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New areas of fixed point results for multi-valued mappings and their applications

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Abstract: In this paper, the notion of limit property (Tayyab kamran, 2004), and some notions and theorems (Tayyab kamran, Calogero Vetro, Muhammad Usman Ali, Mehwish Waheed, 2017) on metric spaces are generalized for multi-valued function on S-m spaces. We also present an application to nonlinear integral equations.

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1 Introduction

Banach's contraction has been generalized in several approaches by many scholars [1, 2, 3, 4, 10, 13]. In 2012, Sedghi et al. generalized metric space called S-metric space (S-ms) [12, 5, 6].
This study, investigate some new common fixed point results for single-valued and multi-valued mappings are proven in S-ms by using a generalization of coincidence point for pair (f, F), in which f is single-valued and F is a multi-valued mapping. Also we generalized the concepts in [8] on an S-ms accomplished with a finite number of graphs and present an application and an example. In this note, (X, S) is an S-ms and $G = \{G_\alpha : \alpha = 1, 2..., l\}$ is the set of all graphs where $G_\alpha = (X, E_\alpha), E_\alpha \subseteq X \times X \times X$ for each $\alpha \in \{1, 2, ..., l\}$.

2 Basic Concepts

Definition 2.1. *Assume X is a nonempty set, $S : X^3 \longrightarrow \mathbb{R}^+$ is called an S-m on X if for all $\zeta, \xi, \eta, a \in X$ the following conditions hold:*

(1) $S(\zeta, \xi, \eta) = 0$ *if and only if* $\zeta = \xi = \eta$*;*

(2) $S(\zeta, \xi, \eta) \leq S(\zeta, \zeta, a) + S(\xi, \xi, a) + S(\eta, \eta, a)$.

The pair (X, S) is called an S-ms.

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Example 2.2. **(1)** *([12]) For any metric space (X,d), the mapping $S : X^3 \longrightarrow \mathbb{R}^+$ defined by $S(\zeta,\xi,\eta) = d(\zeta,\eta) + d(\xi,\eta)$ is an S-m on X.*

(2) *([12]) Assume $X = \mathbb{R}$. The mapping $S : X^3 \longrightarrow \mathbb{R}^+$ defined by $S(\zeta,\xi,\eta) = |\zeta - \xi| + |\zeta + \xi - 2\eta|$ is an S-m on X.*

(3) *Let $\beta \geq 0$ and $X = [\beta,\infty)$. $S : X^3 \longrightarrow [0,\infty)$ defined by*

$$S(\zeta,\xi,\eta) = \begin{cases} 0 & if\ \zeta = \xi = \eta; \\ \max\{\zeta,\xi,\eta\} - \beta & otherwise, \end{cases}$$

is an S-m on X. We call it the max S-m.

Lemma 2.3 ([12])**.** *Assume (X,S) is an S-ms. Then, for all $\zeta,\xi \in X$, $S(\zeta,\zeta,\xi) = S(\xi,\xi,\zeta)$.*

Definition 2.4 ([12])**.** *Assume (X,S) is an S-ms, $\zeta \in X$ and $r > 0$.*

(1) *An open ball $B_s(\zeta,r)$ is defined as*

$$B_s(\zeta,r) = \{\xi \in X : S(\xi,\xi,\zeta) < r\}.$$

(2) *$\{\zeta_n\}$ converges to ζ if $\lim_{n\to\infty} S(\zeta_n,\zeta_n,\zeta) = 0$.*

(3) *$\{\zeta_n\} \subseteq X$ is called a Cauchy sequence if for every $\epsilon > 0$, there exists an $N_0 \in \mathbb{N}$ such that for every $i,j \geq N_0$, $S(\zeta_i,\zeta_i,\zeta_j) < \varepsilon$.*

(4) *An S-ms (X,S) is called complete if every Cauchy sequence converges.*

The set of all closed bounded nonempty subsets of X is denoted by $CB(X)$.

Example 2.5. *Let $X = [0,\infty)$ with the max S-m. Then, we have: $B_s(1,2) = [0,2), B_s(2,1) = \{2\}$ and $B_s(1,1) = \{1\}$.*

Lemma 2.6 ([12])**.** *Let (X,S) be an S-ms and $\{\zeta_n\}$ converges to ζ, then, ζ is unique.*

Lemma 2.7 ([12])**.** *Let (X,S) be an S-ms. if $\{\zeta_n\}$ converges to ζ and $\{\xi_n\}$ converges to ξ, then $S(\zeta_n,\zeta_n,\xi_n) \to S(\zeta,\zeta,\xi)$.*

Definition 2.8 ([11])**.** *Let (X,S) be an S-ms. For $\zeta \in X$ and $A \in CB(X)$, $S(\zeta,\zeta,A)$ is defined as $\inf\{S(\zeta,\zeta,a) : a \in A\}$.*

It can be easily seen that, $S(\zeta,\zeta,A) = 0 \Longleftrightarrow \zeta \in A$.

