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On varieties and the direct limit of groups

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Abstract: Leedham-Green and Mckay [Acta Math. 137(1976) 99-150] introduced the generalized version of the Baer-invariant of a group with respect to two varieties of groups. A group G is called capable if there exists a group H such that $G \cong H/Z(H)$. In this paper, we generalize some properties of capability of direct product of groups with respect to two varieties of groups and direct limits. Moreover, we survey some properties of the Baer-invariant of a pair of groups with respect to two varieties of groups and direct limits.

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1 Introduction and Preliminaries

Let F_∞ be the free group freely generated by the countable set $X = \{x_1, x_2, \ldots\}$ and V be a subset of F_∞. Let V be the veriety of groups defined by the set of laws $\mathcal{V}$. P. Hall [6] introduced the following subgroups of a given group G associated with a variety of groups $\mathcal{V}$, as follows

$$V(G) = \langle v(g_1, \ldots, g_r) \mid g_i \in G, v \in V, 1 \le i \le r\rangle,$$
$$V^*(G) = \{a \in G \mid v(g_1, \ldots, g_i a, \ldots, g_r) = v(g_1, \ldots, g_r), g_i \in G, v \in V, 1 \le i \le r\},$$

which are called the verbal and the marginal subgroups of G, respectively. Let N be a normal subgroup of a groups G. Then we define $[NV^*G]$ to be the subgroup of G generated by the elements of the following set:

$$\{v(g_1, \ldots, g_i n, \ldots, g_r)v(g_1, \ldots, g_r)^{-1} \mid 1 \le i \le r, v \in V, g_i \in G, n \in N\}.$$

Also, we define

$$V^*(N, G) = \{n \in N \mid v(g_1, \ldots, g_i n, \ldots, g_r) = v(g_1, \ldots, g_r), v \in V, g_i \in G, 1 \le i \le r\}.$$

In particular, if $N = G$, then $V(N, G) = [NV^*G] = V(G)$ and $V^*(N, G) = V^*(G)$ are ordinary verbal and marginal subgroups of G.

R. Baer [3] introduced the notion of capable group. A group G is called capable if there exists a group H such that $G \cong H/Z(H)$. F. R. Beyl, U. Felgner and P. Schmid [4] proved that a group G is capable if and only if $Z^*(G) = 1$, where $Z^*(G)$ is the smallest central subgroup of G whose factor group is capable. They showed that the class of all capable groups is closed under the direct products. Then M. R. R. Moghaddam and S. Kayvanfar [11] generalized the concept of capability to $\mathcal{V}$-capability for a group G. (see [9] for more information).

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Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups defined by the sets of laws V and W, respectively. Let G be a group in the variety $\mathcal{W}$ with a free presentation $1 \to R \to F \xrightarrow{\pi} G \to 1$. Clearly $1 \to R/W(F) \to F/W(F) \to G \to 1$ is a $\mathcal{W}$-free presentation of G. If G is in the variety $\mathcal{W}$, then we can define the Baer-invariant of the group G, with respect to two varieties $\mathcal{V}$ and $\mathcal{W}$ as follows

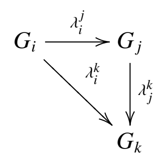
$$\mathcal{W}\mathcal{V}\mathcal{M}(G) = \frac{W(F)(R \cap V(F))}{W(F)[RV^*F]}.$$

We can see that $\mathcal{W}\mathcal{V}\mathcal{M}(G)$ is abelian and independent of the free presentation of G. In particular, when we consider $\mathcal{W}$ to be the variety of all groups, then we obtain the Baer-invariant of the group G, with respect to the variety $\mathcal{V}$ (see [10] for more information).

Let $\{G_i; \lambda_i^j, I\}$ be a system of groups and I a partially ordered set in such a way that for every $i, j \in I$, there exists $k \in I$ such that $i, j \leq k$. For $i \leq j$ there exists a homomorphism $\lambda_i^j : G_i \to G_j$ satisfying the following conditions:

(i) For each $i \in I$, $\lambda_i^i : G_i \to G_i$ is the identity homomorphism;

(ii) If $i \leq j \leq k$, then the following diagram commutes,



that is, $\lambda_i^j \lambda_j^k = \lambda_i^k$.

