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Decomposition theorems and extension principle for complex fuzzy sets

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Abstract: Complex fuzzy set was originally proposed as a mathematical tool to deal with uncertainty by taking amplitude term and phase term memberships of an element of a universal set. In this article, we study (α,θ)-cut sets of the complex fuzzy sets and describe some related properties of them. Based on these (α,θ)-cut sets, some decomposition theorems of the complex fuzzy sets are proposed. Moreover, the concept of Zadeh's extension principle of the fuzzy sets is extended to the complex fuzzy sets and explored various related properties. Finally, some arithmetic operations are demonstrated for the complex fuzzy set by using the extension principle of the complex fuzzy sets. Numerical illustrations for each arithmetic operation are also given.

Keywords: Complex fuzzy set; (α,θ)-cut; Arithmetic Operations
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1 Introduction

In our daily life, there are some data which are hazy than precise. To give modeling these a host of researchers have become involved recently. Fuzzy set theory [12] which was first concept to deal with haziness by allowing partial membership. Fuzzy set theory confine real data, but there are various haziness complex data which cannot describe by fuzzy set theory. To describe such haziness complex data, the notion of complex fuzzy set was introduced by Ramot et al. [7, 8] and a comprehensive study of the mathematical properties of the complex fuzzy set was presented. Basic set theoretic operations on complex fuzzy sets, such as complex fuzzy complement, union, and intersection, are discussed in detail. Two innovative operations, namely set rotation and set reflection, were also introduced. Zhang et al. [14] investigates various operation properties and proposes a distance measure for complex fuzzy sets. The structural properties of complex fuzzy sets are also studied by (see [1-6, 9-11, 13]).
In this article, some structural properties of complex fuzzy set are developed. First, the (α,θ)-cut sets are studied with some related properties of them. By using the(α,θ)-cut sets, some decomposition theorems of complex fuzzy sets are established. Then Zadeh extension principle for fuzzy set is extended into complex fuzzy sets and describes some related properties. Finally, some arithmetic operations of complex fuzzy set are illustrated with the help of the extension principle of complex fuzzy set.

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The paper is organized as follows: In section 2 some preliminaries definitions are given which are useful in rest of the paper. In section 3, the (α,θ)-cut sets are described with some related properties. The decomposition theorems are also described here. In section 4, the extension principle for complex fuzzy sets is established with various properties. In section 5, the arithmetic operations for complex fuzzy sets are illustrated with numerical examples.

2 Preliminaries

In this section, we recall some basic definitions for complex fuzzy sets which are used in later sections.

Definition 2.1. [12] *Let X be non-empty set. A fuzzy set A in X is given by*

$$A = \{(x, \mu_A(x)) : x \in X\}, \text{ where } \mu_A : X \to [0,1].$$

Definition 2.2. [7] *A complex fuzzy set (CFS), defined on a universal set X, is characterized by a membership function $\mu_A(x)$ that assigns any element a complex-valued grade of membership in A. By definition, the values $\mu_A(x)$ may receive all lie within the unit circle in the complex plane, and are thus of the form $r_{A(x)}.\, e^{i\omega_A(x)}$, where $i = \sqrt{-1}$, $r_{A(x)}$ and $\omega_A(x)$ are both real-valued, and $r_A(x) \in [0,1]$ and $\omega_A(x) \in [0,2\pi]$. The complex fuzzy set may be represented as the set of ordered pairs*

$$A = \{(x, \mu_A(x)) : x \in X\} = \{x, r_A(x).\, e^{i\omega_A(x)}) : x \in X\}.$$

Where $'.'$ denotes algebraic product.

The set of all complex fuzzy sets in Xwill be denoted by $CFS(X)$.

Definition 2.3. [7] *Let $A, B \in CFS(X)$, then the subset, equality, union, intersection and complementare defined as follows:*

1. $A \subseteq B$ *iff* $\forall x \in X, r_A(x) \le r_B(x)$, *and* $\omega_A(x) \le \omega_B(x)$;
2. $A = B$ *iff* $\forall x \in X, r_A(x) = r_B(x)$, *and* $\omega_A(x) = \omega_B(x)$;
3. $A \cup B = \{(x, r_{A\cup B}(x) \cdot e^{i\omega_{A\cup B}(x)}) : x \in X\} = \{(x, max(\mu_A(x), \mu_B(x)) \cdot e^{imax(\omega_A(x), \omega_B(x))}) : x \in X\}$;
4. $A \cap B = \{(x, r_{A\cap B}(x) \cdot e^{i\omega_{A\cap B}(x)}) : x \in X\} = \{(x, min(\mu_A(x), \mu_B(x)) \cdot e^{imin(\omega_A(x), \omega_B(x))}) : x \in X\}$;
5. $A^c = \{(x, r_{A^c}(x) \cdot e^{i\omega_{A^c}(x)}) : x \in X\} = \{(x, (1 - r_A(x)) \cdot e^{2\pi - \omega_A(x)}) : x \in X\}$.

3 Decomposition Theorems of Complex Fuzzy Sets

In this section, the concept of (α,θ)-cut and strong (α,θ)-cut of complex fuzzy sets are introduced and some of their properties are described. Decomposition theorems by these (α,θ)-cuts and level set are also established. Throughout this article, we denote $\wedge$ for minimum operator and $\vee$ for maximum operator.

