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Legendre Ritz-Least squares method for the numerical solution of delay differential equations of the multi-pantograph type

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Abstract: This paper is concerned with a Legendre Ritz-Least squares technique for the nonsingular and singular delay differential equations (DDEs) of multi-pantograph type. This technique is based on Legendre polynomials and Least squares. The Legendre Ritz-Least squares technique (LRLS) is used to decrease the problem to a set of the algebraic equation system. The efficiency and reliability of the proposed method are shown by some numerical results. All of the numerical implementations have been performed on a PC using some programs written in MATHEMATICA.

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1 Introduction

In this manuscript, we consider the DDE of multi-pantograph type in the form

$$\begin{aligned}\sum_{k=1}^{r}\frac{f_k(t)}{t^v(t-\lambda)^d}u^k(t)+\sum_{k=1}^{r'}\frac{g_k(t)}{t^v(t-\lambda)^d}u^{(k)}(t)+\sum_{k=1}^{p}\sum_{l=1}^{q}\frac{\alpha_{k,l}(t)}{t^v(t-\lambda)^d}u^k(p_lt)\\+\sum_{k=1}^{p'}\sum_{l=1}^{q'}\frac{\beta_{k,l}(t)}{t^v(t-\lambda)^d}u^{(k)}(q_lt)=h(t),\qquad t\in[0,\lambda]\end{aligned}\tag{1.1}$$

with the boundary conditions

$$u^{(i)}(0)=\alpha_i,\ u^{(j)}(\lambda)=\beta_j,\ i=0,1,...,t_1-1,\ j=0,1,...,t_2-1,t_1+t_2=\max\{r',p'\}\tag{1.2}$$

where $0<p_l,q_l<1,\alpha_k,\beta_l,d,v$, are real constants $u^{(0)}(t)=u(t)$, $u(t)$ from $C^t(0,\lambda)$ is an unknown function and the functions $f_k(t),g_k(t),\alpha_{k,l}(t),\beta_{k,l}(t),h(t)\in C(0,\lambda)$.

In past decade, many researchers became interested in the pantograph problems. This type of equations are charactrized as presence of the linear functional argument [10, 14]. The name pantograph originated from Tayler's work and the Ockendon [15] on the collection of current by the pantograph head of an electric locomotive. These problems are used in applications such as nonlinear dynamical systems, analytic number theory and probability theory[4, 12, 13, 16]. The mentioned problems are usually difficult to solve analytically, therefore researchers used of numerical techniques.

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Recently the pantograph equations together with their systems have been solved using the homotopy methods [26], modified variational iteration method and the variational [10, 18, 3, 30], Taylor polynomial method [20]-[22], the reproducing kernel space method [5], the Bessel collocation method [23, 27, 28] , the Runge- Kutta methods [8], the discontinuous Galerkin methods [2] and [9], the θ-methods [24] and [25], the Hermite method [29], the Chebyshev polynomials technique [19], linear multistep methods [6] and the rational approximation method [7].

This paper is organized as follows: In section 2, we explain the basic formulations of Legendre polynomials required . Section 3 is presented to the solution of Equation (1.1) by the Legendre Ritz-Least squares technique. We prove the convergence of approximate solution in section 4. In section 5, by considering numerical examples, we illustarte the accuracy of the proposed technique. Section 6 is conclusion.

2 Properties of the Legendre polynomials

Legendre functions are solutions to Legendre's differential equation in mathematics:

$$\frac{d}{dt}\left[(1-t^2)\frac{d}{dt}P_k(t)\right]+k(k+1)P_k(t)=0. \tag{2.1}$$

Equation (2.1) is widely used in various sciences, including chemistry and physics.In particular, for solving Laplace's equation in spherical coordinates, differential equation (2.1) happends. Using the standard power series method, the Legendre differential equation may be solved . Equation (2.1) has regular singular points at $t = 1$, so, a series solution about the origin will only converge for $|t| < 1$ in general. The $P_k(t)$ that is solution and also is regular at $t = 1$. This solution is regular at $t = -1$, and for this solution terminates, series when k is an integer number.