In this case, the system $\{G_i; \lambda_i^j, I\}$ is called a direct system of groups. Let $\bigcup_{i \in I} G_i$ be the disjoint union of groups in the direct system. Then we define an equivalence relation on this set as follows

$$x \sim y \quad \Leftrightarrow \quad \lambda_i^k(x) = \lambda_j^k(x), \;\; for\, some\, k \geq i, j.$$

Let $G = \bigcup_{i \in I} G_i / \sim$ be the quotient set and denote the equivalence class of an element x by $\{x\}$. Now, we define the binary operation on G, in the following way. For any $\{x\}$ and $\{y\}$ in G, there exists $i, j \in I$ such that $x \in G_i$ and $y \in G_j$, then for some $k \geq i, j$,

$$\{x\}\{y\} = \{\lambda_i^k(x)\lambda_j^k(y)\}.$$

Clearly, this operation is well-defined and makes G into a group, which is called the direct limit of the direct system $\{G_i; \lambda_i^j, I\}$ and denoted by

$$G = \varinjlim G_i.$$

Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups defined by sets of laws V and W, respectively. Let E be a group and G a group in $\mathcal{W}$. Let $\psi : E \to G$ be an epimorphism such that $\ker \psi \subseteq V^*(E)$. We denoted by $(WV^*)^*(G)$ the intersection of all subgroups of the form $\psi(V^*(E))$. We can see that $(WV^*)^*(G)$ is a characteristic subgroup of G and contained in $V^*(G)$. In particular, if $\mathcal{W}$ is the variety of all groups and $\mathcal{V}$ is the variety of abelian groups, then this subgroup is denoted by $Z^*(G)$.

2 On the direct limit of groups

In this section, we survey some results on capability of groups with respect to two varieties of groups. First, we discuss some results which are needed for the proof of our main results.

Lemma 2.1. *([12], Lemma 2.1 and 2.3) Let $\{G_i, \lambda_i^j, I\}$ be a direct system of groups and N_i a normal subgroup of G_i such that $\lambda_i^j(N_i) \subseteq N_j$, for all $i, j \in I$. Then*

(i) $\{N_i, \lambda_i^j|_{N_i}, I\}$ *and* $\{\frac{G_i}{N_i}, \overline{\lambda_i^j}, I\}$ *are both direct systems, where* $\overline{\lambda_i^j}$ *is the induced homomorphism,*

(ii) If $\mathcal{V}$ is the variety of groups and $N_i \subseteq V^*(G_i)$, for all $i \in I$, then $\varinjlim N_i \subseteq V^*(\varinjlim G_i)$,

(iii) $\varinjlim \frac{G_i}{N_i} = \frac{\varinjlim G_i}{\varinjlim N_i}$,

(iv) The direct limit of exact sequences remain exact.

Theorem 2.2. *([12], Theorem 1.2) Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups and $\{G_i, \lambda_i^j, I\}$ be a direct system of groups in $\mathcal{W}$. Then*

$$\varinjlim \mathcal{WVM}(G_i) = \mathcal{WVM}(\varinjlim G_i).$$

Theorem 2.3. *([5], Lemma 2.7) Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups and G be a group in $\mathcal{W}$. Let N be a normal subgroup of G such that $N \subseteq V^*(G)$. Then $N \subseteq (WV^*)^*(G)$ if and only if the homomorphism $\mathcal{WVM}(G) \to \mathcal{WVM}(G/N)$ is a monomorphism.*

The following theorem is a simple generalization of ([9], Theorem 2.3).