Definition 2.9 ([11])**.** *Let (X,S) be an S-ms, then $S_H : CB(X)^3 \longrightarrow [0,\infty)$, is defined by;*

$$S_H(A,B,C) = H_s(A,C) + H_s(B,C).$$

Where, $H_s(A,B) = \max\{h_S(A,B), h_S(B,A)\}$, $h_s(A,B) = \sup\{S(a,a,B) : a \in A\}$ and $S(a,a,B) = \inf\{S(a,a,b) : b \in B\}$.

Lemma 2.10 ([11])**.** *S_H is an S-m on $CB(X)$.*

Lemma 2.11 ([11])**.** *Let (X,S) be an S-ms. For all $A \in CB(X)$ and $\zeta,\xi \in X$, we have:*

$$S(\zeta,\zeta,A) \leq 2\ S(\zeta,\zeta,\xi) + S(\xi,\xi,A)$$

.

Remark 2.12. *In $(CB(X), S_H)$ for all $\zeta, \xi, \eta \in X$, we have*

$$S_H(\{\zeta\}, \{\xi\}, \{\eta\}) = S(\zeta, \zeta, \eta) + S(\xi, \xi, \eta)$$

.

Remark 2.13. *In Example 2.2(3) assume u is a nondecreasing continuous mapping on X and assume $F(\zeta) = [\beta, u(\zeta)]$, then*

$$H_s(F\zeta, F\xi) = \begin{cases} u(\xi) - \beta & if\ \xi \geq \zeta; \\ u(\zeta) - \beta & if\ \zeta > \xi. \end{cases}$$

Definition 2.14. *Let (X, S) be an S-ms and $g : X \longrightarrow X$ and $G : X \longrightarrow CB(X)$.*

(1) g and G have a coincidence point at a if $g(a) \in G(a)$, also g and G have a common fixed point at a if $g(a) = a \in G(a)$.

(2) $g : X \longrightarrow X$ is G-weakly commuting at $\zeta \in X$ if $g(g(\zeta)) \in G(g(\zeta))$.

(3) We say (g, G) have a limit property if there exist a sequence $\{\zeta_n\}$ in X, a $t \in X$ and an $A \in CB(X)$ such that

$$\lim_{n\to\infty} g\zeta_n = t \in A = \lim_{n\to\infty} G\zeta_n.$$

Lemma 2.15 ([11])**.** *Let (X, S) be an S-ms, $\{\zeta_n\} \subset X$ and let $\{A_n\} \subset CB(X)$. Also $\lim_{n\to\infty} \zeta_n = \zeta$ and $\lim_{n\to\infty} A_n = A \in CB(X)$. Then, $\lim_{n\to\infty} S(\zeta_n, \zeta_n, A_n) = S(\zeta, \zeta, A)$.*

Lemma 2.16 ([11])**.** *Let (X, S) be a S-ms. Let C and D be two distinct subsets in $CB(X)$ and $\gamma > 1$. Then, for every $c \in C$ there exists $d \in D$ with $S(c, c, d) < \frac{\gamma}{2} S_H(C, C, D)$.*

3 Main results

The following theorem is a extension of Theorem 3.4 [7] for S-ms.

Theorem 3.1. *Suppose $g : X \to X$ and $G : X \to CB(X)$ satisfies the following properties:*

(1) *(g, G) have the limit property;*

(2) *For distinct elements $\zeta, \xi \in X$,*

$$\begin{aligned} S_H(G\zeta, G\zeta, G\xi) < \max\{&S(g\zeta, g\zeta, g\xi), \alpha[S(g\zeta, g\zeta, G\zeta) \\ &+ S(g\xi, g\xi, G\xi)], \alpha[S(g\zeta, g\zeta, G\xi) + S(g\xi, g\xi, G\zeta)]\}, \ (1 \leq \alpha < 2). \end{aligned} \tag{3.1}$$

If $g(X)$ is a closed subset of X, then

(a) *g and G have a coincidence point.*

(b) *g and G have a common fixed point, if for every $v \in C(g, G)$, $ggv = gv$, where $C(g, G)$ is the set of all coincidence points of g and G.*

