MATHEMATICS AND COMPUTATIONAL SCIENCES
Vol 4(4), 2023, pp:17-25
DOI:10.30511/MCS.2023.2004315.1126
ISSN:2717-2708
http://mcs.qut.ac.ir/
Qom University of Technology

A numerical method for solving non-linear volterra integro-differential equation of fractional order

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Abstract: In this paper, we develop and implement numerical method for the solution of non-linear Volterra integro-differential equations of fractional order using collocation approximation. We obtain the integral form of the problem and transform it into system of algebraic equations, we solve the algebraic equations using matrix inversion method. The analysis of the developed method is investigated and solution found to be q-contraction and convergent. The uniqueness of the solution also proven. Numerical examples were considered to test the efficiency of the method which shows that the method compete favourably with the existing methods.

Keywords: Collocation; Volterra; Integro-differential; Non-linear; Approximate solution; Fractional order.
2020 Mathematics Subject Classification: 65C30; 65L06; 65C03.
Receive: 8 June 2023, **Accepted:** 22 December 2023

1 Introduction

Fractional differential and integral equations are extremely important in mathematics, physics, chemistry and engineering. Mathematical modeling of real-life problems typically occurs in functional equations such as ordinary and partial differential equations. Vito Volterra developed a new type of equation known as Integro-Differential Equations (IDEs) in the early 1900s to study population growth phenomena. In such equations, one or more derivatives of the unknown function appear under the integral sign. Many mathematical formulations in physical phenomena contain integral differential equations (IDEs); these equations appear in modeling some phenomena in science and engineering [23].

Many methods for determining the numerical solution of integro-differential equations have recently been developed including the Adomian decompositions method by [17, 18], Collocation method by[5, 1, 19, 6], Hybrid non-linear multistep method [7, 8], Chebyshev-Galerkin method [15], Bernoulli matrix method [10], Differential transform method [13], Lagrange Interpolation [22], Bernstein Poly- nomials Method [14], Differential Transformation [12], Haar Wavelet operational matrices [21], Weight- ed Mean-Value theorem [9], Optimal Auxiliary Function Method (OAFM) [24], Block pulse functions operational matrices [20] and Homotopy Analysis Method[11]. [16] presented a new numerical method for solving fractional order Volterra integro-differential equations based on the Bernoulli wavelet approximation. The opera-

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tional matrix of fractional derivatives of order λ in the Caputo sense derived in conjunction with the Gaussian quadrature rule are used to reduce the Volterra integro-differential equations to a system of algebraic equations. The proposed method's convergence analysis and error estimation were discussed. [3] applied the usual collocation approach to first-order Volterra integro-differential equations. The class of integro-differential equations was reformulated to assume an approximate solution in terms of the constructed polynomial. After solving for the unknown, we obtained a system of linear algebraic equations by collocating the resulting equation at various places within the range [0,1]. Collocation approach for the computational solution of fredholm-volterra fractional order of integro-differential equations was presented by [4]. After obtaining the problem in linear integral form, they used typical collocation points to translate it into a set of linear algebraic equations.

In this paper, we consider non-linear Volterra integro-differential equation of fractional order of the form:

$$ {}_0^cD_x^{\alpha} y\left(x\right) = g\left(x\right) + \int_0^x K(x,t) F\left(y(t)\right) dt, \;\; x \in [0,1] \tag{1.1} $$

subject to boundary conditions

$$ y(0) = y_0, y(1) = y_1 \tag{1.2} $$

where ${}_0^cD_x^{\alpha}\left(.\right)$ is the left Caputo derivative operator; $K\left(x,t\right)$ is the Volterra integral kernel function. $g(x)$ is the known function and $y(x)$ is the unknown function

to be determined.

2 Basic Definitions

In this section, we present some definitions and fundamental ideas for the purpose of problem formulation.
Definition 1: Let $\left(a_m\right), m \geq 0$ be a sequence of real numbers. The power series in k with coefficients a_m is an expression.

$$ y(k) = a_0 + a_1 k + a_2 k^2 + a_3 k^3 + \cdots a_M k^M = \sum_{m=0}^{M} a_m k^m = \; \phi(k) \; \mathbf{A} \tag{2.1} $$

where $\phi(k) = [1 \quad k \quad k^2 \quad \cdots \quad k^M] \; and \; \mathbf{A} = [a_0 \quad a_1 \quad \cdots \quad a_M]^T$
Definition 2: The desired collocation points within an interval [a b] are determined using this method.

$$ k_i = a + \frac{(b-a)i}{M}, i = 1, 2, 3, ...M \tag{2.2} $$

Definition 3: Let $z(s)$ be an integrable function, then

$$ {}_0I_x^{\alpha}\left(z(s)\right) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} z(s) ds \tag{2.3} $$