With the normalization $P_k(1) = 1$, these solutions form a polynomial sequence of orthogonal polynomials named the Legendre polynomials for $k = 0, 1, 2, ...$ Each Legendre polynomial $P_k(t)$ is an kth-degree polynomial. Using Rodrigues' formula, it may be shown:

$$P_k(t)=\frac{1}{2^k k!}\frac{d^k}{dt^k}\left[(t^2-1)^k\right]. \tag{2.2}$$

The Legendre polynomials have a important property. This property is that on the interval $[-1, 1]$, with respect to the L^2 inner product, they are orthogonal:

$$\int_{-1}^{1} P_k(t)P_m(t)dt=\frac{2}{2k+1}\delta_{mk} \tag{2.3}$$

(where δ_{mk} equal to 0 if $m \neq k$ and to 1 otherwise that named the Kronecker delta).

The shifted Legendre polynomials are considered as $\tilde{P}_k(t) = P_k(2t - 1)$, on $[0, 1]$, here the shifting function $t \to 2t-1$ is chosen such that it bijectively maps the interval $[0, 1]$ to the interval $[-1, 1]$, by using that the polynomials $\tilde{P}_k(t)$ are orthogonal polynomials:

$$\int_{0}^{1} \tilde{P}_k(t)\tilde{P}_m(t)dt=\frac{1}{2k+1}\delta_{mk}. \tag{2.4}$$

For the shifted Legendre polynomials, The analogue of Rodrigues' formula is

$$\tilde{P}_k(t)=\frac{1}{k!}\frac{d^k}{dt^k}\left[(t^2-t)^k\right]. \tag{2.5}$$

The some shifted Legendre polynomials are in following:

$$\tilde{P}_0(t) = 1,\ \tilde{P}_1(t) = 2t - 1,\ \tilde{P}_2(t) = 6t^2 - 6t + 1. \tag{2.6}$$

3 Legendre Ritz-Least squares method

By a weighted sum of other functions, Least Squares can be used to approximating a function. The best approximation can be determined as that which minimizes the difference between the original function and the approximation; for a Least-Squares approach the quality of the approximation is measured in terms of the squared differences. A generalization to approximation of a data set is the approximation of a function by a sum of other functions, that are usually an orthogonal set:

$$u(t) \approx u_n(t) = c_0\phi_0(t) + c_1\phi_1(t) + ... + c_n\phi_n(t), \qquad t \in [a, b]. \tag{3.1}$$

On $[a, b]$, the set $\phi_j(t)$ is an orthonormal set. The coefficients c_j are selected to make the magnitude of the difference $\|u - u_n\|^2$ as small as possible. For example, L_2 norm of a function $u(t)$ over the interval $[a, b]$ can be considered by

$$\|u\| = \left(\int_a^b u^*(t)u(t)dt\right)^{\frac{1}{2}}, \tag{3.2}$$

above the "*" is complex conjugate in the case of complex functions. The extension of Pythagoras theorem in this manner leads to the notion of Lebesgue measure and function spaces, an idea of "space" more general than the original basis of Euclidean geometry. The $\phi_j(t)$ satisfy orthonormality relations as following

$$\int_a^b \phi_i{}^*(x)\phi_j(x)dx = \delta_{ij}. \tag{3.3}$$

In this manuscript, we are going to obtain an approximated solution of (1.1) in the form

$$u(t) \cong u_n(t) = \sum_{i=0}^{n} c_i t^{t_1}(t - \lambda)^{t_2} P_i(t) + w(t) \qquad t \in [0, \lambda]. \tag{3.4}$$

It can be written as follows

$$u(t) \cong u_n(t) = \sum_{i=0}^{n} c_i P^*{}_i(t) + w(t) \qquad [0, \lambda], \tag{3.5}$$

so that $P_i(t), i = 0, 1, ..., n$ are the Legendre polynomials of the firs kind that was defined in previous section and also $c_i, i = 0, 1, ..., n$ are the unknown coefficients.

