrak{D}^{\nu} F(\tau) = \omega(\tau) - \sum_{k=0}^{n-1} \frac{F^{(k)}(0^+)}{k!} \tau^k,$$

holds on $(0, \mathfrak{b})$.

NOTATIONS. For simplification in proving the next lemma as well as continuing the work, I will introduce the following symbols:

$$\delta_1=\frac{a_1+b_1}{k}+\frac{\gamma_1\varsigma^r}{k\Gamma(r+1)},$$
$$\delta_2=\frac{a_1+b_1(k+1)}{k^2}+\frac{\gamma_1\varsigma^r(1+k\varsigma)}{k^2\Gamma(r+1)}$$
$$\delta_3=a_1+b_1e^{-k}\gamma_1\int_0^\varsigma\frac{(\varsigma-s)^{r-1}}{\Gamma(r)}e^{-ks}ds,$$
$$\delta_4=\frac{a_2+b_2}{k}+\frac{\gamma_2\varsigma^r}{k\Gamma(r+1)}$$
$$\delta_5=-k\left(a_2+b_2e^{-k}\gamma_2\int_0^\varsigma\frac{(\varsigma-s)^{r-1}}{\Gamma(r)}e^{-ks}ds\right)$$
$$\delta_6=k^2\left(a_3+b_3e^{-k}\gamma_3\int_0^\varsigma\frac{(\varsigma-s)^{r-1}}{\Gamma(r)}e^{-ks}ds\right)$$
$$f_1(\tau)=\frac{1}{k\delta_1},\qquad f_2(\tau)=-\frac{1}{k\delta_1}+\frac{1}{k^2\delta_4}+\frac{1}{k\delta_4}\tau,$$
$$f_3(\tau)=-\frac{\delta_2\delta_5+\delta_1\delta_4}{k\delta_1\delta_4\delta_6}-\frac{\delta_5}{k^2\delta_4\delta_6}-\frac{\delta_5}{k\delta_4\delta_6}\tau+\frac{1}{\delta_6}e^{-k\tau}$$
$$\begin{aligned}\phi_1\left(\hbar(1),\hbar(\varsigma)\right)=&-b_1\int_0^1e^{-k(1-s)}\left(\mathfrak{I}^{\alpha-1}\hbar(s)\right)ds\\&-\gamma_1\int_0^\varsigma\frac{(\varsigma-s)^{r-1}}{\Gamma(r)}\left(\int_0^s e^{-k(s-m)}\left(\mathfrak{I}^{\alpha-1}\hbar(m)\right)dm\right)ds+\epsilon_1\end{aligned}$$
$$\begin{aligned}\phi_2\left(\hbar(1),\hbar(\varsigma)\right)=&-b_2\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\hbar(s)ds+kb_2\int_0^1e^{-k(1-s)}\left(\mathfrak{I}^{\alpha-1}\hbar(s)\right)ds\\&-\gamma_2\int_0^\varsigma\frac{(\varsigma-s)^{r-1}}{\Gamma(r)}\left(\mathfrak{I}^{\alpha-1}\hbar(s)\right)ds\\&+k\gamma_2\int_0^\varsigma\frac{(\varsigma-s)^{r-1}}{\Gamma(r)}\left(\int_0^s e^{-k(s-m)}\left(\mathfrak{I}^{\alpha-1}\hbar(m)\right)dm\right)ds+\epsilon_2\end{aligned}$$
$$\begin{aligned}\phi_3\left(\hbar(1),\hbar(\varsigma)\right)=&-b_3\int_0^1\frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}\hbar(s)ds+kb_3\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\hbar(s)ds\\&-k^2b_3\int_0^1e^{-k(1-s)}\left(\mathfrak{I}^{\alpha-1}\hbar(s)\right)ds+\gamma_3\int_0^\varsigma\frac{(\varsigma-s)^{r-1}}{\Gamma(r)}\left(\mathfrak{I}^{\alpha-2}\hbar(s)\right)ds\\&+k\gamma_3\int_0^\varsigma\frac{(\varsigma-s)^{r-1}}{\Gamma(r)}\left(\mathfrak{I}^{\alpha-1}\hbar(s)\right)ds\\&-k^2\gamma_3\int_0^\varsigma\frac{(\varsigma-s)^{r-1}}{\Gamma(r)}\left(\int_0^s e^{-k(s-m)}\left(I^{\alpha-1}\hbar(m)\right)dm\right)ds+\epsilon_3\end{aligned}$$

