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Scalable g-frame in Hilbert spaces

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Abstract: Tight frames and g-frames are extremely useful in applications. A scalable frame was recently introduced as a frame with the property of generating a tight frame by rescaling its frame vectors. In this paper, we generalize this concept to g-frames, introduce scalable g-frames, obtain some characterizations for them, and demonstrate that scalability is stable under unitary operators and isomorphisms between two Hilbert spaces. In addition, we consider Paley-Wiener perturbations of g-frames and achieve some results regarding the preservation of their g-frame property.

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1 Introduction

During the past four decades, the theory of frames has proliferated, resulting in the development of numerous new applications. In addition to traditional applications such as signal processing, image processing, data compression, and sampling theory, frames are currently used to mitigate the effect of packet-based communication system losses, enhance the robustness of data transmission [3, 11], and design high-rate constellations with full diversity in multiple-antenna code design. Various extensions of the frame theory have been studied recently, with several of them incorporated into the elegant theory of g-frames. Sun [13, 14, 7, 18] presented g-frames as an additional generalized frame. The author demonstrated that frames, oblique frames, pseudo-frames, and fusion frames are g-frames special cases. Several authors referred to it as the operator-valued frame. Moreover, Kaftal et al. developed an operator theoretic method for dealing with multiwavelets and multiframes (see[1, 3, 5, 18]).

In this paper, H represents a separable Hilbert space with inner product $\langle .,. \rangle$, J represents a finite or countable subset of $\mathbb{Z}$, and $\{H_j : j \in J\}$ is a sequence of separable Hilbert spaces. Also, for every $j \in J, B(H, H_j)$ is the set of all bounded linear operators from H to H_j, and $B(H, H)$ is denoted by $B(H)$. First, we will review frame s and base s definitions and fundamental properties. For additional information, please refer to Sun [18] and Kutynioks [16]. Survey articles and the book written by Christensen [9]. This article investigates the properties of scalable g-frames. We also consider the Paley-Wiener perturbation of g-frames and obtain some results for scaling operators that preserve the g-frame property. Moreover, we achieve some results regarding preserving the g-frame property of a g-frame and its Paley Wiener

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perturbations.

Definition 1.1. *A frame is a family of vectors* $\{f_j\}_{j\in J}$ *in a Hilbert space* H *if there are constants* $0 < A \le B < \infty$*, such that for every* $f \in H$*,*

$$A\|f\|^2 \le \sum_{j\in J} |\langle f, f_j\rangle|^2 \le B\|f\|^2,$$

where A *and* B *are the lower frame bound and upper frame bound, respectively.*
A frame is referred to as a tight frame when $A = B$*, and a Parseval frame when* $A = B = 1$*. If a sequence* $\{f_j\}_{j\in J}$ *satisfies the upper bound condition, then* $\{f_j\}_{j\in J}$ *is referred to as a Bessel sequence.*

Let $\{f_j\}_{j\in J}$ be a frame in a Hilbert space H. We then obtain the frame operator

$$S : H \longrightarrow H, \qquad Sf = \sum_{j\in J} \langle f, f_j\rangle f_j, \qquad (f \in H).$$

Note that, with respect to the frame operator:

$$\langle Sf, f\rangle = \sum_{j\in J} |\langle f, f_j\rangle|^2, \qquad (f \in H).$$

S is therefore invertible and self-adjoint. In addition, because $S : H \longrightarrow H$ is bijective, the sequence $\{S^{-1}f_j\}_{j\in J}$ is also a frame and is referred to as the canonical dual frame of $\{f_j\}_{j\in J}$.