Definition 3.1. *Let $A = \{(x, \mu_A(x)) : x \in X\} = \{x, r_A(x).\, e^{i\omega_A(x)}) : x \in X\}$ be a complex fuzzy set on X and $\alpha \in [0,1]$ and $\theta \in [0,2\pi]$, then the (α,θ)-cut of A is given by*

$$A_{(\alpha,\theta)} = \{x \in X :\ r_A(x) \ge \alpha\,, \omega_A(x) \ge \theta\}.$$

That is, $A_\alpha = \{x\colon r_A(x) \geq \alpha\}$, *and* $A_\theta = \{x\colon \omega_A(x) \geq \theta\}$ *are* α *and* θ*- cuts of amplitude membership and phase membership of a complex fuzzy set A respectively.*

Example 3.1 (a). *Let* $X = \{-1, 0, 1, 2, 3\}$ *and*

$A = \{(-1, 0.3e^{i0.4\pi}), (0, 0.2e^{1.3\pi}), (1, 0.4e^{i0.9\pi}), (2, 0.6e^{i\pi}), (3, 0.5e^{i1.6\pi})\}$ *be a complex fuzzy set in* X.

Let $\alpha = 0.4,\ \theta = 0.8\pi$, *then*

$$A_{(0.4,1.2\pi)} = \{x \in X\colon\ r_A(x) \geq 0.4\ , \omega_A(x) \geq 0.8\pi\} = \{1,2,3\}.$$

Definition 3.2. *Let* $A = \{(x, \mu_A(x)) \colon x \in X\} = \{x, r_A(x).\, e^{i\omega_A(x)}) \colon x \in X\}$ *be a complex fuzzy set on* X *and* $\alpha \in [0,1]$ *and* $\theta \in [0, 2\pi]$, *then the strong* (α, θ)*-cut of A is given by*

$$A_{(\alpha,\theta)+} = \{x \in X\colon\ r_A(x) > \alpha\ , \omega_A(x) > \theta\}.$$

That is, $A_\alpha = \{x\colon r_A(x) > \alpha\}$, *and* $A_\theta = \{x\colon \omega_A(x) > \theta\}$ *are strong* α *and* θ*- cuts of amplitude membership and phase membership of a complex fuzzy set A respectively.*

Example 3.2 (a). *Let* $X = \{-1, 0, 1, 2, 3\}$ *and*

$A = \{(-1, 0.3e^{i0.4\pi}), (0, 0.2e^{1.3\pi}), (1, 0.4e^{i0.9\pi}), (2, 0.6e^{i\pi}), (3, 0.5e^{i1.6\pi})\}$ *be a complex fuzzy set in* X.

Let $\alpha = 0.4,\ \theta = 1.2\pi$, *then*

$$A_{(0.4,1.2\pi)} = \{x \in X\colon\ r_A(x) \geq 0.4\ , \omega_A(x) \geq 1.2\pi\} = \{2, 3\}.$$

Theorem 3.3. *Let* $A, B \in CFS(X)$, *then* $A \subseteq B$ *implies*

1. $A_{(\alpha,\theta)} \subseteq B_{(\alpha,\theta)}$,
2. $A_{(\alpha,\theta)+} \subseteq B_{(\alpha,\theta)+}$.

Proof: 1. Let $x \in A_{(\alpha,\theta)}$, then

$$r_A(x) \geq \alpha \text{ and } \omega_A(x) \geq \theta$$

As $A \subseteq B$, we have

$$\mu_B(x) \geq \mu_A(x) \geq \alpha\ , \omega_B(x) \geq \omega_A(x) \geq \theta$$

$$\Rightarrow\ \mu_B(x) \geq \alpha\ , \omega_B(x) \geq \theta$$

$$\Rightarrow x \in B_{(\alpha,\theta)}.$$

$$\therefore\ A_{(\alpha,\theta)} \subseteq B_{(\alpha,\theta)}.$$

2. Proof is similar to 1.

Theorem 3.4. *Let* $A, B \in PFS(X)$, *then*

1. $(A \cap B)_{(\alpha,\theta)} = A_{(\alpha,\theta)} \cap B_{(\alpha,\theta)}$,
2. $(A \cap B)_{(\alpha,\theta)+} = A_{(\alpha,\theta)+} \cap B_{(\alpha,\theta)+}$,
3. $(A \cup B)_{(\alpha,\theta)} \supseteq A_{(\alpha,\theta)} \cup B_{(\alpha,\theta)}$;
4. $(A \cup B)_{(\alpha,\theta)+} \supseteq A_{(\alpha,\theta)} \cup B_{(\alpha,\theta)+}$,
5. $(A \cup B)_{(\alpha,\theta)} \supseteq A_{(\alpha,\theta)} \cap B_{(\alpha,\theta)}$,

6. $(A \cup B)_{(\alpha,\theta)+} \supseteq A_{(\alpha,\theta)} \cap B_{(\alpha,\theta)+}$.