Lemma 2.5. *Let $\hbar\in C[0,1]$ and $\omega\in C^2[0,1]$. Then, the following sequential FDE*

$$\mathfrak{D}^\alpha\omega(\tau)+k\mathfrak{D}^{\alpha-1}\omega(\tau)=\hbar(\tau)\quad\tau\in[0,1],\qquad k>0 \tag{2.1}$$

with the boundary conditions

$$\begin{cases}a_1\omega(0)+b_1\omega(1)+\gamma_1\mathfrak{I}^r\omega(\varsigma)=\epsilon_1\\a_2\omega'(0)+b_2\omega'(1)+\gamma_2\mathfrak{I}^r\omega'(\varsigma)=\epsilon_2\\a_3\omega''(0)+b_3\omega''(1)+\gamma_3\mathfrak{I}^r\omega''(\varsigma)=\epsilon_3\end{cases} \tag{2.2}$$

have a unique solution

$$\omega(\tau) = f_1(\tau)\phi_1\left(\hbar(1), \hbar(\varsigma)\right) + f_2(\tau)\phi_2\left(\hbar(1), \hbar(\varsigma)\right) + f_3(\tau)\phi_3\left(\hbar(1), \hbar(\varsigma)\right) + \int_0^\tau e^{-k(\tau-s)}\left(\mathfrak{I}^{\alpha-1}\hbar(s)\right)ds. \tag{2.3}$$

Proof. Let $\omega \in C^2[0,1]$ be a solution of BVP 1.1. As $\omega'' \in C[0,1]$, definition 2.2 reveal that $\mathfrak{D}^{\alpha-1}\omega \in C^1[0,1]$. Moreover, from the relation $\mathfrak{D}^\alpha\omega = \hbar(\tau) - k\mathfrak{D}^{\alpha-1}\omega$ and $\hbar \in C[0,1]$; we have $\mathfrak{D}^\alpha\omega \in C(0,1)$. Therefore, by lemma 2.4 , we have the following relations

$$\mathfrak{I}^\alpha\mathfrak{D}^\alpha\omega(\tau) = \omega(\tau) - a_1 - a_2\tau - a_3\tau^2, \qquad \tau \in C(0,1), \tag{2.4}$$

and

$$\mathfrak{I}^{\alpha-1}\mathfrak{D}^{\alpha-1}\omega(\tau) = \omega(\tau) - b_2 - b_3\tau, \qquad \tau \in C(0,1),$$

So,

$$\mathfrak{I}^\alpha\mathfrak{D}^{\alpha-1}\omega(\tau) = \mathfrak{I}^1\mathfrak{I}^{\alpha-1}\mathfrak{D}^{\alpha-1}\omega(\tau) = \int_0^\tau \omega(s)ds - b_1 - b_2\tau - b_3\frac{\tau^2}{2}. \tag{2.5}$$

Now, from 2.1, 2.4 and 2.5,we get

$$\omega(\tau) + k\int_0^\tau \omega(s)ds = c_0 + c_1\tau + c_2\frac{\tau^2}{2} + \mathfrak{I}^\alpha\hbar(\tau),$$

where $c_0, c_1, c_2 \in \mathbb{R}$. It is easy to see that

$$\omega'(\tau) + k\omega(\tau) = c_1 + c_2\tau + \mathfrak{I}^{\alpha-1}\hbar(\tau),$$

and

$$\omega''(\tau) + k\omega'(\tau) = c_2 + \mathfrak{I}^{\alpha-2}\hbar(\tau).$$

Therefore, we can write the general solution of the fractional differential equation as follows:

$$\omega(\tau) = c_3e^{-k\tau} + \frac{c_1}{k} + c_2\left(\frac{1}{k^2} + \frac{\tau}{k}\right) + \int_0^\tau e^{-k(\tau-s)}\left(\mathfrak{I}^{\alpha-1}\hbar(s)\right)ds, \tag{2.6}$$