Definition 1.2. *We refer to a sequence* $\Lambda = \{\Lambda_j \in B(H, H_j) : j \in J\}$ *as a generalized frame, or simply a g-frame for* H *with respect to* $\{H_j : j \in J\}$ *if there are two positive constants* A *and* B *such that:*

$$A\|f\|^2 \le \sum_{j\in J} \|\Lambda_j f\|^2 \le B\|f\|^2 \qquad (f \in H).$$

We call A *and* B *the lower and upper frame bounds, respectively. We call* $\{\Lambda_j : j \in J\}$ *a tight g-frame if* $A = B$ *and a Parseval g-frame if* $A = B = 1$*. If only the inequality on the right-hand side is required,* Λ *is a g-Bessel sequence.*
If Λ *is a g-Bessel sequence, then the synthesis operator for* Λ *is the linear operator*

$$T_\Lambda : \left(\sum_{j\in J} \oplus H_j\right)_{\ell^2} \longrightarrow H, \qquad T_\Lambda (f_j)_{j\in J} = \sum_{j\in J} \Lambda_j^* f_j.$$

We call the adjoint of the synthesis operator, the analysis operator. The analysis operator is the linear operator

$$T_\Lambda^* : H \longrightarrow \left(\sum_{j\in J} \oplus H_j\right)_{\ell^2}, \qquad T_\Lambda^* f = \{\Lambda_j f\}_{j\in J}.$$

We call $S_\Lambda = T_\Lambda T_\Lambda^*$ *the g-frame operator of* Λ *and* $S_\Lambda f = \sum_{j\in J} \Lambda_j^* \Lambda_j f$*,* $(f \in H)$*, for more details see [13, 18].*

Remark 1.3. *Let* $\{\Lambda_j \in B(H, H_j) : j \in J\}$ *be a g-frame and* S *be the g-frame operator. Then* $\{\Lambda_j S^{-\frac{1}{2}} : j \in J\}$ *is a Parseval g-frame.*

Sun [18] showed that if $\Lambda = \{\Lambda_j \in B(H, H_j : j \in J\}$ and for each $j \in J$, $\{e_{jk} : k \in J_j\}$ is an orthonormal basis for H_j, then $\{\Lambda_j \in B(H, H_j) : j \in J\}$ is a g-frame if and only if $\{\Lambda_j^*(e_{jk}) : j \in J, k \in J_j\}$ is a frame and moreover the g-frame operator of Λ coincides with the frame operator of $\{\Lambda_j^*(e_{jk}) : j \in J, k \in J_j\}$. He showed that every frame, fusion frame, oblique frame, outer frame and pseudo-frame is a g-frame. We generalize his result by using the Parseval frame.

Theorem 1.4. *Let* $\{\Lambda_i \in B(H,H_i) : i \in J\}$ *and for each* $i \in J$, $\{f_{ij} : j \in J_i\}$ *be a Parseval frame for* H_i. *Then* $\{\Lambda_i : i \in J\}$ *is a g-frame (g-Bessel sequence) in* H *w.r.t* $\{H_i : i \in J\}$ *if and only if* $\varphi = \{\Lambda_i^*(f_{ij}) : i \in J, j \in J_i\}$ *is a frame (Bessel sequence) in* H *their frame operators are identical.*

Proof. For every $f \in H$, $i \in J$, $\Lambda_i f \in H_i$ and we have

$$\|\Lambda_i(f)\|^2 = \sum_{j\in J_i} |\langle \Lambda_i(f), f_{ij}\rangle|^2 = \sum_{j\in J_i} |\langle f, \Lambda_i^*(f_{ij})\rangle|^2.$$

So

$$\sum_{i\in J} \|\Lambda_i f\|^2 = \sum_{i\in J}\sum_{j\in J_i} |\langle f, \Lambda_i^*(f_{ij})\rangle|^2.$$

Therefore $\{\Lambda_i : i \in J\}$ is a g-frame (g-Bessel sequence) if and only if $\{\Lambda_i^*(f_{ij}) : i \in J, j \in J_i\}$ is a frame (Bessel sequence). For every $f \in H$, $S_\Lambda(f) = \sum_{i\in J} \Lambda_i^*\Lambda_i(f)$ and since for each $i \in J, \Lambda_i(f) \in H_i$, then $\Lambda_i(f) = \sum_{j\in J_i}\langle \Lambda_i(f), f_{ij}\rangle f_{ij}$. So,

$$S_\Lambda(f) = \sum_{i\in J}\sum_{j\in J_i} \langle f, \Lambda_i^*(f_{ij})\rangle \Lambda_i^*(f_{ij}) = S_\varphi(f),$$

and we have the result. □

2 Scalability

In [16], Kutyniok et al. introduced scalable frames and provided characterizations for them. In this section, we extended this concept to g-frames and applied some of their results to g-frames.