In relation (3.5), $P^*{}_i(t), i = 0, 1, ..., n$ satisfy in homogenous conditions and $w(t)$ satisfies in nonhomogeneous conditions, thus $u_n(t)$ satisfies in conditions (1.2). We can write equation (1.1) in the form

$$\begin{aligned} &\sum_{i=1}^{r} f_i(t)u^i(t) + \sum_{i=1}^{r'} g_i(t)u^{(i)}(t) + \sum_{i=1}^{p}\sum_{j=1}^{q} \alpha_{i,j}(t)u^i(p_j t) \\ &+ \sum_{i=1}^{p'}\sum_{j=1}^{q'} \beta_{i,j}(t)u^{(i)}(q_j t) = x^l(t - \lambda)^d h(t), \qquad 0 \leqslant t \leqslant \lambda \end{aligned} \tag{3.6}$$

also, we can write equation (3.6) in the form

$$F\left(x, u(t), u^{(1)}(t), \ldots, u^{(r')}(t), u(q_j t), u^{(1)}(q_j t), \ldots, u^{(p')}(q_j t)\right) = 0. \tag{3.7}$$

Obviously F is a continuous function on interval $[0, \lambda]$. By substituting $u_n(t)$ in Eq.(3.7) we have

$$F\left(t, u_n(t), u_n^{(1)}(t), \ldots, u_n^{(r')}(t), u_n(q_j t), u_n^{(1)}(q_j t), \ldots, u_n^{(p')}(q_j t)\right) \cong 0. \tag{3.8}$$

Let

$$J(u) = \int_0^\lambda F^2 dt = \|F\|^2 \tag{3.9}$$

the coefficients c_i are obtained by used the Least squares equations

$$\min_{c_i} J(u_n), \ 0 \leqslant i \leqslant n \tag{3.10}$$

where

$$\|.\|^2 = \int_0^\lambda (.)^2 dt. \tag{3.11}$$

Now if we denote by $J(c_0, c_1, \ldots, c_n)$ the expressions which are minimized in (3.10), for minimization, from the necessary conditions, we have to solve the system of following equations that are nonlinear.

$$\frac{\partial J}{\partial c_i} = 0, \ i = 0, ..., n. \tag{3.12}$$

Then we solve system (3.12) by mathematica software.

4 On the convergence of the method

Without loss of generality, we consider $\lambda = 1$. In this section, we debate the convergence of the technique that is presented in previous section. With increase of n, we will present that the approximate value of $\gamma_n \to 0$ where

$$\gamma_n = \min \ \mathrm{J}\left(u_n\right). \tag{4.1}$$

Now we consider function space and then provide some lemmas.
In first, we consider the Banach space $(C^n[0,1], \| \, . \, \|_n)$:

$$C^n[0,1] = \{u(x) \quad | \quad u^{(n)}(x) \in C[0,1]\},$$
$$\| u \|_n = \| u \|_\infty + \| u' \|_\infty + \cdots + \| u^{(n)} \|_\infty .$$

Now let

$$E[0,1] = \{u(x) \in C^n[0,1] \quad | \quad u^{(j)}(0) = u_0^j, \quad u^{(j)}(1) = u_1^j, \quad j = 0, 1, \ldots, n-1\},$$

where u_0^j, u_1^j are constant values. We express a lemma. This lemma has an important role. In following, we consider the lemma that polynomial functions of the metric space $E[0,1]$ are dense in that $E[0,1]$. This is the issue that we will show in Lemma.

Lemma 4.1. *Let $u(t) \in E[0,1]$. There exists a sequence of polynomial functions $\{s_l(t)\}_{l \in N} \subset E[0,1]$ such that $s_l \to u(t)$ with respect to $\| \, . \, \|_n$.*

Proof. **Proof:** [11]. □

Consider $G^n[0,1]$ as follows

$$G^n[0,1] = E[0,1] \bigcap < \{p_j\}_{j=0}^{n} > .$$

In above, $< \{p_j\}_{j=0}^{n} >$ is the Banach subspace of $C^n[0,1]$. This space is generated by the Legendre polynomials that the degree of this space is at most n. $G^n[0,1]$ is a metric subspace of $E[0,1]$. Now consider the functional J in (3.9) as an operator $J : (C^n[0,1], \| \,.\, \|) \to \mathbb{R}$. In Lemma 4.3, we will show that the functional J is continuous on it's domain. We use this important property in Theorem 4.4.