where $c_1, c_2, c_3 \in \mathbb{R}$. Moreover, from this we get

$$\omega'(\tau) = -kc_3e^{-k\tau} + \frac{c_2}{k} + \mathfrak{I}^{\alpha-1}\hbar(\tau) - k\int_0^\tau e^{-k(\tau-s)}\left(\mathfrak{I}^{\alpha-1}\hbar(s)\right)ds, \tag{2.7}$$

and

$$\omega''(\tau) = k^2c_3e^{-k\tau} + \mathfrak{I}^{\alpha-2}\hbar(\tau) - k\mathfrak{I}^{\alpha-1}\hbar(\tau) + k^2\int_0^\tau e^{-k(\tau-s)}\left(\mathfrak{I}^{\alpha-1}\hbar(s)\right)ds. \tag{2.8}$$

By using the boundary condition 2.2 in 2.6- 2.8 we get

$$\begin{cases} \delta_1c_1 + \delta_2c_2 + \delta_3c_3 = \phi_1\left(\hbar(1), \hbar(\varsigma)\right) \\ \delta_4c_2 + \delta_5c_3 = \phi_2\left(\hbar(1), \hbar(\varsigma)\right) \\ \delta_6c_3 = \phi_3\left(\hbar(1), \hbar(\varsigma)\right) \end{cases} \tag{2.9}$$

A simultaneous solution of system 2.9 leads to

$$
\begin{aligned}
c_1 &= \frac{1}{\delta_1}\phi_1\left(\hbar(1),\hbar(\varsigma)\right) - \frac{\delta_2}{\delta_1\delta_4}\phi_2\left(\hbar(1),\hbar(\varsigma)\right) - \frac{\delta_2\delta_5+\delta_3\delta_4}{\delta_1\delta_4\delta_6}\phi_3\left(\hbar(1),\hbar(\varsigma)\right)\\
c_2 &= \frac{1}{\delta_4}\phi_2\left(\hbar(1),\hbar(\varsigma)\right) - \frac{\delta_5}{\delta_4\delta_6}\phi_3\left(\hbar(1),\hbar(\varsigma)\right)\\
c_3 &= \frac{1}{\delta_6}\phi_3\left(\hbar(1),\hbar(\varsigma)\right).
\end{aligned}
$$

Replacing c_1, c_2 and c_3 to 2.6, we obtain the desirable solution 2.3. The converse of the lemma follows by direct computation. The proof is completed. □

Lemma 2.6. *Imagine that $\hbar \in C\left([0,1],\mathbb{R}\right)$. Then, we have*

i. $\left|\phi_1\left(\hbar(1),\hbar(\varsigma)\right)\right| \leq \underbrace{\left(|b_1|\frac{1-e^{-k}}{k\Gamma(\alpha)} + |\gamma_1|\frac{\varsigma^{\alpha+r}(1-e^{-k\varsigma})}{k\Gamma(\alpha)\Gamma(r+1)}\right)}_{L_1}\|\hbar\| + |\epsilon_1| = L_1\|\hbar\| + |\epsilon_1|$

ii. $\left|\phi_2\left(\hbar(1),\hbar(\varsigma)\right)\right| \leq \underbrace{\left(|b_2|\frac{k+1-e^{-k}}{k\Gamma(\alpha)} + |\gamma_2|\frac{\varsigma^{\alpha+r}(k+1-e^{-k\varsigma})}{k\Gamma(\alpha)\Gamma(r+1)}\right)}_{L_2}\|\hbar\| + |\epsilon_2| = L_2\|\hbar\| + |\epsilon_2|$

iii. $\left|\phi_3\left(\hbar(1),\hbar(\varsigma)\right)\right| \leq \underbrace{\left(|b_3|\frac{\alpha-1+k(2-e^{-k})}{k\Gamma(\alpha)} + |\gamma_3|\frac{(\alpha-1)\varsigma^{\alpha+r-1}+\varsigma^{\alpha+r}(2-e^{-k\varsigma})}{\Gamma(\alpha)\Gamma(r+1)}\right)}_{L_3}\|\hbar\| + |\epsilon_3|$

