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C^3-spline Methods for Solving Fractional Integro-differential Equations

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Abstract: Fractional integro-differential equations (FIDEs) constitute an important mathematical tool in modeling many dynamical processes. To solve FIDEs, several analytical and numerical methods have been proposed, namely those based on symmetry and spline approaches. This paper proposes quartic and sextic C^3-spline methods for the numerical solution of FIDEs. The convergence analysis of the proposed strategy is examined in detail. Finally, three numerical examples are given to illustrate the numerical accuracy and efficiency of the proposed strategy.

Keywords: Fractional integro-differential equation; quartic and sextic C^3-splines; convergence analysis
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1 Introduction

Fractional calculus (FC) deals with non-integer order integration and differentiation [18, 35]. The FC plays a significant role in the modeling of various physical phenomena in many areas, such as physics, acoustics, engineering, chemistry, fluid-dynamics traffics, viscoelasticity and biology, nonlinear oscillation of earthquakes, control theory, viscoelastic materials, statistical and solid mechanics, continuum mechanics, signal processing, and economics [28, 38, 2, 25, 5, 19, 32, 6, 14]. Fractional integro-differential equations (FIDEs) are appropriate for the description of hereditary and memory properties of real-world processes. However, solving FIDEs is challenging, and several analytical and numerical methods have been proposed to obtain exact or accurate numerical solutions [24, 27, 26, 16, 17, 8, 4]. Powerful analytic methods include those based on symmetry. Indeed, symmetry plays a crucial role in computational science, and symmetry analysis can lead to systematic determination of exact solutions. Numerical methods are adopted either when no analytic solution can be found, or as a computationally efficient alternative to find approximate solutions.

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In this paper, we develop sextic and quartic C^3-spline methods for computing the numerical solutions of fractional integro-differential equations (FIDEs) of the form:

$$u''(x) + D_x^{-\alpha}u(x) = f(x)\,, \quad x \in [0, b]\,, \quad 0 < \alpha < 1, \tag{1.1}$$

with the initial conditions (ICs)

$$u\,(0) = \gamma_1\,,\; u'\,(0) = \gamma_2\,, \quad \gamma_1, \gamma_2 \in R, \tag{1.2}$$

in which the operator $D_x^{-\alpha}$ represents the Riemann-Liouville fractional integral (RLFI) of order α and $u(x)$ denotes an unknown function to be determined. For the special case $\alpha = 0$, the FIDE (1.1) yields the classical second order initial value problem. The FIDE (1.1) arises when modeling the heat flow in materials with memory. Techniques for solving FIDEs include the Adomian decomposition [21], Homotopy analysis [9], Chebyshev wavelets [39], Haar wavelets [15], CAS wavelets [36], Legendre collocation [34], implicit orthogonal spline collocation [12] methods. Ahmad et al. [1] studied the asymptotic behavior of solutions for a class of FIDEs. Elbeleze et al. [10] implemented the Homotopy perturbation method for approximating FIDEs. The authors of [1, 11] adopted the Taylor expansion technique to approximate FIDEs, while Pedas and Tamme [29] employed the spline collocation method. Mohammadizadeh et al. [22] proposed two unique piecewise C^3-spline techniques to simulate FIDEs. They also studied the existence and uniqueness of the solution of the FIDE (1.1) with ICs (1.2).

The structure of this paper is as follows: Section 2 presents the preliminary concepts of FC. Section 3 develops quartic and sextic C^3-spline methods for the solution of equations (1.1)-(1.2). Section 4 presents the convergence analysis of the proposed strategy. Section 5 gives three numerical examples to illustrate the efficiency and accuracy of the new methods. Finally, Section 6 contains some concluding remarks.

2 Preliminaries and notation

This section introduces some essential definitions, theorems and imperative properties of FC, which will be adopted in the subsequent discussion.

Definition 2.1. *[30] The Riemann-Liouville fractional derivative (RLFD) of order α of a function $f(x)$ is given by:*

$${}^{R}D^{\alpha} f\,(x) = \frac{1}{\Gamma\,(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \int_0^x (x-\tau)^{m-\alpha-1} f\,(\tau)\, d\tau, \quad m-1 < \alpha < m,\, m \in \mathrm{N},$$

with $\Gamma(\cdot)$ denoting the gamma function.