Theorem 4.2. *Let $u : X \to Y$ be a continuous mapping, where the metric spaces X and Y are compact metric space, then u mapping is uniformly continuous.*

Proof. [17]. □

Lemma 4.3. *In the Banach space$(C^t[0,1], \| \,.\, \|)$, the functional J is continuous.*

Proof. We are going to show that $J : C^t[0,1] \to R$ is continuous, where

$$J(u) = \int_0^1 F^2 dx.$$

Let $u^* \in C^t[0,1]$ and $\varepsilon > 0$. Consider $d > 0$ and

$$Q = [0,1] \times \underbrace{[-L-d, L+d] \times \ldots \times [-L-d, L+d]}_{r'+p'-times}$$

where

$$L = \max\left\{ \|u^*\|_\infty, \left\|u^{*(1)}\right\|_\infty, \ldots, \left\|u^{*(r'+p')}\right\|_\infty \right\}.$$

Obviously we have

$$Y^* := (t, u^*(t), u^{*(1)}(t), \ldots, u^{*(r')}(t), u^*(q_j t), u^{*(1)}(q_j t), \ldots, u^{*(p')}(q_j t)) \in Q. \tag{4.2}$$

Let $\|u - u^*\|_{r'+p'} < \delta$, therefore we have $\left\|u^{(j)} - u^{*(j)}\right\|_\infty < \delta$ for $j = 0, 1, \ldots, r' + p'$ and $\varepsilon > 0$ is given and also $\delta > 0$. For small enough value of δ we have

$$Y := (t, u(t), u^{(1)}(t), \ldots, u^{(r')}(t), u(q_j t), u^{(1)}(q_j t), \ldots, u^{(p')}(q_j t))) \in Q. \tag{4.3}$$

Since Q is a compact set and F is continuous on Q with respect to all it's arguments , according to Theorem 4.2 on interval Q, F is uniformly continuous . So if $\delta > 0$ be sufficiently small, then $|F(Y) - F(Y^*)| < \varepsilon$, thus based on the continuity of J, we have $|J(u) - J(u^*)| < \varepsilon$. □

Now we can show the convergence of the approximating technique.

Theorem 4.4. *Let γ_n be the minimum of the functional J on $G^n[0,1]$, then we have*

$$\lim_{n\to\infty} \gamma_n = 0.$$

Proof. For any given $\epsilon > 0$, let $u^* \in E[0,1]$ such that $J(u^*) < \epsilon$. (such u^* exist by the properties of minimum). According to lemma 4.3, J is continuous on $(C^n[0,1], \| . \|)$, so we have

$$| J(u) - J(u^*) | < \epsilon \tag{4.4}$$

provided that $\| u - u^* \| < \delta$. According to lemma 4.1 for large enough value of n there exist $s_n \in G^n[0,1]$ such that $\| s_n - u^* \| < \delta$. Moreover let u_n be the element of $G^n[0,1]$ such that $min J(u_n) = \gamma_n$, then using (4.4) we have

$$0 \leq \gamma_n \leq J(s_n) < 2\epsilon.$$

Since the $\epsilon > 0$ is arbitrary, it follows that

$$\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} min J(u_n) = 0.$$

□

5 Illustrative examples

In this section, we use the technique presented in section 3 for solving the following examples.