Proof. By assumption, there exists a sequence $\{\zeta_n\} \subseteq X$, $t \in X$, and $A \in CB(X)$ with $\lim_{n\to\infty} g(\zeta_n) = t \in A = \lim_{n\to\infty} G\zeta_n$. If for some $a \in X$, $t = g(a)$, Setting $\zeta = \zeta_n$ and $\xi = a$ in relation (3.1) we get:

$$\begin{aligned} S_H(G\zeta_n, G\zeta_n, Ga) < \max\{&S(g\zeta_n, g\zeta_n, ga), \alpha[S(g\zeta_n, g\zeta_n, G\zeta_n) \\ &+ S(ga, ga, Ga)], \alpha[S(g\zeta_n, g\zeta_n, Ga) + S(ga, ga, G\zeta_n)]\}. \end{aligned}$$

By Lemma 2.15, follows, $\lim_{n\to\infty} S_H(G\zeta_n, G\zeta_n, Ga) = S_H(A, A, Ga) \leqslant \alpha\ S(ga, ga, Ga)$.
By definition of S_H:

$$2S(ga, ga, Ga) \leqslant S_H(A, A, Ga) \leqslant \alpha\ S(ga, ga, Ga).$$

That is, $S(ga, ga, Ga) = 0$. Therefore, $ga \in Ga$. Hence, (a) holds. For conclude of (b), by (a), there exist $t, a \in X$ with $t = ga \in Ga$. As $a \in C(g, G)$, thus, $gga = ga$ and $gga \in Gga$. Therefore, $gt = t \in Gt$. □

Example 3.2. *Accomplish $X = [1, \infty)$ with the max S-m.*
Define $g : X \longrightarrow X$, $G : X \longrightarrow CB(X)$ as

$$g(\zeta) = \zeta^3 \quad and \quad G(\zeta) = \left[1, \frac{\zeta^2+1}{2\zeta}\right].$$

The pair (g, G) satisfies the limit property. Indeed;

$$\lim_{n\to\infty} g(1+\frac{1}{n}) = 1 \in \lim_{n\to\infty} G(1+\frac{1}{n}) = \{1\}.$$

For any $\zeta, \xi \in X$, with $\zeta \neq \xi$ the relation (3.1) in Theorem 3.1 holds. For example, in the case $\zeta < \xi$, by Remark 2.13 we get

$$S_H(G\zeta, G\zeta, G\xi) = 2H_S(G\zeta, G\xi) = \frac{\xi^2+1}{\xi} - 2.$$

On the other hand, $S(g\zeta, g\zeta, g\xi) = S(\zeta^3, \zeta^3, \xi^3) = \xi^3 - 1$. So

$$\begin{aligned} S_H(G\zeta, G\zeta, G\xi) < \max\{&S(g\zeta, g\zeta, g\xi), \alpha[S(g\zeta, g\zeta, G\zeta) \\ &+ S(g\xi, g\xi, G\xi)], \alpha[S(g\zeta, g\zeta, G\xi) + S(g\xi, g\xi, G\zeta)]\}. \end{aligned}$$

By Theorem 3.1,g and G have a coincidence point. Indeed, $g(1) \in G(1)$. As $gg(1) = g(1)$ and $gg(1) \in G(1)$, g and G have common fixed point 1.

Now, we generalize the existence of fixed point for multi-valued mappings with values in $(CB(X), S_H)$.

Definition 3.3. *Let (X, S) be an S-ms accomplished with a finite number of graphs $G = \{G_\alpha\}_{\alpha=1}^{l}$. The mapping $T : X \to CB(X)$ is called G-monotone, if $(\zeta, \xi, \eta) \in E_\alpha \Longrightarrow T\zeta \times T\xi \times T\eta \subseteq E_{\alpha+1}$, for each $\alpha \in \{1, 2, \ldots, l\}$, with $E_{l+1} = E_1$.*

Example 3.4. *Let $X = \{1, 2, 3\}$, $E_1 = \{(1, 1, 1), (1, 1, 3), (1, 3, 1), (1, 3, 3), (3, 1, 1),$ $(3, 1, 3), (3, 3, 1), (3, 3, 3)\}$ and $E_2 = \{(2, 2, 2)\}$. The family of graphs $G = \{G_\alpha\}$ is taken as $G_\alpha = (X, E_\alpha), \alpha = 1, 2$. Define a mapping $T : X \to CB(X)$ by:*

$$Tm = \begin{cases} \{2\} & if\ m = 1, 3; \\ \{1, 3\} & if\ m = 2. \end{cases}$$

Let $(m, n, p) \in E_1$ then $Tm = Tn = Tp = \{2\}$ and $Tm \times Tn \times Tp \subseteq E_2$.
Now, if $(m, n, p) \in E_2$, then $Tm = Tn = Tp = \{1, 3\}$ and $Tm \times Tn \times Tp = E_1$. Hence T is G-monotone.