Definition 2.1. *A g-frame* $\Lambda = \{\Lambda_j \in B(H,H_j) : j \in J\}$ *for* H *with respect to* $\{H_j : j \in J\}$ *is scalable, if there exist scalars* $c_j \geq 0,\ j \in J$, *such that* $\{c_j\Lambda_j \in B(H,H_j) : j \in J\}$ *is a Parseval g-frame. If, in addition,* $c_j > 0$, *for all* $j \in J$, *then* $\Lambda = \{\Lambda_j \in B(H,H_j) : j \in J\}$ *is said to be positively scalable. If there exists* $\delta > 0$, *such that* $c_j \geq \delta$, *for all* $j \in J$, *then* $\{\Lambda_j \in B(H,H_j) : j \in J\}$ *is referred to as strictly scalable.*

Example 2.2. *Let* H *be a separable Hilbert space and* $\{f_j : j \in J\}$ *be a scalable frame for* H. *Let* Λ_{f_j} *be the functional induced by* f_j *i.e.*

$$\Lambda_{f_j} f = \langle f, f_j\rangle, \qquad (f \in H).$$

It is easy to check that $\{\Lambda_{f_j} : j \in J\}$ *is a scalable g-frame for* H *with respect to* $\mathbb{C}$.

Clearly, positive and strictly positive scalability coincide for finite g-frames. Moreover, each scaling $\{c_j\Lambda_j \in B(H,H_j) : j \in J\}$ of a g-frame $\Lambda = \{\Lambda_j \in B(H,H_j) : j \in J\}$ with $|J| < \infty$ and positive scalars c_j, is again a g-frame. If J is infinite, then this may not hold. Let $\{e_j : j \in J\}$ be the orthogonal basis for $\oplus_{j\in J}H_j$. The diagonal for an operator $D = D_c : \oplus_{j\in J}H_j \to \oplus_{j\in J}H_j$ corresponding to a sequence $c = \{c_j\}_{j\in J}$, which is defined as

$$D_c\{x_j\}_{j\in J} = \{c_jx_j\}_{j\in J}, \qquad \{x_j\}_{j\in J} \in domD_c,$$

where

$$domD_c := \{\{x_j\}_{j\in J} \in \oplus_{j\in J}H_j : \{c_jx_j\}_{j\in J} \in \oplus_{j\in J}H_j\}.$$

It is well-known that D_c is a selfadjoint operator in $\ell^2(J)$ if and only if $c_j \in \mathbb{R}$ for all $j \in J$. The domain, the kernel, and the range of a linear operator T are denoted by domT, kerT and ranT, respectively. In addition, a closed linear operator T between two Hilbert spaces H and K will be called *ICR* (or an *ICR*-operator), If it is injective and has a closed range, i.e., if there exists $\delta > 0$ such that $\|Tx\| \geq \delta\|x\|$ for every $x \in domT$. We note that the analysis operator of a g-frame is always an *ICR*-operator. Moreover, D_c is an isomorphism if it is a strictly bounded operator.

Lemma 2.3. *Let* $\Lambda = \{\Lambda_j \in B(H, H_j) : j \in J\}$ *be a g-frame and* $\{c_j\}_{j\in J} \subseteq \mathbb{R}^+$. *Consequently,*
(i) *If* $D_c\Lambda$ *is a Parseval g-frame, then for every* $x \in H$:

$$x = \sum_{i\in J} c_j^2 \Lambda_j^* \Lambda_j(x).$$

(ii) *If* $\{c_j^2\Lambda_j \in B(H, H_j) : j \in J\}$ *is a g-dual of* Λ, *then* Λ *is scalable.*