Definition 4: Integration of nth derivatives:
For $\alpha > 0$, Let $y(x)$ be a continuous function, then

$$ {}_0I_x^{\beta}\left(y^{(\beta)} y(x)\right) = y(x) - \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k \tag{2.4} $$

Definition 5: The Caputo derivative with order $\alpha > 0$ of the given function $f(x), \; x \in (a,b)$ is defined as

$$ {}_x^CD_a^{\alpha} y(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-s)^{m-\alpha-1} y^{(m)}(s) ds $$

Definition 5: A metric on a set M is a function $d : M \times M \longrightarrow \mathbb{R}$ with the following properties. For all $x, y \in M$
(a) $d(x, y) \geq 0$;
(b) $d(x, y) = 0 \Longleftrightarrow x = y$
(c) $d(x, y) = d(y, x)$
(d) $d(x, y) \leq d(x, z) + d(x, y)$
If d is a metric on M, then the pair (M, d) is called a metric space.
Definition 6: Let (X, d) be a metric space, A mapping $T : X \longrightarrow X$ is Lipschitzian if there exists a constant $L > 0$ such that $d(Tx, Ty) \leq Ld(x, y) \ \forall \ x, y \in X$.

3 Mathematical Background

Here, we implement collocation approach for the numerical solution of nonlinear fredholm integro- differential equations.

Theorem 3.1. *(Banach Contraction Principle, [2]): Let (X, d) be a complete metric space, then each contraction mapping $T : X \rightarrow X$ has a unique fixed point x of T in X, such that $Tx = x$*

Lemma 3.2. *(Integral form): Let $y \epsilon \mathbb{C}\left((0,1), \mathbb{R}\right)$ be the solution to (1.1) with (1.2), then it is equivalent to*

$$y(x) = U(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} (
$$

$\int_0^x K(s,t)F\left(y(t)\right)dtds(3.1)$
where

$$U(x) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(s) ds$$

Proof. Getting ${}_0I_x^{\alpha}(.)$ operator from both side of equation (1.1) yields

$${}_0I_x^{\alpha}\left(y^{(\alpha)}(x)\right) = {}_0I_x^{\alpha}\left(g(x)\right) + {}_0I_x^{\alpha}\left(\int_0^x K(x,t)F\left(y(t)\right)dt\right)$$

using (2.4) gives

$$y(x) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + {}_0I_x^{\alpha}\left(g(x)\right) + {}_0I_x^{\alpha}\left(\int_0^x K(x,t)F\left(y(t)\right)dt\right)$$

using (2.3) gives

$$y(x) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(g(x)\right) ds + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^x K(x,t)F\left(y(t)\right)dt\right) ds$$

$$y(x) = U(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} (
$$

$\int_0^x K(x,t)F\left(y(t)\right)dt\ ds$
where

$$U(x)=\sum_{k=0}^{N}\frac{y^{(k)}(0)}{k!}x^k+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}g(s)ds$$

□

3.1 Method of Solution

We approximate the solution of (3.2) by the polynomial approximate solution in the form

$$y_N\left(x\right)=\phi\left(x\right)\mathbf{A} \tag{3.2}$$

where $\phi\left(x\right)$ is an interpolating polynomial and $\mathbf{A}$ are parameters to be determined,

$$\phi\left(x\right)=\left[\begin{array}{cccc}\phi_0\left(x\right) & \phi_1\left(x\right) & \phi_2\left(x\right) & \cdots\phi_N\left(x\right)\end{array}\right]$$
$$\mathbf{A}=\left[\begin{array}{ccccc}a_0 & a_1 & a_2 & \cdots & a_N\end{array}\right]^T$$

substituting(3.2) into(3.2) gives

$$\phi\left(x\right)\mathbf{A}=U(x)+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\int_0^x K(s,t)F\left(\phi\left(t\right)\mathbf{A}\right)dt\right)ds \tag{3.3}$$

where

$$U(x)=\sum_{k=0}^{N}\frac{y^{(k)}(0)}{k!}x^k+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}g(s)ds$$

collecting the like terms

$$\left(\phi\left(x\right)-\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\int_0^x K(s,t)F\left(\phi\left(t\right)\right)dt\right)ds\right)\mathbf{A}=U(x) \tag{3.4}$$

equation (3.4) can be written in this form

$$W(x)\mathbf{A}=U(x) \tag{3.5}$$

where

$$W(x)=\phi\left(x\right)-\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\int_0^x K(s,t)F\left(\phi\left(t\right)\right)dt\right)ds_{1\times[N+1]}$$