Theorem 2.4. *Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups, A and B be two groups in $\mathcal{W}$ with*

$$\mathcal{WVM}(A \times B) \cong \mathcal{WVM}(A) \times \mathcal{WVM}(B).$$

Then $(WV^*)^*(A \times B) = (WV^*)^*(A) \times (WV^*)^*(B)$. Consequently, $A \times B$ is $\mathcal{W}$-$\mathcal{V}$-capable if and only if A and B are both $\mathcal{W}$-$\mathcal{V}$-capable.

Lemma 2.5. *([9], Lemma 2.5) For any family of groups $\{G_i\}_{i\in I}$, consider the directed system $\{\mathcal{G}_{I_\lambda}, \phi_\lambda^{\lambda'}, \Lambda\}$ consisting of all finite direct products $\mathcal{G}_{I_\lambda} = \prod_{i\in I_\lambda} G_i$ (I_λ is a finite subset of I), with the natural embedding homomorphisms $\phi_\lambda^{\lambda'} : \mathcal{G}_{I_\lambda} \to \mathcal{G}_{I_{\lambda'}}$ ($I_\lambda \subseteq I_{\lambda'}$). Also, the index set Λ is ordered in a directed way so that for any $\lambda, \lambda' \in \Lambda$, $\lambda \leq \lambda'$ if and only if $I_\lambda \subseteq I_{\lambda'}$. Then the direct product $\mathcal{G}_I = \prod_{i\in I} G_i$ is a direct limit of this directed system.*

Theorem 2.6. *Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups, $\{G_i\}_{i\in I}$ be a family of groups in $\mathcal{W}$ such that for any $i, j \in I$, $\mathcal{WVM}(G_i \times G_j) \cong \mathcal{WVM}(G_i) \times \mathcal{WVM}(G_j)$. Then $(WV^*)^*(\prod_{i\in I} G_i) = \prod_{i\in I}(WV^*)^*(G_i)$. Consequently, $\prod_{i\in I} G_i$ is $\mathcal{W}$-$\mathcal{V}$-capable if and only if each G_i is $\mathcal{W}$-$\mathcal{V}$-capable.*

Proof. By the notations of Lemma 2.5, we conclude that $\prod_{i\in I} G_i$, $\prod_{i\in I}(WV^*)^*(G_i)$ and $\prod_{i\in I} \frac{G_i}{(WV^*)^*(G_i)}$ are direct limits of directed systems $\{\prod_{i\in I_\lambda} G_i, \phi_\lambda^{\lambda'}, \Lambda\}$, $\{\prod_{i\in I_\lambda}(WV^*)^*(G_i), \bar{\phi}_\lambda^{\lambda'}, \Lambda\}$, and $\{\prod_{i\in I_\lambda} G_i/(WV^*)^*(G_i), \psi_\lambda^{\lambda'}, \Lambda\}$ Assume that $\{G_i\}_{i\in I}$ is a family of groups in which for any G_i and G_j ($i, j \in I$), $\mathcal{WVM}(G_i \times G_j) \cong \mathcal{WVM}(G_i) \times \mathcal{WVM}(G_j)$. By Theorem 2.4, $\prod_{i\in I_\lambda}(WV^*)^*(G_i) \subseteq (WV^*)^*(\prod_{i\in I_\lambda} G_i)$, for any finite subset I_λ of I. Thus, by using Theorem 2.3 the following map is a monomorphism

$$\mathcal{WVM}(\prod_{i\in I_\lambda} G_i) \to \mathcal{WVM}(\frac{\prod_{i\in I_\lambda} G_i}{\prod_{i\in I_\lambda}(WV^*)^*(G_i)}).$$

Now, by Lemma 2.1, we obtain the following monomorphism

$$\varinjlim \mathcal{WVM}(\prod_{i\in I_\lambda} G_i) \to \varinjlim \mathcal{WVM}(\frac{\prod_{i\in I_\lambda} G_i}{\prod_{i\in I_\lambda}(WV^*)^*(G_i)}).$$

Thus, we obtain the following monomorphism

$$\mathcal{WVM}(\prod_{i\in I} G_i) \to \mathcal{WVM}(\frac{\prod_{i\in I} G_i}{\prod_{i\in I}(WV^*)^*(G_i)}).$$