Proof.1. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, so from the theorem 3.3 we have,

$(A)_{(\alpha,\theta)} \subseteq (A \cup B)_{(\alpha,\theta)}$ and $B_{(\alpha,\theta)} \subseteq (A \cup B)_{(\alpha,\theta)}$

$\Rightarrow (A \cup B)_{(\alpha,\theta)} \subseteq A_{(\alpha,\theta)} \cap B_{(\alpha,\theta)}$.

Again, let $x \in A_{(\alpha,\theta)} \cap B_{(\alpha,\theta)}$

$\Rightarrow x \in A_{(\alpha,\theta)}$ and $x \in B_{(\alpha,\theta)}$

$\Rightarrow\ r_A(x) \geq \alpha\ ,\ r_B(x) \geq \alpha \Rightarrow min\{\ r_A(x),\ r_B(x)\} \geq \alpha$

$\omega_A(x) \geq \theta\ , \omega_B(x) \geq \theta \Rightarrow min\{\omega_A(x), \omega_B(x)\} \geq \theta$

$\Rightarrow x \in (A \cap B)_{(\alpha,\theta)}$.

Therefore, $A_{(\alpha,\theta)} \cap B_{(\alpha,\theta)} \subseteq (A \cap B)_{(\alpha,\theta)}$.

Hence, $(A \cap B)_{(\alpha,\theta)} = A_{(\alpha,\theta)} \cap B_{(\alpha,\theta)}$.

3. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, so from the theorem 3.3 we have,

$A_{(\alpha,\theta)} \subseteq (A \cup B)_{(\alpha,\theta)}$ and $B_{(\alpha,\theta)} \subseteq (A \cup B)_{(\alpha,\theta)}$

$\Rightarrow (A \cup B)_{(\alpha,\theta)} \supseteq A_{(\alpha,\theta)} \cup B_{(\alpha,\theta)}$.

Theorem 3.5. *Let $A, B \in PFS(X)$, then $A\ = B$ implies*

1. $A_{(\alpha,\theta)} = B_{(\alpha,\theta)}$,
2. $A_{(\alpha,\theta)+} = B_{(\alpha,\theta)+}$.

Proof: Trivial

Theorem 3.6. *Let $A \in PFS(X)$.*

1. *If $\alpha_1 \leq \alpha_2$, $\theta_1 \leq \theta_2$, then $A_{(\alpha_1,\theta_1)} \subseteq A_{(\alpha_2,\theta_2)}$;*
2. *If $\alpha_1 \leq \alpha_2$, $\theta_1 \leq \theta_2$, then $A_{(\alpha_1,\theta_1)+} \subseteq A_{(\alpha_2,\theta_2)}+$.*

Definition 3.7. *Let $A = \{(x, \mu_A(x)) : x \in X\} = \{x, r_A(x). e^{i\omega_A(x)}) : x \in X\}$ be a complex fuzzy set on X. Then the $A^*_{(\alpha,\theta)}$ with respect to $A_{(\alpha,\theta)}$ is defined by*

$$A^*_{(\alpha,\theta)}(x) = \begin{cases} \alpha e^{i\theta} & ;\ x \in\ A_{(\alpha,\theta)} \\ 0e^{i0} & ;\ otherwise \end{cases}.$$

Definition 3.8. *Let $A = \{(x, \mu_A(x)) : x \in X\} = \{x, r_A(x). e^{i\omega_A(x)}) : x \in X\}$ be a complex fuzzy set on X. Then the $A^*_{(\alpha,\theta)+}$ with respect to $A_{(\alpha,\theta)+}$is defined by*

$$A^*_{(\alpha,\theta)+}(x) = \begin{cases} \alpha e^{i\theta} & ;\ x \in\ A_{(\alpha,\theta)+} \\ 0 & ; otherwise \end{cases}.$$

Definition 3.9. *Let A be a complex fuzzy set, then the level set for amplitude membership and phase membership are defined as*

$$\curlywedge (A) = \left\{\alpha e^{i\theta} \colon\ r_A(x) = \alpha\ , \alpha \in [0,1]\ and\ \ \omega_A(x) = \theta\ , \theta \in [0,2\pi]\right\}.$$

Theorem 3.10. *(First decomposition theorem): Let X be a non-empty set. For a complex fuzzy set,* $A = \left\{\left(x, \mu_A(x)\right) \colon x \in X\right\} = \left\{x, r_A(x). e^{i\omega_A(x)}\right) \colon x \in X\right\}$ *in* X,

$$A = \bigcup_{\substack{\alpha\in[0,1] \\ \theta\in[0,2\pi]}} A^*_{(\alpha,\theta)}.$$

Proof. Let x be an arbitrary element in X and let $r_A(x) = a$ and $\omega_A(x) = b$.

Then

$$\bigcup_{\substack{\alpha\in[0,1] \\ \theta\in[0,2\pi]}} A^*_{(\alpha,\theta)} = \sup_{\substack{\alpha\in[0,1] \\ \theta\in[0,2\pi]}} A^*_{(\alpha,\theta)} = max\left[\sup_{\substack{\alpha\in[0,a] \\ \theta\in[0,b]}} A^*_{(\alpha,\theta)}, \sup_{\substack{\alpha\in(a,1] \\ \theta\in(b,2\pi]}} A^*_{(\alpha,\theta)}\right]$$

For each$\alpha \in (a, 1]$ and $\theta \in (b, 2\pi]$, we have $r_A(x) = a < \alpha$ and $\omega_A(x) = b < \theta$.