Definition 3.5. *Let (X, S) be an S-ms accomplished with a finite number of graphs $G = \{G_\alpha\}_{\alpha=1}^{l}$. The pair (G, S) is called regular if the following conditions satisfy:*
if $\{\zeta_n\} \subseteq X$ and $\zeta \in X$ such that
(a) for all $\alpha \in \{1, 2, \ldots, l\}$, there exists a subsequence $\{\zeta_{n_{\alpha,i}}\} \subseteq \{\zeta_n\}$ with $(\zeta_{n_{\alpha,i}}, \zeta_{n_{\alpha,i}}, \zeta_{n_{\alpha,i+1}}) \in E_\alpha$, for all i,
(b) $S(\zeta_n, \zeta_n, \zeta) \to 0$,
then there exists a subsequence $\{\zeta_{n_i}\} \subseteq \{\zeta_n\}$ and $\beta \in \{1, 2, \ldots, l\}$ such that $(\zeta_{n_i}, \zeta_{n_i}, \zeta) \in E_\beta$, for all i.

Example 3.6. *Assume $G = (X, E)$ is a graph where $X = C([a, b])$ is the family of all continuous functions on $[a, b]$, and $E \subseteq X \times X \times X$ is given by $(\zeta, \xi, \eta) \in E$ if and only if $\zeta(r) \leq \xi(r) \leq \eta(r)$ for all $r \in [a, b]$. Accomplish X with the following S-m.*

$$S(\zeta, \xi, \eta) = \| \zeta - \eta \|_\infty + \| \xi - \eta \|_\infty = \sup_{r\in[a,b]} |\zeta(r) - \eta(r)| + \sup_{r\in[a,b]} |\xi(r) - \eta(r)|.$$

Assume $\{\zeta_n\}$ is a sequence and ζ is a point in X such that
(i) there exists a subsequence $\{\zeta_{n_i}\}$ of $\{\zeta_n\}$ such that $\zeta_{n_i}(r) \leq \zeta_{n_{i+1}}(r)$, holds for all $i \in \mathbb{N}$ and $r \in [a, b]$,
(ii) $S(\zeta_n, \zeta_n, \zeta) \to 0$.
Then $(\zeta_{n_i}, \zeta_{n_i}, \zeta) \in E$ for all $i \in \mathbb{N}$. Thus, (G, S) is regular.

Let Θ be the set of all functions $\theta : [0, \infty) \to [0, \infty)$ satisfying the following conditions:
(i) θ is increasing;
(ii) $\sum_{n=1}^{\infty} \theta^n(r) < \infty$ for each $r > 0$.

Remark 3.7. *Each function $\theta \in \Theta$ satisfies condition $\theta(r) < r$, for each $r > 0$.*

Theorem 3.8. *Let (X, S) be a complete S-ms accomplished with a finite number of graphs $G = \{G_\alpha\}_{\alpha=1}^{l}$ and assume $T : X \to CB(X)$ is a G-monotone multi-valued mapping such that*

(i) there exist an $\omega \in X$ and a $\eta \in T\omega$ such that $(\omega, \omega, \eta) \in E_1$;

(ii) (G, S) is regular;

(iii) there exist $\theta \in \Theta$ and a lower semi-continuous function $\lambda : X \to \mathbb{R}^+$ such that

$$S_H(T\zeta, T\zeta, T\xi) + \lambda(v_1) + \lambda(v_2) \leqslant \theta(S(\zeta, \zeta, \xi) + \lambda(\zeta) + \lambda(\xi)), \tag{3.2}$$

for all $\zeta, \xi \in X$, and $(v_1, v_2) \in T\zeta \times T\xi$, whenever $(\zeta, \zeta, \xi) \in E_\alpha$, for any $\alpha \in \{1, 2, \cdots, l\}$, with $E_{l+1} = E_1$.

Then T has a fixed point.