$= L_3\|\hbar\| + |\epsilon_3|$

Proof. Obviously, we have

$$
\begin{aligned}
&\left|\Im^{\alpha-1}\hbar(s)\right| = \frac{1}{\Gamma(\alpha-1)}\int_0^s (s-m)^{\alpha-1}\hbar(m)dm = \frac{s^\alpha}{\Gamma(\alpha)}\|\hbar\|,\\
&\left|\Im^{\alpha-1}\hbar(1)\right| \leq \frac{1}{\Gamma(\alpha)}\|\hbar\|,\\
&\left|\int_0^s e^{-k(s-m)}\left(\Im^{\alpha-1}\hbar(m)\right)dm\right| \leq \frac{s^\alpha}{\Gamma(\alpha)}\frac{1-e^{-ks}}{k}\|\hbar\|,\\
&\left|\int_0^1 e^{-k(1-s)}\left(\Im^{\alpha-1}\hbar(s)\right)ds\right| \leq \frac{1}{k\Gamma(\alpha)}\frac{1-e^{-k}}{k}\|\hbar\|,\\
&\left|\int_0^\varsigma \frac{(\varsigma-s)^{r-1}}{\Gamma(r)}\left(\Im^{\alpha-1}\hbar(s)\right)ds\right| \leq \frac{\varsigma^{\alpha+r}}{\Gamma(\alpha)\Gamma(r+1)}\|\hbar\|,\\
&\left|\int_0^\varsigma \frac{(\varsigma-s)^{r-1}}{\Gamma(r)}\left(\int_0^s e^{-k(s-m)}\left(\Im^{\alpha-1}\hbar(s)\right)dm\right)ds\right| \leq \frac{\varsigma^{\alpha+r}(1-e^{-k\varsigma})}{k\Gamma(\alpha)\Gamma(r+1)}\|\hbar\|,
\end{aligned}
$$

Hence,

$$\begin{aligned}|\phi_1(\hbar(1),\hbar(\varsigma))| &\leq \left|-b_1\int_0^1 e^{-k(1-s)}\left(\mathfrak{I}^{\alpha-1}\hbar(s)\right)ds\right| \\ &\quad+\left|\gamma_1\int_0^{\varsigma}\frac{(\varsigma-s)^{r-1}}{\Gamma(r)}\left(\int_0^s e^{-k(s-m)}\left(\mathfrak{I}^{\alpha-1}\hbar(s)\right)dm\right)ds+\epsilon_1\right| \\ &\leq\left(|b_1|\frac{1-e^{-k}}{k\Gamma(\alpha)}+|\gamma_1|\frac{\varsigma^{\alpha+r}(1-e^{-k\varsigma})}{k\Gamma(\alpha)\Gamma(r+1)}\right)\|\hbar\|+|\epsilon_1| \\ &= L_1\|\hbar\|+|\epsilon_1|\end{aligned}$$

In case the proof (ii) and (iii) looks like (i) it is removed. □

Set $C([0,1])$ is all the continuous functions on $[0,1]$. Consider $\mathfrak{E}=C([0,1],\mathbb{R})$ and it is considered as the all continuous functions of $[0,1]$ within $\mathbb{R}$ denotes the Banach space endowed with the norm given by $\|\omega\|=\sup_{0\leq\tau\leq1}|\omega(\tau)|$. With regards to lemma 2.5, substituting $g(t)$ by $\digamma(\tau,\omega(\tau))$ in 2.1, the solution of problem 1.1 is changed into the fixed point of operator equation $\omega=\mathfrak{P}\omega$, where in operator $\mathfrak{P}:\mathfrak{E}\to\mathfrak{E}$ is defined as:

$$\begin{aligned}\mathfrak{P}\omega(\tau) &= f_1(\tau)\phi_1\left(\digamma(1,\omega(1)),\digamma(\varsigma,\omega(\varsigma))\right) \\ &\quad+f_2(\tau)\phi_2\left(\digamma(1,\omega(1)),\digamma(\varsigma,\omega(\varsigma))\right) \\ &\quad+f_3(\tau)\phi_3\left(\digamma(1,\omega(1)),\digamma(\varsigma,\omega(\varsigma))\right) \\ &\quad+\int_0^{\tau}e^{-k(\varsigma-s)}\left(\mathfrak{I}^{\alpha-1}\digamma(s,\omega(s))\right)ds\end{aligned} \tag{2.10}$$

Theorem 2.7. *(see [16]). Let $\mathbb{F}:\mathfrak{E}\to\mathfrak{E}$ be completely continuous. Let*

$$\mathfrak{U}=\{\omega\in\mathfrak{E}:\ \omega=\lambda\mathbb{F}\omega,\ for\ some\ 0<\lambda<1\}$$

Then,either set $\mathfrak{U}$ is unbounded or $\mathbb{F}$ has at least one fixed point.