Therefore, $A^*_{(\alpha,\theta)} = 0e^{i0}$.

On the other hand, for each $\alpha \in [0\ , a]$ and $\theta \in [0\ , b]$ we have $r_A(x) = a \geq \alpha$ and $\omega_A(x) = b \geq \theta$.

Therefore, $A^*_{(\alpha,\theta)} = \alpha e^{i\theta}$.

Thus, $\bigcup_{\substack{\alpha\in[0,1] \\ \theta\in[0,2\pi]}} A^*_{(\alpha,\theta)} = \sup_{\substack{\alpha\in[0,a] \\ \theta\in[0,b]}} A^*_{(\alpha,\theta)}$

$= ae^{i\theta}$

$= A.$

Example 3.10 (a). *Let A be any complex fuzzy set in X, given by*

$$A = \left\{\left(x_1, 0.5e^{i0.3\pi}\right), \left(x_2, 0.2, e^{i0.4\pi}\right), \left(x_3, 0.6, e^{i1.2\pi}\right), \left(x_4, 0.3, e^{i1.4\pi}\right)\right\}.$$

Let us denote A for convenience as

$$A = \frac{0.5e^{i0.3\pi}}{x_1} + \frac{0.2,e^{i0.4\pi}}{x_2} + \frac{0.6,e^{i1.2\pi}}{x_3} + \frac{0.3,e^{i1.4\pi}}{x_4}.$$

For $\alpha = 0.5$ *and* $\theta = 0.3\pi$, *the complex fuzzy sets* $A^*_{(0.5,0.3\pi)}$, *with respect to complex fuzzy set* $A_{(0.5,0.3\pi)}$ *is given by*

$$A^*_{(0.5,0.3\pi)} = \frac{0.5e^{i0.3\pi}}{x_1} + \frac{0e^{i0}}{x_2} + \frac{0.5e^{i0.3\pi}}{x_3} + \frac{0e^{i0}}{x_4},$$

Similarly,

for $\alpha = 0.2$ *and* $\theta = 0.4\pi$,

$$A^*_{(0.2,0.4\pi)} = \frac{0e^{i0}}{x_1} + \frac{0.2,e^{i0.4\pi}}{x_2} + \frac{0.2,e^{i0.4\pi}}{x_3} + \frac{0.2,e^{i0.4\pi}}{x_4},$$

for $\alpha = 0.6$ *and* $\theta = 1.2\pi$,

$$A^*_{(0.6,1.2\pi)} = \frac{0e^{i0}}{x_1} + \frac{0e^{i0}}{x_2} + \frac{0.6,e^{i1.2\pi}}{x_3} + \frac{0e^{i0}}{x_4},$$

for $\alpha = 0.3$ *and* $\theta = 1.4\pi$,

$$A^*_{(0.3,1.4\pi)} = \frac{0e^{i0}}{x_1} + \frac{0e^{i0}}{x_2} + \frac{0e^{i0}}{x_3} + \frac{0.3,e^{i1.4\pi}}{x_4}.$$

Consequently, $A = \bigcup_{\substack{\alpha\in[0,1] \\ \theta\in[0,2\pi]}} A^*_{(\alpha,\theta)}$.

Theorem 3.11. *(Second decomposition theorem): Let* X *be a non-empty set. For a complex fuzzy set* $A = \{(x,\mu_A(x)): x\in X\} = \{x, r_A(x).\, e^{i\omega_A(x)}): x \in X\}$ *in* X,

$$A = \bigcup_{\substack{\alpha\in[0,1] \\ \theta\in[0,2\pi]}} A^*_{(\alpha,\theta)+}.$$

Proof. Let x be an arbitrary element in X and let $r_A(x) = a$ and $\omega_A(x) = b$.

Then

$$\begin{aligned}
\bigcup_{\substack{\alpha\in[0,1] \\ \theta\in[0,2\pi]}} A^*_{(\alpha,\theta)+} &= \sup_{\substack{\alpha\in[0,1] \\ \theta\in[0,2\pi]}} A^*_{(\alpha,\theta)+} \\
&= max\left[\sup_{\substack{\alpha\in[0,a) \\ \theta\in[0,b)}} A^*_{(\alpha,\theta)+},\ \sup_{\substack{\alpha\in[a,1] \\ \theta\in[b,2\pi]}} A^*_{(\alpha,\theta)+}\right] \\
&= \sup_{\substack{\alpha\in[0,a) \\ \theta\in[0,b)}} A^*_{(\alpha,\theta)+} \\
&= ae^{i\theta} \\
&= A.
\end{aligned}$$

Theorem 3.12. *(Third decomposition theorem): Let* X *be a non-empty set. For a complex fuzzy set* $A = \{(x,\mu_A(x)): x\in X\} = \{x, r_A(x).\, e^{i\omega_A(x)}): x \in X\}$ *in* X,

$$A = \bigcup_{\substack{\alpha\in\wedge(A) \\ \theta\in\wedge(A)}} A^*_{(\alpha,\theta)}.$$

Proof. Trivial.