Definition 2.2. *[30] The Caputo fractional derivative of order α of a function $f(x)$ is given by:*

$$D^{\alpha} f\,(x) = \frac{1}{\Gamma\,(m-\alpha)} \int_0^x (x-\tau)^{m-\alpha-1} f^{(m)}\,(\tau)\, d\tau, \;\; m-1 < \alpha < m,\, m \in \mathrm{N}.$$

Definition 2.3. *[7] The two parameter Mittag-Leffler function can be represented by:*

$$E_{\nu,\beta}\,(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\,(\nu k + \beta)}, \quad \nu > 0, \;\; \beta > 0.$$

Definition 2.4. *[30] The RLFI of order α of a function $f(x)$ is given by:*

$$D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} f(\tau)\, d\tau, \qquad \alpha > 0.$$

We have the following relation:

$$D^{-\alpha} x^m = \frac{\Gamma(m+1)}{\Gamma(m+1+\alpha)} x^{m+\alpha}, \qquad m > -1.$$

Theorem 2.5. *[20] (Leibniz's formula) Assume that $f(x)$ is a continuous function over $[0,t]$ and for all $a \in [0,t]$ $g(x)$ is analytical. Then, for $\alpha > 0$ and $0 < a \leq t$, we have:*

$$D^{-\alpha} f(x)\, g(x) = \sum_{n=0}^{\infty} (-1)^n C_\alpha^n D^n f(x)\, {}^R D^{-\alpha-n} g(x),$$
$${}^R D^{\alpha} f(x)\, g(x) = \sum_{n=0}^{\infty} C_\alpha^n D^n f(x)\, {}^R D^{\alpha-n} g(x),$$

where D^n is the ordinary differential operator and $C_\alpha^n = \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)}$.

Theorem 2.6. *[13] Let $f(x) \in C^m[0,1]$, $g(x) \in C[0,1]$ and $\alpha \in (m-1, m)$, with $m \in N$. Then, the following relation holds for $x \in [0,1]$:*

$$(1)\ D^{\alpha} D^{-\alpha} g(x) = g(x),$$
$$(2)\ D^{-\alpha} D^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} \frac{x^k}{k!} f^{(k)}(0),$$
$$(3)\ \lim_{x \to 0} D^{\alpha} f(x) = \lim_{x \to 0} D^{-\alpha} f(x) = 0.$$

3 Description of the method

3.1 Quartic C^3- spline method

This subsection employs the quartic C^3- spline method developed by Sallam and Karaballi [37] for obtaining the numerical solutions of the FIDE (1.1).

For a positive integer n, we partition the interval $[0,b]$ into n equal subintervals $I_i = [x_{i-1}, x_i]$, $i = 1, \ldots, n$, with the spatial step size equal to $h = \frac{b}{n}$. Let Π_4 be the collection of all polynomials of degree at most 4 and

$$S_{n,4}^{(3)} = \left\{ s(x) \ : \ s \in C^3[0,b],\ s \in \Pi_4, \text{ for } x \in I_i,\ i = 1, \ldots, n \right\}.$$

Now, we establish a piecewise polynomial $s(x) \in S_{n,4}^{(3)}$ satisfying (1.1) and (1.2), namely

$$s''(x) = -D_x^{-\alpha} s(x) + f(x), \qquad s(0) = \gamma_1\ , s'(0) = \gamma_2, \tag{3.1}$$

where $s(x)$ satisfies the following properties:
(1) $s''(x_i) = -D_x^{-\alpha} s(x_i) + f(x_i)$.

(2) for $x \in [0,b]$, $s(x)$ and its derivatives up to order 3 are continuous.
For any real numbers $s''_i, i=0,\ldots,n,\ s_0, s'_0$ and s'''_0, we define the unique quartic spline $s \in S^{(3)}_{n,4}$ over the interval I_i

$$s(x) = s_{i-1} + hs'_{i-1}\mathcal{B}(t) + h^2 s''_{i-1}\mathcal{C}(t) + h^2 s''_i \mathcal{D}(t) + h^3 s'''_{i-1}\mathcal{E}(t), \tag{3.2}$$

in which $t = \frac{x-x_{i-1}}{h}$ and $\mathcal{B}(t), \mathcal{C}(t), \mathcal{D}(t)$ and $\mathcal{E}(t)$ represent the polynomials of degree at most 4. To obtain $\mathcal{B}(t), \mathcal{C}(t), \mathcal{D}(t)$ and $\mathcal{E}(t)$, we perform

$$\begin{aligned}
&s'(x) = s'_{i-1}\mathcal{B}'(t) + hs''_{i-1}\mathcal{C}'(t) + hs''_i\mathcal{D}'(t) + h^2 s'''_{i-1}\mathcal{E}'(t),\\
&s''(x) = \tfrac{1}{h}s'_{i-1}\mathcal{B}''(t) + s''_{i-1}\mathcal{C}''(t) + s''_i\mathcal{D}''(t) + hs'''_{i-1}\mathcal{E}''(t),\\
&s'''(x) = \tfrac{1}{h^2}s'_{i-1}B'''(t) + \tfrac{1}{h}s''_{i-1}\mathcal{C}'''(t) + \tfrac{1}{h}s''_i\mathcal{D}'''(t) + s'''_{i-1}\mathcal{E}'''(t).
\end{aligned}$$