Thus, by Theorem 2.2 we have

$$\prod_{i\in I}(WV^*)^*(G_i) \subseteq (WV^*)^*(\prod_{i\in I} G_i).$$

□

Corollary 2.7. *Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups and $\{G_i\}_{i\in I}$ be a family of perfect groups in $\mathcal{W}$. Then $\prod_{i\in I} G_i$ is $\mathcal{W}$-$\mathcal{V}$-capable if and only if each G_i is $\mathcal{W}$-$\mathcal{V}$-capable, where $\mathcal{V}$ may be variety of alelian groups, variety of nilpotent groups or variety of polynilpotent groups.*

In the following, we provide some nontrivial groups which are not capable:

1. There is no G such that $\frac{G}{Z(G)} \cong Q_8$. Thus, quaternion group Q_8 is not capable.
2. In [16] it is proved that if a finite group G contains a normal subgroup H which is either generalized quaternion of order 2^n, $n > 2$, or semidihedral of order 2^n, $n > 3$. Then G is not capable. So, $Q_8 \times Q_8$ is not capable. In [16], also it is proved that if G is a finite nilpotent group and contains a normal subgroup H which is an extraspecial p-group of order p^3 and exponent p, p odd, then G is not capable.

Theorem 2.8. *([13], Theorem 2.5) Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups and $\{G_i,\ \lambda_i^j, I\}$ be a direct system of groups in the variety $\mathcal{W}$. Let N_i be a normal subgroup of G_i such that $\lambda_i^j(N_i) \subseteq N_j$ for all $i, j \in I$ $(i < j)$. Then*

$$\mathcal{WVM}(\varinjlim N_i, \varinjlim G_i) = \varinjlim \mathcal{WVM}(N_i, G_i).$$

Lemma 2.9. *([13], Corollary 3.6) Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups and $\{G_i, \lambda_i^j, I\}$ be a direct system of groups in the variety $\mathcal{W}$. Let N_i and K_i be normal subgroups of G_i such that $K_i \subseteq V^*(G_i)$, $\lambda_i^j(N_i) \subseteq N_j$ and $\lambda_i^j(K_i) \subseteq K_j$, for all $i, j \in I$ $(i < j)$. If $G = \varinjlim G_i$, $N = \varinjlim N_i$ and $K = \varinjlim K_i$, then the following sequence is exact.*

$$1 \to \mathcal{WVM}(K, G) \to \mathcal{WVM}(N, G) \to \mathcal{WVM}(\frac{N}{K}, \frac{G}{K}) \to \frac{K \cap [NV^*G]}{[KV^*G]} \to 1.$$

Proposition 2.10. *Under assumption of Lemma 2.9,*

(i) The following sequence is exact:

$$\mathcal{WVM}(N, G) \to \mathcal{WVM}(\frac{N}{K}, \frac{G}{K}) \to \frac{K}{[KV^*G]} \to \frac{N}{[NV^*G]} \to \frac{N}{[NV^*G]K} \to 1,$$

(ii) The following conditions are equivalent:

(a) the sequence $1 \to \mathcal{WVM}(\frac{N}{K}, \frac{G}{K}) \to \frac{K}{[KV^*G]} \to \frac{N}{[NV^*G]} \to \frac{N}{[NV^*G]K} \to 1$ *is exact,*

(b) $\mathcal{WVM}(K, G) = \mathcal{WVM}(N, G)$,

(c) $\mathcal{WVM}(\frac{N}{K}, \frac{G}{K}) \cong \frac{K \cap [NV^*G]}{[KV^*G]}$.

Proof. (*i*) By Lemma 2.1, Theorem 2.8 and a simple generalization of Lemma 3.1 of [15], it is clear.