4 Extension Principle for Complex Fuzzy Sets

Definition 4.1. *Let* X *and* Y *be two non-empty sets and* $f: X \to Y$ *be a mapping. Two mappings can be induced by* f *as the following:*

$$f: CFS(X) \to CFS(Y) \text{ and } f^{-1}: CFS(Y) \to CFS(X)$$

which are defined by

$$f(A)(y) = r_{f(A)}(y)e^{i\omega_{f(A)}(y)}, \text{ where } A \in CFS(X)$$

and

$$r_{f(A)}(y) = \begin{cases} \vee\{r_A(x): x \in f^{-1}(y)\} & ; f^{-1}(y) \neq \phi \\ 0 & ; Otherwise \end{cases},$$

$$\omega_{f(A)}(y) = \begin{cases} \vee\{\omega_A(x): x \in \bar{f}^{-1}(y)\} & ; f^{-1}(y) \neq \phi \\ 0 & ; Otherwise \end{cases}.$$

And

$$f^{-1}(B)(x) = \left(r_{f^{-1}(B)}(x)e^{i\omega_{f^{-1}(B)}(x)}\right), \text{ where } B \in PFS(Y)$$

and

$$r_{f^{-1}(B)}(x) = r_B(f(x)),$$

$$\omega_{f^{-1}(B)}(x) = \omega_B(f(x)).$$

Theorem 4.2. *Let $f: X \to Y$ and $A, B, A_i \in CFS(X)$, then induced mapping f satisfies that*

1. $A \subseteq B$ *implies* $f(A) \subseteq f(B)$,
2. $f(\cup_{i\in I}(A_i)) = \left(\cup_{i\in I}(f(A_i))\right)$,
3. $f(\cap_{i\in I}(A_i)) \subseteq \left(\cap_{i\in I}(f(A_i))\right)$,
4. $f(A_{(\alpha,\theta)}) \subseteq (f(A))_{(\alpha,\theta)}$,
5. $f(A_{(\alpha,\theta)+}) = (f(A))_{(\alpha,\theta)+}$.

Proof.

1. It is trivial.
2. For each $y \in Y$, if $f^{-1}(y)$ is not empty, then

$$f(\cup_{i\in I}(A_i))(y) = r_{f(\cup_{i\in I}A_i)}(y)e^{i\omega_{f(\cup_{i\in I}A_i)}(y)},$$

where

$$\begin{aligned} r_{f(\cup_{i\in I}A_i)}(y) &= \vee_{x\in f^{-1}(y)}\{r_{\cup_{i\in I}A_i}(x)\} \\ &= \vee_{x\in f^{-1}(y)}\{\vee_{i\in I}\{r_{A_i}(x)\}\} \\ &= \vee_{i\in I}\{\vee_{x\in f^{-1}(y)}\{r_{A_i}(x)\}\} \\ &= \vee_{i\in I}\{r_{f(A_i)}(y)\} \end{aligned}$$

$\therefore r_{f_{(\cup_{i\in I}A_i)}}(y) = r_{\cup_{i\in I}f(A_i)}(y).$

Again,

$$\begin{aligned} \omega_{f(\cup_{i\in I}A_i)}(y) &= \vee_{x\in f^{-1}(y)}\{\omega_{\cup_{i\in I}A_i}(x)\} \\ &= \vee_{x\in f^{-1}(y)}\{\vee_{i\in I}\{\omega_{A_i}(x)\}\} \end{aligned}$$

$= \vee_{i\in I}\{\vee_{x\in f^{-1}(y)}\{\omega_{A_i}(x)\}\}$

$= \vee_{i\in I}\{\omega_{f(A_i)}(y)\}$

$\therefore \omega_{f(\cup_{i\in I}A_i)}(y) = \omega_{\cup_{i\in I}f(A_i)}(y)$.

Hence, $f(\cup_{i\in I}(A_i)) = \left(\cup_{i\in I}(f(A_i))\right)$.

3. Similar proof of 2.
4. Let $y \in f(A_{(\alpha,\theta)})$, if $f^{-1}(y)$ is not empty, then there exists $x \in A_{(\alpha,\theta)}$ such that $f(x) = y$ and $r_A(x) \geq \alpha$, $\omega_A(x) \geq \theta$

$\Rightarrow \vee\{r_A(x): x \in \bar{f}^{-1}(y)\} \geq \alpha$, $\vee\{\omega_A(x): x \in \bar{f}^{-1}(y)\} \geq \theta$

i,e $r_{f(A)}(y) \geq \alpha$, $\omega_{f(A)}(y) \geq \theta$

$\Rightarrow y \in (f(A))_{(\alpha,\theta)}$.

$\therefore (A_{(\alpha,\theta)}) \subseteq (f(A))_{(\alpha,\theta)}$.