Theorem 2.8. *(see [12]). Let $\mathfrak{E}$ be a Banach space, $\mathfrak{D}\subseteq\mathfrak{E}$ be closed and $\mathbb{F}:\mathfrak{D}\to\mathfrak{D}$ a strict contraction, i.e., $|\mathbb{F}\omega_2-\mathbb{F}\omega_1|\leq k|\omega_2-\omega_1|$ for some $k\in(0,1)$ and all $\omega_1,\omega_2\in\mathfrak{D}$. Then, $\mathbb{F}$ has a unique fixed point.*

3 Main Results

Our assumption for $\digamma$ will be outlined before we start and introduce the main results

(a) $\digamma:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous.

(b) There are positive constants $a_{11},a_{12}\in\mathbb{R}^+$ such that
$|\digamma(\tau,\omega)|\leq a_{11}|\omega|^{\sigma_1}+a_{12},\quad\forall\tau\in[0,1]\quad\omega\in\mathbb{R},\quad 0<\sigma_1<1.$

(c) there exist positive constants $a_{21}\in\mathbb{R}^+$ such that
$|\digamma(\tau,\omega_2)-\digamma(\tau,\omega_1)|\leq a_{21}|\omega_2-\omega_1|,\quad\forall\tau\in[0,1]\quad\omega_1,\omega_2\in\mathbb{R}.$

Theorem 3.1. *Imagine that* (**a**) *and* (**b**) *hold. After that problem 1.1 has at least one solution.*

Proof. First the definition of a ball in $\mathfrak{E}$ as $B_R=\{\omega|\ \omega\in\mathfrak{E};\|\omega\|\leq R\}$ is defined, where

$$R\geq\max\left\{(2L_4a_{11})^{\frac{1}{1-\sigma_1}},2\left(2L_4a_{12}+L_5\right)\right\}.$$

It is shown that $\mathfrak{P} : B_R \to B_R$. For $\omega \in B_R$, using lemma 2.6 and the condition (**b**), we have

$$\begin{aligned}
|\mathfrak{P}\omega(t)| &= |f_1(\tau)|\,|\phi_1(F(1,\omega(1)), F(\varsigma,\omega(\varsigma)))| \\
&\quad + |f_2(\tau)|\,|\phi_2(F(1,\omega(1)), F(\varsigma,\omega(\varsigma)))| \\
&\quad + |f_3(\tau)|\,|\phi_3(F(1,\omega(1)), F(\varsigma,\omega(\varsigma)))| \\
&\quad + \int_0^\tau e^{-k(\tau-s)} \left|\mathfrak{I}^{\alpha-1} F(s,\omega(s))\right| ds \\
&\le M_1\left(|b_1|\frac{1-e^{-k}}{k\Gamma(\alpha)} + |\gamma_1|\frac{\varsigma^{\alpha+r}(1-e^{-k\varsigma})}{k\Gamma(\alpha)\Gamma(r+1)}\right)(a_{11}R^{\sigma_1}+a_{12}) + M_1|\epsilon_1| \\
&\quad + M_2\left(|b_2|\frac{k+1-e^{-k}}{k\Gamma(\alpha)} + |\gamma_2|\frac{\varsigma^{\alpha+r}(k+1-e^{-k\varsigma})}{k\Gamma(\alpha)\Gamma(r+1)}\right)(a_{11}R^{\sigma_1}+a_{12}) + M_2|\epsilon_2| \\
&\quad + M_3\left(|b_3|\frac{\alpha-1+k(2-e^{-k})}{\Gamma(\alpha)} + |\gamma_3|\frac{(\alpha-1)\varsigma^{\alpha+r-1}+k\tau^{\alpha+r}(2-e^{-k\varsigma})}{\Gamma(\alpha)\Gamma(r+1)}\right) \\
&\qquad \times (a_{11}R^{\sigma_1}+a_{12}) \\
&\quad + M_3|\epsilon_3| + \frac{1-e^{-k}}{k\Gamma(\alpha)}(a_{11}R^{\sigma_1}+a_{12}) \\
&\le \left(\sum_{i=1}^3 M_i L_i + \frac{1}{k\Gamma(\alpha)}\right)(a_{11}R^{\sigma_1}+a_{12}) + \sum_{i=1}^3 M_i|\epsilon_i| \\
&= L_4(a_{11}R^{\sigma_1}+a_{12}) + L_5 \\
&= L_4 a_{11}R^{\sigma_1} + (L_4 a_{12} + L_5) \\
&\le \frac{R}{2} + \frac{R}{2} = R,
\end{aligned}$$