Proof. By assumption, there exists $\zeta_0 \in X$ and $\zeta_1 \in T\zeta_0$ with $(\zeta_0, \zeta_0, \zeta_1) \in E_1$. If $\zeta_1 = \zeta_0$, then ζ_0 is a fixed point of T. Otherwise assume

$$t_0 = S(\zeta_0, \zeta_0, \zeta_1) + \lambda(\zeta_0) + \lambda(\zeta_1).$$

By relation (3.2) and Lemma 2.16, for $\gamma > 2$, there exists $\zeta_2 \in T\zeta_1$ such that

$$\begin{aligned} S(\zeta_1, \zeta_1, \zeta_2) + \lambda(\zeta_1) + \lambda(\zeta_2) &< \frac{\gamma}{2}(S_H(T\zeta_0, T\zeta_0, T\zeta_1) + \lambda(\zeta_1) + \lambda(\zeta_2)) \\ &\leq \frac{\gamma}{2}(\theta(S(\zeta_0, \zeta_0, \zeta_1) + \lambda(\zeta_0) + \lambda(\zeta_1))) \\ &= \frac{\gamma}{2}\theta(t_0). \end{aligned}$$

Since $\theta \in \Theta$ is increasing, we have

$$\theta(S(\zeta_1, \zeta_1, \zeta_2) + \lambda(\zeta_1) + \lambda(\zeta_2)) < \theta(\frac{\gamma}{2}\theta(t_0)).$$

Put $\gamma_1 = \frac{2\theta(\frac{\gamma}{2}\theta(t_0))}{\theta(S(\zeta_1,\zeta_1,\zeta_2)+\lambda(\zeta_1)+\lambda(\zeta_2))}$. Then $\gamma_1 > 2$. If $\zeta_1 = \zeta_2$, then ζ_1 is a fixed point. Otherwise since T is G-monotone, we have $(\zeta_1, \zeta_1, \zeta_2) \in E_2$. Again by relation (3.2) and Lemma 2.16 there exists $\zeta_3 \in T\zeta_2$ such that

$$\begin{aligned} S(\zeta_2, \zeta_2, \zeta_3) + \lambda(\zeta_2) + \lambda(\zeta_3) &< \frac{\gamma_1}{2}(S_H(T\zeta_1, T\zeta_1, T\zeta_2) + \lambda(\zeta_2) + \lambda(\zeta_3)) \\ &\leq \frac{\gamma_1}{2}\theta(S(\zeta_1, \zeta_1, \zeta_2) + \lambda(\zeta_1) + \lambda(\zeta_2)) \\ &= \theta(\frac{\gamma}{2}\theta(t_0)). \end{aligned}$$

We continue this process to obtain a sequence $\{\zeta_m\} \subset X$ such that $\zeta_{m+1} \in T\zeta_m$, and also $\zeta_{m+1} \neq \zeta_m$ for each $m \in \{0, 1, 2, \cdots\}$. So for each $m \in \{0, 1, 2, \cdots\}$, there exists $\alpha = \alpha(m) \in \{1, 2, \cdots, l\}$ such that, $(\zeta_m, \zeta_m, \zeta_{m+1}) \in E_\alpha$ and

$$S(\zeta_m, \zeta_m, \zeta_{m+1}) + \lambda(\zeta_m) + \lambda(\zeta_{m+1}) < \theta^{m-1}(\frac{\gamma}{2}\theta(t_0)).$$

Therefore, $\lim_{m\to\infty} S(\zeta_m, \zeta_m, \zeta_{m+1}) + \lambda(\zeta_m) + \lambda(\zeta_{m+1}) = 0$.
Consequently, $S(\zeta_m, \zeta_m, \zeta_{m+1}) \to 0$ and $\lambda(\zeta_m) \to 0$. From

$$\begin{aligned}\sum_{m=1}^{\infty} S(\zeta_m, \zeta_m, \zeta_{m+1}) &\leq \sum_{m=1}^{\infty}[S(\zeta_m, \zeta_m, \zeta_{m+1}) + \lambda(\zeta_m) + \lambda(\zeta_{m+1})] \\ &\leq \sum_{m=1}^{\infty} \theta^{m-1}(\frac{\gamma}{2}\theta(t_0)) < \infty,\end{aligned}$$

we get that $\{\zeta_m\}$ is a Cauchy sequence. Since (X, S) is complete, there exists $\hat{\zeta} \in X$ such that $\zeta_m \to \hat{\zeta}$. Since (G, S) is regular, there exist a subsequence $\{\zeta_{m_i}\}$ of $\{\zeta_m\}$ and $\beta \in \{1, 2, \cdots, l\}$ such that $(\zeta_{m_i}, \zeta_{m_i}, \hat{\zeta}) \in E_\beta$ for all i. Therefore, $\lambda(\hat{\zeta}) \leq \liminf_{m\to\infty} \lambda(\zeta_m) = 0$ and hence $\lambda(\hat{\zeta}) = 0$.
As $(\zeta_{m_i}, \zeta_{m_i}, \hat{\zeta}) \in E_\beta$ for all i, so, by Lemma 2.11, Remark 3.7 and relation (3.2), we have