Example 5.1. *Consider the linear multi-pantograph differential equation that is a delay differential equation:*

$$u^{(2)}(x) - \frac{3}{4}u(x) - u(\frac{x}{2}) - u^{(1)}(\frac{x}{2}) - \frac{1}{2}u^{(2)}(\frac{x}{2}) = -x^2 - x + 1, \ 0 < x < 1$$

With the initial conditions $u(0) = u^{(1)}(0) = 0$.
The exact solution is $u(x) = x^2$. *Here* $t = 2, t_1 = 2, t_2 = 0, g_2(x) = x(x-1), f_1(x) = -\frac{3}{4}x(x-1), \alpha_{1,1}(x) = -x(x-1), \beta_{1,1}(x) = -x(x-1), \beta_{2,2}(x) = -\frac{x}{2}(x-1), h(x) = -x^2 - x + 1, \mu_0 = 0, \mu_1 = 0,$ *and* $w(x) = 0$.

Now, we find the approximate solution

$$u(x) \cong u_n(x) = \sum_{i=0}^{n} c_i x^2 \tilde{P}_i(x) = \sum_{i=0}^{n} c_i \tilde{P}_i^*(x). \tag{5.1}$$

We determine the following values of $c_i s$ *in approximation (5.1) for* $n = 1$: $c_0 = 1, c_1 = 0$. *Thus we have* $u_1(x) = x^2$ *that is exact solution.*

Example 5.2. *In this example we consider the singular multi-Pantograph delay differential equation*

$$u^{(2)}(x) - \frac{1}{x}u^{(1)}(\frac{x}{2}) - \frac{1}{x^2}u^{(1)}(\frac{x}{4}) - \frac{1}{x-1}u(x) = h(x), \ 0 < x \leqslant 1.$$

With the boundary conditions $u(0) = 1, u(1) = e$ *and the exact solution* $u(x) = e^x$, *so that* $t = 2, t_1 = 1, t_2 = 1, g_1(x) = x^2(x-1), f_1(x) = -x^2, \beta_{1,1}(x) = x(x-1), \beta_{1,2}(x) = (x-1), \mu_0 = 1, \gamma_0 = e,$ *and* $w(x) = 1 + x(e-1)$ *and*

$$h(x) = e^x + \frac{1}{x}e^{\frac{x}{2}} + \frac{1}{x^2}e^{\frac{x}{4}} - \frac{1}{x-1}e^x.$$

Using the method presented in section 3, for different values of n in approximation, we get the following values of $c_i s$ *(3.5)*

$$n = 1 : \quad c_0 = 0.85934, c_1 = 0.14841,$$

$$n = 2 : \quad c_0 = 0.847276, c_1 = 0.139969, c_2 = 0.0115461,$$

$$n = 3 : \quad c_0 = 0.847431, c_1 = 0.140094, c_2 = 0.0116594, c_3 = 0.000871596$$

Table 1: The approximate values of μ_n with n=1,2 and 3 for example 5.2

n	μ_n
1	9.44192×10^{-6}
2	1.44782×10^{-7}
3	6.01778×10^{-8}

Table 2: Absolute error with n=1,2 and 3 for example 5.2

x	Absolute error, n=1	Absolute error, n=2	Absolute error, n=3
0	0	0	0
0.1	2.20561E-6	2.2336E-6	1.14761E-6
0.3	0.00236921	0.000085674	0.0000107518
0.5	0.00441539	0.0000439641	0.0000193498
0.7	0.00388333	0.0000103449	0.0000333011
0.9	0.00117563	0.0000399172	5.9857E-6
1	0	0	0

The approximate values of μ_n, for different number of basis functions n, are demonstrated in table 1. In the following table the absolute error for $n = 1$, $n = 2$ and $n = 3$ are demonstrated.

Example 5.3. *In this example, we consider the linear multi-pantograph differential equation that is delay:*

$$u^{(1)}(x) + u(x) + \mu_1(x)u(\frac{x}{2}) + \mu_2(x)u(\frac{x}{4}) = 0,\ 0 < x \leqslant 1$$