(*ii*) By Lemma 2.9 and Lemma 2.1 (*iv*), the following sequence is exact:

$$1 \to \mathcal{WVM}(K,G) \to \mathcal{WVM}(N,G) \to \mathcal{WVM}(\frac{N}{K},\frac{G}{K}) \to \frac{K\cap[NV^*G]}{[KV^*G]} \to 1.$$

We can see that (b) and (c) are equivalent. On the other hand, by first part sequence and Lemma 2.1 and Theorem 2.8, the following sequence is exact:

$$\mathcal{WVM}(\frac{N}{K},\frac{G}{K}) \xrightarrow{\alpha} \frac{K}{[KV^*G]} \to \frac{N}{[NV^*G]} \to \frac{N}{[NV^*G]K} \to 1.$$

Now, by the proof of Theorem 2.6 of [1], we have $\frac{\left|\mathcal{WVM}(\frac{N}{K},\frac{G}{K})\right|}{\left|\ker\alpha\right|} = \left|\frac{K\cap[NV^*G]}{[KV^*G]}\right|$ Hence, (*a*) and (*c*) are equivalent by Lemma 2.1 and Theorem 2.8. □

In finall, we improve Theorem 5 of [1].

Theorem 2.11. *Let (N,G) be a nilpotent pair of groups of class $c\geq 2$, $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups defined by the sets of laws V and W, respectively such that G be in W. Also, let $1\to R\to F\to G\to 1$ be a free presentation of G with $N\cong S/R$ for a normal subgroup S of F. If K is a normal subgroup of G such that $K\subseteq N$ and $K\cong T/R$ for a normal subgroup T of F, then*

(i) $\left|V_{c-1}(N,G)\cap[NV^*G]\right|\left|\mathcal{WVM}(N,G)\right| = \left|\mathcal{WVM}(\frac{N}{K},\frac{G}{K})\right|\left|\frac{W(F)[TV^*F]}{W(F)[RV^*F]}\right|$,

(ii) $d(\mathcal{WVM}(N,G)) \leq d(\mathcal{WVM}(\frac{N}{V_{c-1}(N,G)},\frac{G}{V_{c-1}(N,G)}) + d\left(\frac{W(F)[TV^*F]}{W(F)[RV^*F]}\right)$,

(iii) $e(\mathcal{WVM}(N,G))$ *divides* $e\left(\mathcal{WVM}\left(\frac{N}{V_{c-1}(N,G)},\frac{G}{V_{c-1}(N,G)}\right)\right)e\left(\frac{W(F)[TV^*F]}{W(F)[RV^*F]}\right)$.

Proof. By Theorem 2.2 of [5], the following sequence is exact:

$$1 \to \frac{W(F)[TV^*F]}{W(F)[RV^*F]} \to \mathcal{WVM}(N,G) \xrightarrow{\alpha} \mathcal{WVM}(\frac{N}{K},\frac{G}{K}) \xrightarrow{\beta} \frac{K\cap[NV^*G]}{[KV^*G]} \to 1.$$

Thus,

$$|\mathcal{WVM}(N,G)| = |Im\,\alpha|\left|\frac{W(F)(R\cap[TV^*F])}{W(F)[RV^*F]}\right|$$

and

$$\frac{\mathcal{WVM}(\frac{N}{K},\frac{G}{K})}{Im\,\alpha} \cong \frac{K\cap[NV^*G]}{[KV^*G]},$$

where, $K = V_{c-1}(N,G)$. Hence,

$$\left|K\cap[NV^*G]\right|\left|\mathcal{WVM}(N,G)\right| = \left|\mathcal{WVM}(\frac{N}{K},\frac{G}{K})\right|\left|[KV^*G]\right|\left|\frac{W(F)(R\cap[TV^*F])}{W(F)[RV^*F]}\right|.$$

On the other hand,

$$[KV^*G] = [V_{c-1}(N,G)V^*G] = V_c(N,G) = \langle e\rangle.$$

But $[KV^*G] \cong \frac{[TV^*F]}{R\cap[TV^*F]}$. Thus, $R\cap[TV^*F] = [TV^*F]$. This implies part (*i*). Similarly, we can prove (*ii*) and (*iii*). □

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