$$\begin{aligned}S(\hat{\zeta}, \hat{\zeta}, T\hat{\zeta}) &\leq 2S(\hat{\zeta}, \hat{\zeta}, \zeta_{m_{i+1}}) + S(\zeta_{m_{i+1}}, \zeta_{m_{i+1}}, T\hat{\zeta}) \\ &\leq 2S(\hat{\zeta}, \hat{\zeta}, \zeta_{m_{i+1}}) + \frac{1}{2}S_H(T\zeta_{m_i}, T\zeta_{m_i}, T\hat{\zeta}) \\ &\leq 2S(\hat{\zeta}, \hat{\zeta}, \zeta_{m_{i+1}}) + \frac{1}{2}\theta(S(\zeta_{m_i}, \zeta_{m_i}, \hat{\zeta}) + \lambda(\zeta_{m_i}) + \lambda(\hat{\zeta})) \\ &< 2S(\hat{\zeta}, \hat{\zeta}, \zeta_{m_{i+1}}) + \frac{1}{2}[S(\zeta_{m_i}, \zeta_{m_i}, \hat{\zeta}) + \lambda(\zeta_{m_i}) + \lambda(\hat{\zeta})],\end{aligned}$$

for all i. It follows $S(\hat{\zeta}, \hat{\zeta}, T\hat{\zeta}) = 0$. That is, $\hat{\zeta} \in T\hat{\zeta}$. □

Corollary 3.9. *Assume (X, S) is a complete S-ms, $T : X \to CB(X)$ is a multi-valued mapping and there exist a lower semi-continuous mapping*
$\lambda : X \to \mathbb{R}^+$, and a $t \in [0, 1)$ such that

$$S_H(T\zeta, T\zeta, T\xi) + \lambda(v_1) + \lambda(v_2) \leqslant t[(S(\zeta, \zeta, \xi) + \lambda(\zeta) + \lambda(\xi)], \tag{3.3}$$

for all $\zeta, \xi \in X$ and $(v_1, v_2) \in T\zeta \times T\xi$.
Then T has a fixed point.

Proof. Assume $l = 1$, $E_1 = X \times X \times X$ and $\lambda(\zeta) = t\zeta$, then the result follows from Theorem 3.8. □

Corollary 3.10. *Let (X, S) be a complete S-ms and let $T : X \longrightarrow CB(X)$ be an a-contraction $(0 < a < 1)$, that is, for all $\zeta, \xi \in X$, $S_H(T\zeta, T\zeta, T\xi) \leq a\ S(\zeta, \zeta, \xi)$. Then, there exists $\tilde{\zeta} \in X$ such that $\tilde{\zeta} \in T\tilde{\zeta}$.*

Proof. Assume $\lambda(\zeta) = 0$ for each $\zeta \in X$, then the result follows from Corollary 3.9. □

Example 3.11. *Accomplish $X = [0, \frac{\pi}{2}]$ with the max S-m. Define $T : X \longrightarrow CB(X)$ as*

$$T\zeta = \begin{cases} \{0\} & \text{if } \zeta \in \left[0, \frac{\pi}{4}\right); \\ \{0, \frac{1}{2}\} & \text{if } \zeta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right); \\ \{0, \frac{1}{3}\} & \text{if } \zeta = \frac{\pi}{2}. \end{cases}$$

For $\zeta = 0$ *and* $\xi = \frac{\pi}{4}$, *we have* $S_H(T\zeta, T\zeta, T\xi) = 2H_S(\{0\}, \{0, \frac{1}{2}\}) = 1$ *and* $S(\zeta, \zeta, \xi) = \max\{\zeta, \xi\} = \frac{\pi}{4}$. *So,* $S_H(T\zeta, T\zeta, T\xi) > S(\zeta, \zeta, \xi)$. *Therefore we can not apply Corollary 3.10 to this example. Now, define* $\lambda : X \longrightarrow \mathbb{R}^+$ *as* $\lambda(\zeta) = \begin{cases} \zeta & if\ \zeta < \frac{\pi}{2}, \\ 1 & if\ \zeta = \frac{\pi}{2}. \end{cases}$ *By following cases, relation (3.3) holds for* $t = \frac{3}{\pi}$.