Proof. Let $x \in H$. Since $\{c_j\Lambda_j \in B(H, H_j) : j \in J\}$ is a Parseval g-frame, then

$$x = \sum_{i\in J} (c_j\Lambda_j)^*(c_j\Lambda_j)(x) = \sum_{i\in J} c_j^2\Lambda_j^*\Lambda_j(x).$$

(ii) Let $x \in H$. Then

$$x = \sum_{i\in J} (c_j^2\Lambda_j)^*\Lambda_j(x) = \sum_{i\in J} (c_j\Lambda_j)^*(c_j\Lambda_j)(x).$$

As a result, it is a Parseval g-frame. □

Remark 2.4. *We note that* Λ *is scalable with a scaling sequence* $c = \{c_j\}_{j\in J}$ *if and only if* $\{c_j^2\Lambda_j \in B(H, H_j) : j \in J\}$ *is a g-dual of* Λ, *when* $c = \{c_j\}_{j\in J}$ *is bounded.*

The following result describes when a scaling preserves the g-frame property.

Proposition 2.5. *Let* $\Lambda = \{\Lambda_j \in B(H, H_j) : j \in J\}$ *be a g-frame with the frame operator* S_Λ, *analysis operator* T_Λ^*, $\{c_j\}_{j\in J} \subseteq \mathbb{R}^+$ *and* $\Gamma = \{c_j\Lambda_j \in B(H, H_j) : j \in J\}$. *Then the following conditions are equivalent.*
(i) Γ *is a g-frame.*
(ii) $ranT_\Lambda^* \subset domD_c$ *and* $D_c|_{ranT_\Lambda^*}$ *is* ICR.
Moreover, in this case, the g-frame operator of the g-frame Γ *is given by*

$$S_\Gamma = \overline{T_\Lambda D_c} D_c T_\Lambda^*,$$

where $\overline{T_\Lambda D_c}$ *denotes the closure of the operator* $T_\Lambda D_c$.

Proof. $(i \Rightarrow ii)$ Let Γ be a g-frame. Then for every $f \in H$

$$T_\Gamma^* f = \{(c_j\Lambda_j)(f)\}_{j\in J} = D_c\{\Lambda_j f\}_{j\in J} = D_cT_\Lambda^* f.$$

Hence, $T_\Gamma^* = D_cT_\Lambda^*$. Also, $S_\Gamma = T_\Gamma T_\Gamma^* = (D_cT_\Lambda^*)^* D_cT_\Lambda^*$, since $(D_cT_\Lambda^*)^* = T_\Lambda D_c$ on $\text{dom}(D_c)$. However, $\text{dom}(D_c)$ is dense in $\oplus H_j$ and $(D_cT_\Lambda^*)^*$ is a bounded operator, ($T_\Gamma^* = D_cT_\Lambda^*$ is bounded) $(D_cT_\Lambda^*)^*$ is bounded, and consequently $(D_cT_\Lambda^*)^*$ is the closure of $T_\Lambda D_c$, and we obtain $S_\Gamma = \overline{T_\Lambda D_c} D_c T_\Lambda^*$.
$(ii \Rightarrow i)$ Let $ranT_\Lambda^* \subseteq domD_c$ and $D_c|_{ranT_\Lambda^*}$ be ICR. Since Λ is a g-frame, then T_Λ^* is ICR, especially with a closed range. Thus, by the closed graph Theorem $D_c|_{ranT_\Lambda^*}$ is bounded, and since $D_c|_{ranT_\Lambda^*}$ is ICR; we conclude that there exists $A', B' > 0$ such that

$$A'\|g\| \le \|D_c g\| \le B'\|g\|, \qquad (g \in ranT_\Lambda^*).$$

If we take A, B the g-frame bounds of $\{\Lambda_j \in B(H, H_j) : j \in J\}$, then

$$A\|f\|^2 \le \|T_\Lambda^* f\|^2 \le B\|f\|^2, \qquad (f \in H).$$

Hence for every $f \in H$, we have:

$$AA'^2\|f\|^2 \le A'^2\|T_\Lambda^* f\|^2 \le \|D_cT_\Lambda^*(f)\|^2$$

$$\leq B^{'^2}\|T_\Lambda^* f\|^2 \leq BB^{'^2}\|f\|^2, \qquad (f \in H).$$

Hence

$$AA^{'^2}\|f\|^2 \leq \|T_\Gamma^*(f)\|^2 \leq BB^{'^2}\|f\|^2, \qquad (f \in H).$$