5. For all $y \in Y$, $y \in (f(A))_{(\alpha,\theta)+}$

$\Leftrightarrow r_{f(A)}(y) > \alpha$, $\omega_{f(A)}(y) > \theta$
$\Leftrightarrow \vee_{x\in f^{-1}(y)} r_A(x) > \alpha$, $\vee_{x\in f^{-1}(y)} \omega_A(x) > \theta$
$\Leftrightarrow (\exists x_0 \in X)(y = f(x_0))$ and $r_A(x_0) > \alpha$, $\omega_A(x_0) \geq \theta$
$\Leftrightarrow (\exists x_0 \in X)(y = f(x_0))$ and $x_0 \in A_{(\alpha,\theta)+}$

$\Leftrightarrow y \in f(A_{(\alpha,\theta)+})$.

Thus, $f(A_{(\alpha,\theta)+}) = (f(A))_{(\alpha,\theta)+}$.

Theorem 4.3. *Let $f: X \to Y$ and $A, B, B_i \in PFS(Y)$, then induced mapping f^{-1} satisfies that*

1. $A \subseteq B$ *implies* $f^{-1}(A) \subseteq f^{-1}(B)$,
2. $f^{-1}(B^c) = (f^{-1}(B))^c$,
3. $f^{-1}(\cup_{i\in I}(B_i)) = (\cup_{i\in I} f^{-1}(B_i))$,
4. $f^{-1}(\cap_{i\in I}(B_i)) = (\cap_{i\in I} f^{-1}(B_i))$,
5. $f^{-1}(B_{(\alpha,\theta)}) = (f^{-1}(B))_{(\alpha,\theta)}$,
6. $f^{-1}(B_{(\alpha,\theta)+}) = (f^{-1}(B))_{(\alpha,\theta)+}$.

Proof. The proofs of 1 and 2 are trivial.

3. For all $x \in X$, we have

$$f^{-1}(\cup_{i\in I}(B_i))(x) = r_{f^{-1}(\cup_{i\in I}B_i)}(x)e^{i\omega_{f^{-1}(\cup_{i\in I}B_i)}(x)}$$

where

$r_{f^{-1}(\cup_{i\in I}B_i)}(x) = r_{\cup_{i\in I}B_i}(f(x))$

$= \vee_{i\in I}\{r_{B_i}(f(x))\}$

$= \vee_{i\in I}\{r_{f^{-1}(B_i)}(x)\}$

$\therefore r_{f^{-1}{}_{(\cup_{i\in I}B_i)}}(x) = r_{\cup_{i\in I}f^{-1}(B_i)}(x)$.

Again,

$\omega_{f^{-1}(\cup_{i\in I}B_i)}(x) = \omega_{\cup_{i\in I}B_i}(f(x))$

$= \vee_{i\in I}\{\omega_{B_i}(f(x))\}$

$= \vee_{i\in I}\{\omega_{f^{-1}(B_i)}(x)\}$

$\therefore \omega_{f^{-1}{}_{(\cup_{i\in I}B_i)}}(x) = \omega_{\cup_{i\in I}f^{-1}(B_i)}$.

Hence, $f^{-1}(\cup_{i\in I}(B_i)) = (\cup_{i\in I} f^{-1}(B_i))$.

4. Similar proof of the proof 3.

5. For all $x \in X$, we have,
$x \in (f^{-1}(B))_{(\alpha,\theta)}$
$\Leftrightarrow \{x \in X: r_{f^{-1}(B)}(x) \geq \alpha, \omega_{f^{-1}(B)}(x) \geq \theta\}$
$\Leftrightarrow \{x \in X: r_B(f(x)) \geq \alpha, \omega_B(f(x)) \geq \theta\}$
$\Leftrightarrow \{x \in X: f(x) \in B_{(\alpha,\theta)}\}$.

Therefore, $x \in f^{-1}(B_{(\alpha,\theta)})$.

Thus, $f^{-1}(B_{(\alpha,\theta)}) = (f^{-1}(B))_{(\alpha,\theta)}$.

6. Similar proof of 5

5 Arithmetic Operations on Complex Fuzzy Sets

In this section, arithmetic operations for complex fuzzy sets by extension principle are described.

Definition 5.1. *Let $A, B \in CFS(X)$. Then $A * B$ (where $* \in (+, -, ., \div)$) is defined by*