where in $M_i = \max_{\tau\in[0,1]}|f_1(\tau)|$, $L_4 = \sum_{i=1}^3 M_i L_i + \frac{1}{k\Gamma(\alpha)}$ and $L_5 = \sum_{i=1}^3 M_i|\epsilon_i|$. This means $\mathfrak{P} : B_R \to B_R$. From the relation 2.10, it is easy to know that operator $\mathfrak{P}$ is continuous on $[0,1]$. Now, we show that $\mathfrak{P}$ is equicontinuous operator. Let $\tau_1, \tau_2 \in [0,1]$ *with* $\tau_1 < \tau_2$. Set $M = \max_{\tau\in[0,1]}|F(\tau,\omega(\tau))|, \ \forall\omega \in B_R$.

Ergo

$$\begin{aligned}
|\mathfrak{P}\omega(\tau_2)-\mathfrak{P}\omega(\tau_1)| &= |f_2(\tau_2)-f_2(\tau_1)|\,|\phi_2\left(F(1,\omega(1)),F(\varsigma,\omega(\varsigma))\right)| \\
&\quad + |f_3(\tau_2)-f_3(\tau_1)|\,|\phi_3\left(F(1,\omega(1)),F(\varsigma,\omega(\varsigma))\right)| \\
&\quad + \left|\int_0^{\tau_2} e^{-k(\tau_2-s)}\left(\mathfrak{I}^{\alpha-1}F(s,\omega(s))\right)ds - \int_0^{\tau_1} e^{-k(\tau_1-s)}\left(\mathfrak{I}^{\alpha-1}F(s,\omega(s))\right)ds\right| \\
&\le \frac{1}{k|\delta_4|}\left(|b_2|\frac{k+1-e^{-k}}{k\Gamma(\alpha)} + |\gamma_2|\frac{\varsigma^{\alpha+r}(k+1-e^{-k\varsigma})}{k\Gamma(\alpha)\Gamma(r+1)} + |\epsilon_2|\right)|\tau_2-\tau_1| \\
&\quad + \frac{1}{|\delta_6|}\left(e^{-k\tau_1}-e^{-k\tau_2}\right) + \frac{|\delta_5|M}{k|\delta_4\delta_6|} \\
&\qquad \times\left(|b_3|\frac{\alpha-1+k(2-e^{-k})}{\Gamma(\alpha)} + |\gamma_3|\frac{(\alpha-1)\varsigma^{\alpha+r-1}+k\varsigma^{\alpha+r}(2-e^{-k\varsigma})}{\Gamma(\alpha)\Gamma(r+1)}\right)|\tau_2-\tau_1| \\
&\quad + \frac{|\delta_5|M}{k|\delta_4\delta_6|}|\epsilon_2|\;|\tau_2-\tau_1| + \int_0^{\tau_1} e^{-k(\tau_2-s)}\left|\mathfrak{I}^{\alpha-1}F(s,\omega(s))\right|ds \\
&\quad + \int_0^{\tau_1}\left(e^{-k(\tau_1-s)}-e^{-k(\tau_2-s)}\right)\left|\mathfrak{I}^{\alpha-1}F(s,\omega(s))\right|ds \\
&\le M\left(\frac{L_2}{k|\delta_4|} + \frac{L_2|\delta_5|}{k|\delta_4||\delta_6|+|\epsilon_2|+|\epsilon_3|}\right)|\tau_2-\tau_1| + \frac{1}{|\delta_6|}\left(e^{-k\tau_1}-e^{-k\tau_2}\right) \\
&\quad + \frac{M}{k\Gamma(\alpha)}\left\{2\left(1-e^{-k(\tau_2-\tau_1)}\right) + \left(e^{-k\tau_1}-e^{-k\tau_2}\right)\right\}
\end{aligned}$$