And consequently, Γ is a g-frame, moreover

$$S_\Gamma = (D_c T_\Lambda^*)^*(D_c T_\Lambda^*) = \overline{T_\Lambda D_c} D_c T_\Lambda^*.$$

□

Proposition 2.6. *Let* $\Lambda = \{\Lambda_j \in B(H, H_j) : j \in J\}$ *be a g-frame for* H*. Then* $\Gamma = \{c_j\Lambda_j \in B(H, H_j) : j \in J\}$ *is a g-frame if and only if* $D|_{ranT_\Lambda^*}$ *is a bounded* ICR*-operator.*

Proof. Let Λ be a g-frame with bounds $A, B > 0$. Thus, for every $f \in H$

$$\sqrt{A}\|f\| \leq \|T_\Lambda^* f\| \leq \sqrt{B}\|f\|.$$

If Γ is a g-frame with bounds $A^{'}, B^{'} > 0$, then for every $f \in H$,

$$\sqrt{A'}\|f\| \leq \|T_\Gamma^* f\| = \|D_c T_\Lambda^*(f)\| \leq \sqrt{B'}\|f\|.$$

So for every $f \in H$,

$$\begin{array}{rcl} \frac{\sqrt{A'}}{\sqrt{B}}\|T_\Lambda^*(f)\| & \leq & \sqrt{A'}\|f\| \\ & \leq & \|D_c T_\Lambda^*(f)\| \\ & \leq & \frac{\sqrt{B'}}{\sqrt{A}}\|T_\Lambda^*(f)\|, \end{array}$$

and therefore $D_c|_{ranT_\Lambda^*}$ is a bounded ICR-operator.
Conversely, if $D_c|_{ranT_\Lambda^*}$ is a bounded ICR operator, then there exist $A^{''}, B^{''} > 0$ such that for every $f \in H$,

$$\begin{array}{rcl} A^{''}\|T_\Lambda^*(f)\| & \leq & \|D_c T_\Lambda^*(f)\| \\ & = & \|T_\Gamma^*(f)\| \\ & \leq & B^{''}\|T_\Lambda^* f\|. \end{array}$$

Hence

$$\begin{array}{rcl} A^{''}\sqrt{A'}\|f\| & \leq & A^{''}\|T_\Lambda^*(f)\| \\ & \leq & \|T_\Gamma^*(f)\| \\ & \leq & B^{''}\|T_\Lambda^*(f)\| \\ & \leq & B^{''}\sqrt{B'}\|f\|, \end{array}$$

and we have the result. □

In the next proposition, we will present an equivalent condition for the scalability of g-frames in Hilbert spaces.

Proposition 2.7. *Let $\Lambda = \{\Lambda_j \in B(H, H_j) : j \in J\}$ be a g-frame with frame operator S_Λ and analysis operator T^*_Λ. Consequently, the following conditions are equivalent:*
(i) Λ is (positively, strictly) scalable.
(ii) There exists a non-negative (positive, strictly positive, respectively) diagonal operator D in $\oplus_{j\in J} H_j$ such that

$$\overline{T_\Lambda D}(DT^*_\Lambda) = I_H.$$

Proof. $(i \Rightarrow ii)$ Let Λ be scalable with sequence scalars $\{c_j\}_{j\in J} \subset \mathbb{R}^+$. Therefore, $\Gamma = \{c_j \Lambda_j \in B(H, H_j) : j \in J\}$ is a Parseval g-frame. Hence, by Proposition 2.5 $ranT^*_\Lambda \subset domD_c$ and $S_\Gamma = \overline{T_\Lambda D_c}(D_c T^*_\Lambda)$ is the g-frame operator of Γ. Since g-frame operator of Parseval g-frame is the identity operator, then $\overline{T_\Lambda D_c}(D_c T^*_\Lambda) = I_H$.
$(ii \Rightarrow i)$ Let D be a non-negative diagonal operator in $\oplus_{j\in J} H_j$ such that