1. *If* $\zeta, \xi \in \left[0, \frac{\pi}{4}\right)$ *or* $\zeta, \xi \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$ *or* $\zeta = \xi = \frac{\pi}{2}$, *then for each* $(v_1, v_2) \in T\zeta \times T\xi$, $S_H(T\zeta, T\zeta, T\xi) + \lambda(v_1) + \lambda(v_2) = 0$.
2. *If* $\zeta \in \left[0, \frac{\pi}{4}\right)$ *and* $\xi \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$, *then for each* $(v_1, v_2) \in T\zeta \times T\xi$, $S_H(T\zeta, T\zeta, T\xi) = 2H_S(\{0\}, \{0, \frac{1}{2}\}) = 1, \lambda(v_1) + \lambda(v_2) = 0$ *or* $\frac{1}{2}$, *and* $S(\zeta, \zeta, \xi) + \lambda(\zeta) + \lambda(\xi) = 2\xi + \zeta \geq \frac{\pi}{2}$.
3. *If* $\zeta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$ *and* $\xi = \frac{\pi}{2}$, *then for each* $(v_1, v_2) \in T\zeta \times T\xi$, $S_H(T\zeta, T\zeta, T\xi) = 2H_S(\{0, \frac{1}{2}\}, \{0, \frac{1}{3}\}) = 2\max\{0, \frac{1}{2}\} = 1$, $\lambda(v_1) + \lambda(v_2) = 0$ *or* $\frac{1}{3}$ *or* $\frac{1}{2}$ *or* $\frac{5}{6}$, *and* $S(\zeta, \zeta, \xi) + \lambda(\zeta) + \lambda(\xi) = \xi + \zeta + 1 \geq \frac{3\pi+4}{4}$.
4. *If* $\zeta \in \left[0, \frac{\pi}{4}\right)$ *and* $\xi = \frac{\pi}{2}$, *then for each* $(v_1, v_2) \in T\zeta \times T\xi$, $S_H(T\zeta, T\zeta, T\xi) = 2H_S(\{0\}, \{0, \frac{1}{3}\}) = \frac{2}{3}$, $\lambda(v_1) + \lambda(v_2) = 0$ *or* $\frac{1}{3}$, *and* $S(\zeta, \zeta, \xi) + \lambda(\zeta) + \lambda(\xi) = \xi + \zeta + 1 \geq \frac{\pi+2}{2}$.

So, all conditions of Corollary 3.9 are satisfied. Hence T *has a fixed point.*

4 Application

In this section, we investigate the existence of solution for certain nonlinear integral equations. Consider the integral equation

$$\zeta(r) = \int_a^r \int_a^v q(r, v, w)\ g(r, v, \zeta(w))\ dw\ dv + k(r), \quad r \in [a, b] \tag{4.1}$$

where $g : [a, b] \times [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a closed bounded continuous function, $q : [a, b] \times [a, b] \times [a, b] \to [0, \infty)$ is a continuous function and $k \in X = C([a, b])$.
The purpose of this section is the existence of a solution for the integral equation (4.1) by using Theorem 3.8.
Accomplish $X = C([a, b])$ with the following S-m

$$S(\zeta, \xi, \eta) = \parallel \zeta - \eta \parallel_\infty + \parallel \xi - \eta \parallel_\infty .$$

Clearly, (X, S) is a complete S-ms. We accomplish (X, S) with the graph $G = (X, E)$ such that $E \subseteq X \times X \times X$ is defined by

$$(\zeta, \xi, \eta) \in E \ \text{ if and only if } \ \zeta(r) \leq \xi(r) \leq \eta(r) \ \text{ for all } r \in [a, b].$$

By Example 3.6, (G, S) is regular.
Now, consider the given operator $\mu : X \to X$,

$$\mu(\zeta)(r) = \int_a^r \int_a^v q(r, v, w)\ g(r, v, \zeta(w))\ dw\ dv + k(r), \quad r \in [a, b]. \tag{4.2}$$

Each fixed point of μ is a solution of the integral equation (4.1).
For this purpose, we state the following theorem.