That is, as $\tau_1 \to \tau_2$,

$$|\mathfrak{P}\omega(\tau_2)-\mathfrak{P}\omega(\tau_1)| \to 0$$

Thus, $\mathfrak{P}(B_R) \subseteq B_r$ is an equicontinuous set. Moreover, it is uniformly bounded due to $\mathfrak{P}(B_R) \subseteq B_r$. Using the Arzela-Ascoli theorem, it can be concluded that $\mathfrak{P}$ is a completely continuous operator. Consider $\mathfrak{U} = \{\omega \in B_R |\ \omega = \mu\mathfrak{P}\omega, 0 < \mu < 1\}$ and show that $\mathfrak{U}$ is bounded. For $\omega \in \mathfrak{U}$, we know $\|\omega\| < \|\mathfrak{P}\omega\| \le R$. This is associated with theorem 2.7 to validate that the 1.1 has at least one solution in B_R, hence, we proved. □

Theorem 3.2. *Imagine that* (**a**) *and* (**c**) *hold. If* $a_{21}L_4 < 1$*, then, problem 1.1 has unique solution.*

Proof. Define $sup_{\tau\in[0;1]}|F(\tau,0)| = N < \infty$ such that $r \ge \frac{N+L_5}{1-a_{21}L_4}$. Firstly, it is revealed that $\mathfrak{P}(B_r) \subseteq B_r$, where $B_r = \{\omega|\ \omega \in E; \|u\| \le r\}$. For $\omega \in B_r$, by computing directly, we have

$$\begin{aligned}
|\mathfrak{P}\omega(\tau)| &= |f_1(\tau)|\,|\phi_1\left(F(1,\omega(1))-F(1,0)+F(1,0),F(\varsigma,\omega(\varsigma))-F(\varsigma,0)+F(\varsigma,0)\right)| \\
&\quad + |f_2(\tau)|\,|\phi_2\left(F(1,\omega(1))-F(1,0)+F(1,0),F(\varsigma,\omega(\varsigma))-F(\varsigma,0)+F(\varsigma,0)\right)| \\
&\quad + |f_3(\tau)|\,|\phi_3\left(F(1,\omega(1))-F(1,0)+F(1,0),F(\varsigma,\omega(\varsigma))-F(\varsigma,0)+F(\varsigma,0)\right)| \\
&\quad + \int_0^{\tau} e^{-k(\tau-s)}\left|\mathfrak{I}^{\alpha-1}\left(F(s,\omega(s))-F(s,0)+F(s,0)\right)\right|ds \\
&\le \left(M_1L_1+M_2L_3+M_3L_3+\frac{1}{k\Gamma(\alpha)}\right)(a_{21}\|\omega\|+N) + (M_1|\epsilon_1|+M_2|\epsilon_2|+M_3|\epsilon_3|) \\
&= L_4\left(a_{21}\|\omega\|+N\right)+L_5 \\
&\le r.
\end{aligned}$$

Moreover, for any $\omega_1, \omega_2 \in B_r$, we have

$$
\begin{aligned}
|\mathfrak{P}\omega_2(\tau) - \mathfrak{P}\omega_1(\tau)| \leq & |f_1(\tau)| \left|\phi_1\Big(F(1,\omega_2(1)), F(\varsigma,\omega_2(\varsigma))\Big) - \phi_1\Big(F(1,\omega_1(1)), F(\varsigma,\omega_1(\varsigma))\Big)\right| \\
& + |f_2(t)| \left|\phi_2\Big(F(1,\omega_2(1)), F(\varsigma,\omega_2(\varsigma))\Big) - \phi_2\Big(F(1,\omega_1(1)), F(\varsigma,\omega_1(\varsigma))\Big)\right| \\
& + |f_3(t)| \left|\phi_3\Big(F(1,\omega_2(1)), F(\varsigma,\omega_2(\varsigma))\Big) - \phi_3\Big(F(1,\omega_1(1)), F(\varsigma,\omega_1(\varsigma))\Big)\right| \\
& + \int_0^\tau e^{-k(\tau-s)}\mathfrak{J}^{\alpha-1} |F(s,\omega_2(s)) - F(s,\omega_1(s))|\, ds \\
\leq & \left(M_1L_1 + M_2L_3 + M_3L_3 + \frac{1}{k\Gamma(\alpha)}\right) a_{51}\|\omega_2 - \omega_1\| \\
= & L_4 a_{21}\|\omega_2 - \omega_1\|.
\end{aligned}
$$