$$\overline{T_\Lambda D}(DT^*_\Lambda) = I_H.$$

Then DT^*_Λ is defined on H, especially, $ranT^*_\Lambda \subset domD$. Since T^*_Λ is bounded and D is closed, the operator DT^*_Λ is also closed. Hence, following the closed graph theorem, DT^*_Λ is a bounded operator from H into $\oplus_{j\in J} H_j$. In fact, $(T_\Lambda D)(DT^*_\Lambda) = I_H$ implies that DT^*_Λ is even isometric. Thus, based on the boundedness of T^*_Λ we can conclude that $D \mid ranT^*_\Lambda$ is *ICR*. Let $\{c_j\}_{j\in J}$ be the sequence of non-negative scalars such that $D = D_c$. As a consequence of Proposition 2.5, $\Gamma = \{c_j \Lambda_j\}_{j\in J}$ is a g-frame with frame operator $S_\Gamma = I_H$, which implies that Γ is a Parseval g-frame. The proofs for positive and strict scalability of Λ follow similarly. □

We provide a highly useful implication of Proposition 2.5, which shows that scalability is stable under unitary transformations.

Corollary 2.8. *Let H and K be Hilbert spaces, $\Lambda = \{\Lambda_j \in B(H, H_j) : j \in J\}$ be a g-frame and $U \in B(K, H)$ be an isomorphism i.e $UU^* = I_H$ and $U^*U = I_K$. Then Λ is scalable if and only if $\Lambda U = \{\Lambda_j U \in B(K, H_j) : j \in J\}$ is scalable.*

Proof. Let Λ be a scalable g-frame for H with diagonal operator D. Since the analysis operator of ΛU is given by $T^*_{\Lambda U} = T^*_\Lambda U$, we have

$$\begin{aligned}
\overline{(T_{\Lambda U} D)}(DT^*_{\Lambda U}) &= \overline{(U^* T_\Lambda D)}(DT^*_\Lambda U) \\
&= U^* \overline{(T_\Lambda D)}(DT^*_\Lambda) U \\
&= U^* U \\
&= I_K.
\end{aligned}$$

Hence ΛU is scalable.
For the converse, it is enough to note that if U is an isomorphism, then U^* is an isomorphism and $\Lambda = (\Lambda U)U^*$. □

The following result is about the preserving of g-frame property of a g-frame and its Paley Wiener perturbations.

Proposition 2.9. *Let $\Lambda = \{\Lambda_j \in B(H, H_j) : j \in J\}$ be a g-frame and $c = \{c_j\}_{j\in J} \subseteq \mathbb{R}$ such that $D_c \Lambda$ is a g-frame. Then*
(i) Let $\Gamma = \{\Gamma_j \in B(H, H_j) : j \in J\}$ such that for every $j \in J$ and $x \in H$,

$$\|\Gamma_j(x) - \Lambda_j(x)\| \le \lambda_1 \|\Lambda_j(x)\| + \lambda_2 \|\Gamma_j(x)\|, \tag{2.1}$$

where $\lambda_1, \lambda_2 \in (0,1)$. *Then,* Γ *and* $D_c\Gamma$ *are g-frames.*
(ii) *Let* $T_j \in B(H_j, K_j)$ *such that for every* $j \in J$ *and* $x_j \in H_j$,

$$A^{'}\|x_j\| \leq \|T_jx_j\| \leq B^{'}\|x_j\|, \tag{2.2}$$

where $A^{'}, B^{'} > 0$. *Then* $\eta = \{T_j\Lambda_j \in B(H, K_j) : j \in J\}$ *and* $D_c\eta$ *are g-frames.*
(iii) *Let* $U : \oplus_{j\in J}H_j \to \oplus_{j\in J}K_j$ *be a bounded* ICR*-operator, such that* $UH_j \subseteq K_j$ *for every* $j \in J$. *Then* $U\Lambda$ *and* $D_cU\Lambda$ *are g-frames.*