$$A * B = \{z, r_{A*B}(z)e^{i\omega_{A*B}(z)}\},$$

where

$$r_{A*B}(z) = \vee_{z=x*y}[r_A(x) \wedge r_B(y)]$$

and

$$\omega_{A*B}(z) = \vee_{z=x*y}[\omega_A(x) \wedge \omega_B(y)].$$

Definition 5.2. *(Addition operation) Let* $A, B \in CFS(X)$, *then*

$$A + B = \{z, r_{A+B}(z)e^{i\omega_{A+B}(z)}\},$$

where

$$r_{A+B}(z) = \vee_{z=x+y}[r_A(x) \wedge r_B(y)]$$

and

$$\omega_{A+B}(z) = \vee_{z=x+y}[(\omega_A(x) \wedge \omega_B(y))].$$

Example 5.2(a). *Let* $A, B \in CFS(X)$, *where*

$$A = \{(1, 0.5e^{i0.6\pi}), (2, 0.4e^{i1.2\pi})\}$$

and

$$B = \{(2, 0.5e^{i0.8\pi}), (3, 0.2e^{i1.8\pi}), (4, 0.6e^{i0.5\pi})\}.$$

Therefore,

$$A + B = \{(1 + 2, min(0.5,0.5)e^{imin(0.6\pi,0.8\pi)}),$$

$$(1 + 3, min(0.5,0.2)e^{imin(0.6\pi,1.8\pi)}),$$

$$(1 + 4, min(0.5,0.6)e^{imin(0.6\pi,0.5\pi)}),$$

$$(2 + 2, min(0.4,0.5)e^{imin(1.2\pi,0.8\pi)}),$$

$$(2 + 3, min(0.4,0.2)e^{imin(1.2\pi,1.8\pi)}),$$

$$(2 + 4, min(0.4,0.6)e^{imin(1.2\pi,0.5\pi)})\}$$

$$= \{(3, 0.5e^{i0.6\pi}), (4, 0.2e^{i0.6\pi}), (5, 0.5e^{i0.5\pi}), (4, 0.4e^{i0.8\pi}), (5, 0.2e^{i1.2\pi}), (6, 0.4e^{i0.5\pi})\}$$

$$= \{(3, 0.5e^{i0.6\pi}), (4, max(0.2, 04)e^{imax(0.6\pi,0.8\pi)}), (5, max(0.5, 0.2)e^{imax(0.5\pi,1.2\pi)}), (6, 0.4e^{i0.5\pi})\}$$

$$= \{(3, 0.5e^{i0.6\pi}), (4, 0.4e^{i0.8\pi}), (5, 0.5e^{i1.2\pi}), (6, 0.4e^{i0.5\pi})\}.$$

Definition 5.3. *(Subtraction operation) Let* $A, B \in CFS(X)$, *then*

$$A - B = \{z, r_{A-B}(z)e^{i\omega_{A-B}(z)}\},$$

where

$$r_{A-B}(z) = \vee_{z=x-y}[r_A(x) \wedge r_B(y)]$$

and

$$\omega_{A-B}(z) = \vee_{z=x-y}[\omega_A(x) \wedge \omega_B(y)].$$

Example 5.3 (a). *Let* $A, B \in CFS(X)$, *where*

$$A = \{(1, 0.5e^{i0.6\pi}), (2, 0.4e^{i1.2\pi})\}$$

and

$$B = \{(2, 0.5e^{i0.8\pi}), (3, 0.2e^{i1.8\pi}), (4, 0.6e^{i0.5\pi})\}.$$

Therefore,

$$A - B = \{(1-2, min(0.5,0.5)e^{imin(0.6\pi,0.8\pi)}),$$
$$(1-3, min(0.5,0.2)e^{imin(0.6\pi,1.8\pi)}),$$
$$(1-4, min(0.5,0.6)e^{imin(0.6\pi,0.5\pi)}),$$
$$(2-2, min(0.4,0.5)e^{imin(1.2\pi,0.8\pi)}),$$
$$(2-3, min(0.4,0.2)e^{imin(1.2\pi,1.8\pi)}),$$
$$(2-4, min(0.4,0.6)e^{imin(1.2\pi,0.5\pi)})\}$$

$$= \{(-1, 0.5e^{i0.6\pi}), (-2, 0.2e^{i0.6\pi}), (-3, 0.5e^{i0.5\pi}), (0, 0.4e^{i0.8\pi}), (-1, 0.2e^{i1.2\pi}), (-2, 0.4e^{i0.5\pi})\}$$
$$= \{(-1, max(0.5, 0.2)e^{imax(0.6\pi,1.2\pi)}), (-2, max(0.2, 0.4)e^{imax(0.6\pi 0.5\pi)}), (-3, 0.5e^{i0.5\pi}), (0, 0.4e^{i0.8\pi})\}$$
$$= \{(-1, 0.5e^{i1.2\pi}), (-2, 0.4e^{i0.6\pi}), (-3, 0.5e^{i0.5\pi}), (0, 0.4e^{i0.8\pi})\}.$$

Definition 5.4. *(Multiplication operation) Let $A, B \in CFS(X)$, then*

$$A \cdot B = \{z, r_{A \cdot B}(z)e^{i\omega_{A \cdot B}(z)}\},$$

where

$$r_{A \cdot B}(z) = \vee_{z=x \cdot y}[r_A(x) \wedge r_B(y)]$$

and

$$\omega_{A \cdot B}(z) = \vee_{z=x \cdot y}[\omega_A(x) \wedge \omega_B(y)].$$

Example 5.4(a). *Let $A, B \in CFS(X)$, where*

$$A = \{(1, 0.5e^{i0.6\pi}), (2, 0.4e^{i1.2\pi})\}$$

and

$$B = \{(2, 0.5e^{i0.8\pi}), (3, 0.2e^{i1.8\pi}), (4, 0.6e^{i0.5\pi})\}.$$

Therefore,

$$A \cdot B = \{(1 \cdot 2, min(0.5,0.5)e^{imin(0.6\pi,0.8\pi)}),$$
$$(1 \cdot 3, min(0.5,0.2)e^{imin(0.6\pi,1.8\pi)}),$$
$$(1 \cdot 4, min(0.5,0.6)e^{imin(0.6\pi,0.5\pi)}),$$
$$(2 \cdot 2, min(0.4,0.5)e^{imin(1.2\pi,0.8\pi)}),$$