with the conditions $u(0) = 1$ *and the exact solution* $u(x) = e^{-x}cos(x)$. *Here* $t = 1, t_1 = 1, t_2 = 0, g_1(x) = x(x-1), f_1(x) = x(x-1), \alpha_{1,1}(x) = \mu_1(x)x(x-1), \alpha_{1,2}(x) = \mu_2(x)x(x-1), \mu_1 = 0,$ *and* $w(x) = 1, \mu_1(x) = e^{-\frac{x}{2}}\sin(\frac{x}{2}),\ \mu_2(x) = 2e^{-\frac{3x}{4}}\cos(\frac{x}{2})\sin(\frac{x}{4})$.
For different values of n *in approximation, we get the following values of* $c_i s$ *(3.5)*

$$n = 2 : c_0 = -0.925611, c_1 = 0.109401, c_2 = 0.0149859,$$

$$n = 3 : c_0 = -0.92417, c_1 = 0.104889, c_2 = 0.0225399,$$
$$c_3 = -0.00449883,$$

$$n = 4 : c_0 = -0.924073, c_1 = 0.104601, c_2 = 0.0230318,$$
$$c_3 = -0.00518268, c_4 = 0.00038924.$$

The table 3 represents the values of μ_n for various values of approximations. In the table 4, we present the absolute error for $n = 2$, $n = 3$, $n = 4$.

Example 5.4. *Consider the nonlinear Pantograph delay differential equation*

$$u^{(1)}(x) + 2u^2(\frac{x}{2}) = 1,\ 0 < x \leqslant 1$$

Table 3: The approximate values of μ_n with n=2,3 and 4 for example 5.3

n	μ_n
2	0.0000467903
3	4.23784×10^{-7}
4	5.02496×10^{-10}

Table 4: Absolute error with n=2,3 and 4 for example 5.3

x	Absolute error, n=2	Absolute error, n=3	Absolute error, n=4
0	0	0	0
0.2	0.000541123	0.0000251695	1.91049E-6
0.4	0.000960244	0.0000645165	9.82443E-7
0.6	0.000851248	0.0000635897	1.17069E-6
0.8	0.00054761	0.0000225268	1.91205E-6
1	9.28635E-6	5.65977E-6	7.30549E-8

with the conditions $u(0) = 0$ and the exact solution $u(x) = sin(x)$, so that $t = 1, t_1 = 1, t_2 = 0, g_1(x) = x(x-1), \alpha_{2,1}(x) = 2x(x-1), \mu_0 = 0$, and $w(x) = 0$. For this problem, we get the following values of c_is, for different values of n in approximation (3.5)

$$n = 2 : c_0 = 0.955515, c_1 = -0.0183804, c_2 = -0.0955685,$$

$$n = 3 : c_0 = 0.938408, c_1 = 0.0172705, c_2 = -0.122128,$$

$$c_3 = 0.00790985,$$

$$n = 4 : c_0 = 0.94557, c_1 = 0.00118783, c_2 = -0.107592,$$

$$c_3 = 0.000652679, c_4 = 0.00165208.$$

The following table shows the values of μ_n for various values of approximations. In the table 6, we present the absolute error for $n = 2$, $n = 3$ and $n = 4$.

Table 5: The approximate values of μ_n with n=2,3 and 4 for example 5.4

n	μ_n
2	2.19831×10^{-6}
3	2.97473×10^{-8}
4	2.27915×10^{-11}

Table 6: Absolute error with n=2,3 and 4 for example 5.4

x	Absolute error, n=2	Absolute error, n=3	Absolute error, n=4
0	0	0	0
0.1	0.000169395	0.0000133775	1.15691E-7
0.3	0.0000550626	0.0000218773	3.08618E-7
0.5	0.000289892	1.26838E-6	5.4947E-7
0.7	0.0000843005	0.000025639	1.61022E-7
0.9	0.000250521	3.30438E-6	1.6063E-7
1	0.000095541	0.0000111601	3.97656E-8

6 Conclusion

In this manuscript an efficient and accurate technique for the singular and nonsingular delay differential equations (DDEs) of multi-pantograph type is developed. We reduce the problem to solving a system of algebraic systems by special type of polynomial basis functions. The polynomial functions has the great flexibility in satisfying boundary and initial conditions. The convergence of the technique has been presented and illustrative examples to show applicability and validity of the new technique are included.

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