Proof. (i) If $\{\Lambda_j \in B(H, H_j) : j \in J\}$ is a g-frame with bounds A, B, then for every $j \in J$

$$\begin{aligned} \|\Gamma_jx\| - \|\Lambda_jx\| &\leq \|\Gamma_jx - \Lambda_jx\| \\ &\leq \lambda_1\|\Lambda_jx\| + \lambda_2\|\Gamma_jx\| \qquad (x \in H). \end{aligned}$$

Therefore

$$\|\Gamma_jx\| \leq \frac{1+\lambda_1}{1-\lambda_2}\|\Lambda_jx\|.$$

So,

$$\sum_{j\in J}\|\Gamma_jx\|^2 \leq \left(\frac{1+\lambda_1}{1-\lambda_2}\right)^2\sum_{j\in J}\|\Lambda_jx\|^2 \leq B\left(\frac{1+\lambda_1}{1-\lambda_2}\right)^2\|x\|^2.$$

Similarly,

$$\frac{1-\lambda_1}{1+\lambda_2}\|\Lambda_jx\| \leq \|\Gamma_jx\|.$$

So,

$$A\left(\frac{1-\lambda_1}{1+\lambda_2}\right)^2\|x\|^2 \leq \sum_{j\in J}\|\Gamma_j(x)\|^2.$$

Therefore $\{\Gamma_j \in B(H, H_j) : j \in J\}$ is a g-frame with bounds $A\left(\frac{1-\lambda_1}{1+\lambda_2}\right)^2$, $B\left(\frac{1+\lambda_1}{1-\lambda_2}\right)^2$. Since $D_c\Lambda$ is a g-frame and $D_c\Lambda, D_c\Gamma$ satisfy in Condition (2.1), we get that $D_c\Gamma$ is a g-frame.
(ii) For every $x \in H, j \in J, \Lambda_jx \in H_j$ by (2.2) we have

$$\begin{gathered} AA^{'}\|x\|^2 \leq A\sum_{j\in J}\|\Lambda_j(x)\|^2 \leq \sum_{j\in J}\|T_j\Lambda_jx\|^2 \\ \leq B\sum_{j\in J}\|\Lambda_j(x)\|^2 \leq BB^{'}\|x\|^2. \end{gathered}$$

Hence, we have the result.
(iii) Since U is a bounded ICR-operator, there exist $\delta, M > 0$ such that

$$\delta\|\{x_j\}_{j\in J}\|_2 \leq \|U\{x_j\}_{j\in J}\| \leq M\|\{x_j\}_{j\in J}\|_2, \qquad (\{x_j\}_{j\in J} \in \oplus_{j\in J}H_j).$$

Then for every $x \in H$

$$\begin{aligned} A\delta^2\|x\|^2 &\leq \delta^2\sum_{j\in J}\|\Lambda_j(x)\|^2 \\ &\leq \sum_{j\in J}\|U\Lambda_j(x)\|^2 \\ &\leq M^2\sum_{j\in J}\|\Lambda_j(x)\|^2 \\ &\leq M^2B\|x\|^2. \end{aligned}$$

Hence we have the result. □

A projection is an idempotent linear transformation $P : K \to K$ of a linear space K into itself. An orthogonal projection on a Hilbert space K is a projection $P : K \to K$, such that $kerP = (ranP)^{\perp}$. Hence, $K = kerP \oplus ranP$.

Theorem 2.10. *Let $\Lambda = \{\Lambda_j \in B(H, H_j) : j \in J\}$ be a g-frame for H, with analysis operator T_Λ^*. Then, the following conditions are equivalent:*
(i) Λ is strictly scalable.
(ii) There exists a strictly positive bounded diagonal operator D in $\oplus_{j\in J}H_j$ such that DT_Λ^* is an isometry (that is, $T_\Lambda D^2 T_\Lambda^* = I_H$).
(iii) There exists a Hilbert space K and a bounded ICR-operator $L : K \to \oplus_{j\in J}H_j$ such that $T_\Lambda^* T_\Lambda + LL^*$ is a strictly positive bounded diagonal operator.