Theorem 4.1. *Suppose* $\mu : X \to X$ *is the integral operator given by* (4.2) *and assume*

(i) There exists $\omega \in X$ *such that* $(\omega, \omega, \mu(\omega)) \in E$;

(ii) The mapping $g(r, v, \cdot) : \mathbb{R} \to \mathbb{R}$ *is increasing, for every* $r, v \in [a, b]$;

(iii) For all $\zeta, \xi \in X$ with $(\zeta, \zeta, \xi) \in E$, and all $r, v \in [a, b]$ we have

$$g(r, v, \xi(w)) - g(r, v, \zeta(w)) \leq \frac{1}{8} \ln [1 + 2(\xi(v) - \zeta(v))];$$

(iv) $\sup_{r \in [a,b]} \left| \int_a^b \int_a^b q(r, v, w) dw \ dv \right| \leq 1.$

Then μ has a fixed point.

Proof. By assumption (ii), μ is increasing and it follows that μ is G-monotone. By assumption (iii), for all $\zeta, \xi \in X$ with $(\zeta, \zeta, \xi) \in E$, we have

$$\begin{aligned} |\mu(\xi)(r) - \mu(\zeta)(r)| &= \left| \int_a^r \int_a^v q(r, v, w)[g(r, v, \xi(w)) - g(r, v, \zeta(w))] \ dw \ dv \right| \\ &= \int_a^r \int_a^v q(r, v, w)[g(r, v, \xi(w)) - g(r, v, \zeta(w))] \ dw \ dv \\ &\leq \int_a^r \int_a^v q(r, v, w) \frac{\ln (1 + 2(\xi(v) - \zeta(v))}{8} \ dw \ dv \\ &\leq \frac{1}{8} \ln (1 + 2\|\zeta - \xi\|_\infty). \end{aligned}$$

So, for all $\zeta, \xi \in X$ with $(\zeta, \zeta, \xi) \in E$, it follows that

$$S_H(\{\mu\zeta\}, \{\mu\zeta\}), \{\mu\xi\}) = 2S(\mu\zeta, \mu\zeta, \mu\xi) = 4\|\mu\zeta - \mu\xi\|_\infty \leq \frac{1}{2} \ln (1 + S(\zeta, \zeta, \xi)) = \theta(S(\zeta, \zeta, \xi)),$$

where $\theta(r) = \frac{1}{2} \ln (1 + r)$. Set $\lambda(\zeta) = 0$ for each $\zeta \in X$. Hence, by Theorem 3.8, μ has a fixed point. □

Example 4.2. *In the integral equation (4.1) assume $X = C([0, 1], \mathbb{R}^+)$.*
for all $r, t_1, t_2, v \in [0, 1]$, $r_1 \in [0, \infty)$, $\alpha \in (0, \infty)$ and $w \in [0, \alpha]$,
$g : [0, 1] \times [0, 1] \times [0, \alpha] \longrightarrow \mathbb{R}$ is defined by
$g(r, v, w) = \cos r + e^{v^2} - \dfrac{1}{8} \arctan \frac{1}{1+w} + \frac{\pi}{2}$, $q(t_1, t_2, v) = \sin \frac{v}{1+t_1+t_2}$. *For all $\zeta, \xi \in X$ with $(\zeta, \zeta, \xi) \in E$ we have:*

$$\begin{aligned} g(r, v, \xi(w)) - g(r, v, \zeta(w)) &= \frac{1}{8}[\arctan \frac{1}{1 + \zeta(w)} - \arctan \frac{1}{1 + \xi(w)}] \\ &\leq \frac{1}{8} \ \frac{\frac{1}{1+\zeta(w)} - \frac{1}{1+\xi(w)}}{1 + \frac{1}{(1+\xi(w))^2}} \leq \frac{1}{8} \ \frac{\xi(w) - \zeta(w)}{1 + \xi(w) - \zeta(w)} \\ &\leq \frac{1}{8} \ ln[1 + 2(\xi(v) - \zeta(v))]. \end{aligned}$$

So, all conditions of Theorem 4.1 are satisfied. Therefore, the following integral equation for $k \in X$ has a solution.

$$\zeta(r) = \int_0^r \int_0^v \sin \frac{w}{1 + r + v} \ (\cos r + e^{v^2} - \frac{1}{8} \arctan \frac{1}{1 + \zeta(w)} + \frac{\pi}{2}) \ dw \ dv + k(r), \quad r \in [0, 1].$$

5 Conclusions

We generalized some related results in literature. Theorem 3.1 is a extension of Theorem 3.4 of Tayyab Kamran, 2004. Theorem 3.8 and Theorem 4.1 are extensions of Theorem 2.3 and Theorem 3.2 of Tayyab kamran, Calogero Vetro, Muhammad Usman Ali and Mehwish Waheed, 2017, for multi-valued and single-valued mappings on S_H-metric and S-ms, respectively.

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