$$\left(2\cdot 3, min(0.4{,}0.2)e^{imin(1.2\pi,1.8\pi)}\right),$$

$$\left(2\cdot 4, min(0.4{,}0.6)e^{imin(1.2\pi,0.5\pi)}\right)\}$$

$$=\left\{\left(2, 0.5e^{i0.6\pi}\right), \left(3, 0.2e^{i0.6\pi}\right), \left(4, 0.5e^{i0.5\pi}\right), \left(4, 0.4e^{i0.8\pi}\right), \left(6, 0.2e^{i1.2\pi}\right), \left(8, 0.4e^{i0.5\pi}\right)\right\}$$

$$=\left\{\left(2, 0.5e^{i0.6\pi}\right), \left(3, 0.2e^{i0.6\pi}\right), \left(4, max(0.5, 0.4)e^{imax(0.5\pi,0.8\pi)}\right), \left(6, 0.2e^{i1.2\pi}\right), \left(8, 0.4e^{i0.5\pi}\right)\right\}$$

$$=\left\{\left(2, 0.5e^{i0.6\pi}\right), \left(3, 0.2e^{i0.6\pi}\right), \left(4, 0.5e^{i0.8\pi}\right), \left(6, 0.2e^{i1.2\pi}\right), \left(8, 0.4e^{i0.5\pi}\right)\right\}.$$

Definition 5.5. *(Division operation) Let $A, B \in CFS(X)$, then*

$$A \div B = \left\{z, r_{A\div B}(z)e^{i\omega_{A\div B}(z)}\right\},$$

where

$$r_{A\div B}(z) = \vee_{z=x\div y}[r_A(x) \wedge r_B(y)]$$

and

$$\omega_{A\div B}(z) = \vee_{z=x\div y}[\omega_A(x) \wedge \omega_B(y)].$$

Example 5.5 (a). *Let $A, B \in CFS(X)$, where*

$$A = \left\{\left(1, 0.5e^{i0.6\pi}\right), \left(2, 0.4e^{i1.2\pi}\right)\right\}$$

and

$$B = \left\{\left(2, 0.5e^{i0.8\pi}\right), \left(3, 0.2e^{i1.8\pi}\right), \left(4, 0.6e^{i0.5\pi}\right)\right\}.$$

Therefore,

$$A + B = \{\left(1 \div 2, min(0.5{,}0.5)e^{imin(0.6\pi,0.8\pi)}\right),$$

$$\left(1 \div 3, min(0.5{,}0.2)e^{imin(0.6\pi,1.8\pi)}\right),$$

$$\left(1 \div 4, min(0.5{,}0.6)e^{imin(0.6\pi,0.5\pi)}\right),$$

$$\left(2 \div 2, min(0.4{,}0.5)e^{imin(1.2\pi,0.8\pi)}\right),$$

$$\left(2 \div 3, min(0.4{,}0.2)e^{imin(1.2\pi,1.8\pi)}\right),$$

$$\left(2 \div 4, min(0.4{,}0.6)e^{imin(1.2\pi,0.5\pi)}\right)\}$$

$$=\left\{\left(\frac{1}{2}, 0.5e^{i0.6\pi}\right), \left(\frac{1}{3}, 0.2e^{i0.6\pi}\right), \left(\frac{1}{4}, 0.5e^{i0.5\pi}\right), \left(1, 0.4e^{i0.8\pi}\right), \left(\frac{2}{3}, 0.2e^{i1.2\pi}\right), \left(\frac{1}{2}, 0.4e^{i0.5\pi}\right)\right\}$$

$$=\left\{\left(\frac{1}{2}, max(0.5, 0.4)e^{imax(0.6\pi,0.5\pi)}\right), \left(\frac{1}{3}, 0.2e^{i0.6\pi}\right), \left(\frac{1}{4}, 0.5e^{i0.5\pi}\right), \left(1, 0.4e^{i0.8\pi}\right), \left(\frac{2}{3}, 0.2e^{i1.2\pi}\right)\right\}$$

$$=\left\{\left(\frac{1}{2}, 0.5e^{i0.6\pi}\right), \left(\frac{1}{3}, 0.2e^{i0.6\pi}\right), \left(\frac{1}{4}, 0.5e^{i0.5\pi}\right), \left(1, 0.4e^{i0.8\pi}\right), \left(\frac{2}{3}, 0.2e^{i1.2\pi}\right)\right\}.$$

6 Conclusion

In science and engineering there are many problems which cannot be characterized by a single membership function of Zadeh's fuzzy sets. To overcome this problem, complex fuzzy sets play an important rule. But still now, the structural properties of complex fuzzy sets are not studied broadly. In this article, we have tried to develop some structural properties of complex fuzzy sets. It would be very helpful to reader as well as researcher for further developing and application of complex fuzzy sets. In future, we will consider the application of decomposition theorems and extension principle in decision making problems.

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