collocating (3.5) using the standard collocation points

$$x_i=a+\frac{b-a}{N}i$$

$$W(x_i)\mathbf{A}=U(x_i) \tag{3.6}$$

where

$$W(x_i)=\begin{bmatrix} W_0(x_0) & W_1(x_0) & W_2(x_0) & \cdots & W_N(x_0)\\ W_0(x_1) & W_1(x_1) & W_2(x_1) & \cdots & W_N(x_1)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ W_0(x_N) & W_1(x_N) & W_2(x_N) & \cdots & W_N(x_N)\end{bmatrix}, U(x_i)=\begin{bmatrix} U(x_0)\\ U(x_1)\\ \vdots\\ U(x_N)\end{bmatrix}$$

using the boundary conditions

$$y(0)=y_0,\ y(1)=y_1 \tag{3.7}$$

hence, (1.2) becomes

$$\phi^{(\alpha)}(0)=y_\alpha,\ \phi^{(\alpha)}(1)=y_\alpha \tag{3.8}$$

substituting (3.8) into equation (3.6) gives

$$W^*(x_i)\mathbf{A}=U^*(x_i) \tag{3.9}$$

which is a $(N+1)\times(N+1)$ nonlinear equations. We solved for $\mathbf{A}$ in (3.9) and substituted the result into the approximate solution to obtain the numerical solution.

$$y(x)=\phi(x_i)W^{*-1}(x_i)U^*(x_i) \tag{3.10}$$

4 Uniqueness of the Method

In this section, we established the uniqueness of the method by introducing the following theorem and hypothesis:
H_1 : There exist a constant, $L>0$, such that for any y_N and $y\in C([0,1],\mathbb{R})$

$$|F(y_N)-F(y)|\le L|y_N-y|$$

H_2 : There exist a function $K^*\in C([0,1]\times[0,1],\mathbb{R})$, the set of all positive functions such that

$$K^*=\max_{x\in[0,1]}\int_0^x|K(x,t)|\,dt<\infty$$

H_3 : The function $g\in\mathbb{R}$ is continuous.

Theorem 4.1. *Assume the H_1- H_3 hold. If*

$$\left(\frac{Lk^*}{\Gamma(\alpha+1)}\right)<1 \tag{4.1}$$

then there exist a unique solution $y(x)\in T$

Proof.

$$(Ty_N)(x)=U(x)+\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(\int_0^x K(s,t)F(y_N(t))\,dt\right)ds \tag{4.2}$$

and

$$(Ty)(x)=U(x)+\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(\int_0^x K(s,t)F(y(t))\,dt\right)ds \tag{4.3}$$

Substract (4.3) from (4.2) gives

$$(Ty_N)(x)-(Ty)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(\int_0^x|K(s,t)|\,|[F(y_N(t))-F(y(t))]|\,dt\right)ds$$

Taking the absolute value gives

$$|(Ty_N)(x) - (Ty)(x)| \leq \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} ($$

$\int_0^x K(s,t)\left[F(y_N(t)) - F(y(t))\right] dtds$
Taking maximum of both sides and using H_1 *and* H_2

$$d(Ty_N(x), Ty(x)) \leq \left[\frac{LK^*}{\Gamma(\alpha+1)}\right] d(y_N, y)$$

Based on the inequality (4.1) we have

$$d(Ty_N(x), Ty(x)) \leq d(y_N, y)$$

By the Banach contraction principle, we can conclude that T has a unique fixed point. □

5 Numerical Examples

In this section, we present numerical examples to evaluate the effectiveness and accuracy of the method. Let $y_n(x)$ and $y(x)$ be the approximate and exact solutions respectively. $\text{Error}_N = |y_n(x) - y(x)|$

Example 5.1. *[16] Considering Fractional Volterra integro-differential equation*

$$D^{1.2}y(x) - \int_0^x (x-t)^2 y^3(t)dt = g(x) \tag{5.1}$$

subject to initial conditions

$$y(0) = 0, \ y(0) = 1$$

where

$$g(x) = \frac{2.5}{\Gamma 0.8} x^{0.8} - \frac{x^9}{252}$$

Exact solution: $y(x) = x^2$
The approximate solution of equation(5.1) *at* $N = 4$ *gives*

$$y_4 = \left(\begin{array}{c} 1.709743458000 \times 10^{-14} + 2.889777306000 \times 10^{-11}x + 0.999999999739x^2 \\ +4.119158348000 \times 10^{-10}x^3 - 1.798809990000 \times 10^{-10}x^4 \end{array} \right)$$