Thus

$$\|\mathfrak{P}\omega_2 - \mathfrak{P}\omega_1\| \leq= L_4 a_{21}\|\omega_2 - \omega_1\|.$$

As $L_4 a_{21} < 1$, $\mathfrak{P}$ is a constriction operator. In view of theorem 2.8, operator $\mathfrak{P}$ has a unique fixed point, which means that the system 1.1 has a unique solution, and this terminates the proof. □

4 Illustrative Examples

After we succeeded in investigating the results of existence and uniqueness, we provide two examples to show the effectiveness of studied results in this paper. Consider

$$
\begin{cases}
\mathfrak{D}^{2.5}\omega(\tau) + k\mathfrak{D}^{1.5}\omega(\tau) = F(\tau,\omega(\tau)) \quad \tau \in [0,1], \quad k > 0 \\
0.01\omega(0) + 0.02\omega(1) - \mathfrak{I}^2\omega(0.5) = 0.001 \\
0.02\omega'(0) + 0.01\omega'(1) + 2\mathfrak{I}^2\omega'(0.5) = 0.002 \\
0.12\omega''(0) + 0.18\omega''(1) - 3\mathfrak{I}^2\omega''(0.5) = 0.003
\end{cases}
\tag{4.1}
$$

According to problem 1.1, it is clear $\alpha = 2.5, k = 1, r = 2, \varsigma = 0.5, a_1 = 0.01, a_2 = 0.02, a_3 = 0.12$, $b_1 = 0.02, b_2 = 0.01, b_3 = 0.18$, $\gamma_1 = -1, \gamma_2 = 2, \gamma_3 = -3$, $\epsilon_1 = 0.001, \epsilon_2 = 0.002, \epsilon_3 = 0.003$.

Example 4.1. *Let $F(\tau,\omega(\tau)) = 0.01\omega(\tau)^{0.03} + 0.1e^{-t}$. So, we observe*

$$|F(\tau,\omega(\tau))| \leq 0.01\, |\omega(\tau)|^{0.03} + 0.1$$

Ergo $a_{11} = 0.01, a_{12} = 0.1$. Now, theorem 3.1 assure that system 4.1 is resolves.

Example 4.2. *Let $F(\tau,\omega(\tau)) = 0.001\omega(\tau) + 0.15e^{-t}$. So, we have*

$$|F(\tau,\omega_2(\tau)) - F_1(\tau,\omega(\tau))| \leq 0.001\, |\omega_2(\tau) - \omega_1(\tau)|\, .$$

where $a_{21} = 0.001$. Also it is easy to see

$$M_1 = 10.53,\ M_2 = 23.434,\ M_3 = 21.899, L_1 = 0.0749,\ L_2 = 1.0638,\ L_3 = 4.9031.$$

Therefore $L_4 = \sum_{i=1}^{3} M_iL_i + \frac{1}{k\Gamma(\alpha)} = 133.843066$. Now, we can obtain that

$$a_{21}L_4 = 0.001 \times 133.843066 = 0.133843 < 1.$$

Hence, theorem 3.2 implies that problem 4.1 has a unique solution.

5 Conclusions

We have acquired some existence results for novel sequential Caputo FDE by applying nonlinear growth conditions given in assumption (**b**). Clearly, it is distinct from the existing linear condition. Firstly, using theorem 2.7 existence of the solution to the proposed class of problem is obtained, then utilizing the fixed point theory and some new equalities of norm form uniqueness of the solution is proved(theorem 2.8). Clearly, examplea 4.1 and 4.2 show that obtained results are easy to verify and apply. Also, some special cases are included in our results. For example, if $a_1 = a_2 = a_3 = 1, b_1 = b_2 = b_3 = 1$, and $\gamma_1 = \gamma_1 = \gamma_1 = 0$

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