Proof. $(i) \Leftrightarrow (ii)$ This equivalence follows from Proposition 2.5.
$(ii) \Rightarrow (iii)$ Let D be a strictly positive bounded diagonal operator in $\oplus_{j\in J}H_j$ such that $T_\Lambda D^2 T_\Lambda^* = I_H$. Now, we choose $K := ran(DT_\Lambda^*)^{\perp} = kerT_\Lambda D \subset \oplus_{j\in J}H_j$. Let $L : K \to \oplus_{j\in J}H_j$ be defined by $L := D^{-1} \mid_K$. Hence L is a bounded ICR-operator and $L^* = P_K D^{-1}$. So

$$\begin{aligned}(DT_\Lambda^* T_\Lambda D)^2 &= DT_\Lambda^*(T_\Lambda D^2 T_\Lambda^*)T_\Lambda D\\ &= DT_\Lambda^* T_\Lambda D.\end{aligned}$$

So $DT_\Lambda^* T_\Lambda D$ is an idempotent and since it is self-adjoint, then $DT_\Lambda^* T_\Lambda D$ is an orthogonal projection, and its kernel coincides with $K = ker(T_\Lambda D)$, therefore $DT_\Lambda^* T_\Lambda D = P_{K^\perp}$. Hence

$$\begin{aligned}T_\Lambda^* T_\Lambda + LL^* &= D^{-1}(DT_\Lambda^* T_\Lambda D)D^{-1} + D^{-1}P_K D^{-1}\\ &= D^{-1}(P_{K^\perp} + P_K)D^{-1}\\ &= D^{-1}(I_{\oplus_{j\in J}H_j})D^{-1}\\ &= D^{-2},\end{aligned}$$

which is strictly positive bounded diagonal operator in $\oplus_{j\in J}H_j$.
$(iii) \Rightarrow (ii)$ Suppose that (iii) holds for a Hilbert space K and a bounded ICR-operator $L : K \to \oplus_{j\in J}H_j$ such that $T_\Lambda^* T_\Lambda + LL^* = D^{-2}$. We note that D^{-1} is a bounded strictly positive operator in $\oplus_{j\in J}H_j$. Now we define $G : H \oplus K \to \oplus_{j\in J}H_j$ by

$$G\begin{pmatrix}x\\ y\end{pmatrix} := T_\Lambda^*(x) + L(y), \qquad \begin{pmatrix}x\\ y\end{pmatrix} \in H \oplus K.$$

Then

$$G^*(\nu) = \begin{pmatrix}T_\Lambda \nu\\ L^*\nu\end{pmatrix}, \qquad (\nu \in \oplus_{j\in J}H_j),$$

and hence $GG^* = T_\Lambda^* T_\Lambda + LL^* = D^{-2}$. Hence $DG(DG)^* = I_{\oplus H_j}$ and therefore DG is an isomorphism. Therefore $G^* D^2 G = (DG)^*(DG) = I_{H\oplus K}$. This implies that

$$\begin{aligned}\begin{pmatrix}I_H & 0\\ 0 & I_K\end{pmatrix} &= \begin{pmatrix}T_\Lambda\\ L^*\end{pmatrix}(D^2 T_\Lambda^*, D^2 L)\\ &= \begin{pmatrix}T_\Lambda D^2 T_\Lambda^* & T_\Lambda D^2 L\\ L^* D^2 T_\Lambda^* & L^* D^2 L\end{pmatrix},\end{aligned}$$

or, equivalently,

$$T_\Lambda D^2 T_\Lambda^* = I_H, \qquad T_\Lambda D^2 L = 0, \qquad L^* D^2 T_\Lambda^* = 0, \ \ \textit{and} \qquad L^* D^2 L = I_K,$$

we have the result. □

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