Table 1: Exact, approximate and absolute error values for example 5.1

x	Exact	Our method$_{N=4}$	Error$_4$	Error $_{[24]=4}$
0.2	0.200000000000	0.200000000000	0.00	6.43e-18
0.4	0.400000000000	0.400000000000	0.00	3.93e-18
0.6	0.600000000000	0.600000000000	0.00	5.12e-19
0.8	0.800000000000	0.800000000000	0.00	8.30e-19
1.0	1.000000000000	1.000000000000	0.00	1.45e-19

Example 5.2. *[16] Considering Fractional differential equation*

$$D^{\frac{3}{2}}y(x) + y(x) = g(x) \tag{5.2}$$

subject to boundary conditions

$$y(0) = 0, y(1) = 0$$

where

$$g\left(x\right) = x^5 - x^4 + \frac{128}{7\sqrt{\pi}}x^{3.5} - \frac{64}{5\sqrt{\pi}}x^{2.5}$$

Exact solution: $y\left(x\right) = x^4\left(x - 1\right)$

The approximate solution of equation(5.2) *at* $N = 5$ *gives*

$$y_5 = \left(x^5 - 1.0x^4 + 3.681035366e - 18x^3 + 1.235294746e - 18x^2 - 1.251278496e - 17x\right)$$

Table 2: Exact and approximate values for example 5.2

x	Exact	$N = 5$	Error_5
0.2	-0.1280000000e-2	-0.1280000000e-2	0.00
0.4	-0.15360000000e-1	-0.15360000000e-1	0.00
0.6	-0.51840000000e-1	-0.51840000000e-1	0.00
0.8	-0.81920000000e-1	-0.81920000000e-1	0.00
1.0	0.000000000000	0.000000000000	0.00

Example 5.3. *[21] Considering Fractional differential equation*

$$D^{\alpha}y(x) + ay(x) = g(x), 1 \leq x \leq 2 \tag{5.3}$$

$$y(0) = 0,\ y(1) = -\frac{1}{40},\ \alpha = \frac{3}{2},\ a = \frac{e^{-3\pi}}{\sqrt{\pi}}$$

$$g(x) = \frac{e^{-3\pi}}{40\sqrt{\pi}}\left(x^2\left(40x^2 - 74x + 33\right) + 4e^{3\pi}\sqrt{x}\left(128x^2 - 148x + 33\right)\right)$$

Exact solution:

$$y(x) = \left(x^2 - \frac{37}{20}x + \frac{33}{40}\right)x^2$$

The approximate solution of equation(5.3) *at* $N = 8$ *gives*

$$y_8 = \begin{pmatrix} -6.772990005e - 22x^8 + 3.14953473e - 21x^7 \\ -6.066384787e - 21x^6 + 6.916096771e - 21x^5 \\ +x^4 - 1.85x^3 + 0.825x^2 - 3.066883108e - 16x \end{pmatrix}$$

6 Discussion of Results

In this section, we discuss the numerical results obtained from the solved examples using the derived numerical method.

Based on the result obtained for example 5.1 as shown in the Table 1 that the approximate solution at N=4 gives $y_4 = 1.709743458000 \times 10^{-14} + 2.889777306000 \times 10^{-11}x + 0.999999999739x^2$

Table 3: Exact and approximate values for example 5.3

x	Exact	Our method$_{N=8}$	error$_8$	error $_{[21]=8}$
0.2	0.019800000000	0.019800000000	0.00	4.59278e-5
0.4	0.039200000000	0.039200000000	0.00	1.13310e-4
0.6	0.027000000000	0.027000000000	0.00	1.13306e-4
0.8	-0.9600000000e-2	-0.9600000000e-2	0.00	6.67933e-5
1.0	-0.25000000000e-1	-0.25000000000e-1	0.00	6.27542e-4

$+4.119158348000 \times 10^{-10}x^3 - 1.798809990000 \times 10^{-10}x^4$. The numerical result converge to exact solution and this confirmed that our method performed better than the method proposed by [16].
The results of the numerical example 5.2 as shown in Table 2 that the approximate solution at N=5 gives $y_5 = x^5 - 1.0x^4 + 3.681035366e - 18x^3 + 1.235294746e - 18x^2 - 1.251278496e - 17x$. The numerical result also converge to exact solution.
The approximate solution obtained in example 5.3 at $N = 8$ gives $y_8 = -6.772990005e - 22x^8 + 3.14953473e - 21x^7 - 6.066384787e - 21x^6 + 6.916096771e - 21x^5 + x^4 - 1.85x^3 + 0.825x^2$ $-3.066883108e - 16x$.The numerical result converge to exact solution and this confirmed that our method performed better than the method proposed by [21] as shown in Table 3.

7 Conclusion

The collocation method was examined for the numerical solution of nonlinear Volterra integro- differential equations of Fractional order. This method is found to be reliable, effective and straight forward to compute. Maple 18 is used for all of the computations in this work.

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