Meanwhile, by taking the nodal points, we have

$$\begin{aligned}
&s(x_{i-1}) = s_{i-1} + hs'_{i-1}\mathcal{B}(0) + h^2 s''_{i-1}\mathcal{C}(0) + h^2 s''_i\mathcal{D}(0) + h^3 s'''_{i-1}\mathcal{E}(0),\\
&s'(x_{i-1}) = s'_{i-1}\mathcal{B}'(0) + hs''_{i-1}\mathcal{C}'(0) + hs''_i\mathcal{D}'(0) + h^2 s'''_{i-1}\mathcal{E}'(0),\\
&s''(x_{i-1}) = \tfrac{1}{h}s'_{i-1}\mathcal{B}''(0) + s''_{i-1}\mathcal{C}''(0) + s''_i\mathcal{D}''(0) + hs'''_{i-1}\mathcal{E}''(0),\\
&s'''(x_{i-1}) = \tfrac{1}{h^2}s'_{i-1}\mathcal{B}'''(0) + \tfrac{1}{h}s''_{i-1}\mathcal{C}'''(0) + \tfrac{1}{h}s''_i\mathcal{D}''(0) + s'''_{i-1}\mathcal{E}''(0),\\
&s''(x_i) = \tfrac{1}{h}s'_{i-1}\mathcal{B}''(1) + s''_{i-1}\mathcal{C}''(1) + s''_i\mathcal{D}''(1) + hs'''_{i-1}\mathcal{E}''(1).
\end{aligned}$$

Now, we obtain

$$\mathcal{B}(t) = t,\ \mathcal{C}(t) = \frac{t^2}{12}\left(6 - t^2\right),\ \mathcal{D}(t) = \frac{t^4}{12},\ \mathcal{E}(t) = \frac{t^3}{12}(2-t). \tag{3.3}$$

By means of the Definition 2.4 and equation (3.2), and using $x = x_i$, we can conclude that

$$\begin{aligned}
D^{-\alpha}(s(x))|_{x=x_i} &= \tfrac{t^\alpha}{\Gamma(\alpha+1)}s_{i-1}|_{t=1} + h\,D^{-\alpha}(\mathcal{B}(t))|_{t=1}s'_{i-1} + h^2\,D^{-\alpha}(\mathcal{C}(t))|_{t=1}s''_{i-1}\\
&+h^2\,D^{-\alpha}(\mathcal{D}(t))|_{t=1}s''_i + h^3\,D^{-\alpha}(\mathcal{E}(t))|_{t=1}s'''_{i-1},
\end{aligned} \tag{3.4}$$

in which

$$\begin{aligned}
&D^{-\alpha}(\mathcal{B}(t))|_{t=1} = \tfrac{1}{\Gamma(\alpha+2)}, \quad D^{-\alpha}(\mathcal{C}(t))|_{t=1} = \tfrac{1}{\Gamma(\alpha+3)} - \tfrac{2}{\Gamma(\alpha+5)},\\
&D^{-\alpha}(\mathcal{D}(t))|_{t=1} = \tfrac{2}{\Gamma(\alpha+5)}, \quad D^{-\alpha}(\mathcal{E}(t))|_{t=1} = \tfrac{1}{\Gamma(\alpha+4)} - \tfrac{2}{\Gamma(\alpha+5)}.
\end{aligned}$$

Finally, we formulate the spline solution $s(x)$ to estimate the numerical solution $y(x)$ of (1.1)-(1.2), which satisfies the conditions over the first interval $[x_0, x_1]$, such as $s_0 = \gamma_1$, $s'_0 = \gamma_2$, $s''_0 = f(0)$, $s'''_0 = y'''_0 = f'(0)$, and also satisfies the following properties:

$$\begin{aligned}
&s_i = s_{i-1} + hs'_{i-1} + \tfrac{h^2}{12}[5s''_{i-1} + s''_i] + \tfrac{1}{12}h^3 s'''_{i-1},\\
&s'_i = s'_{i-1} + \tfrac{h}{3}[2s''_{i-1} + s''_i] + \tfrac{1}{6}h^2 s'''_{i-1},\\
&s'''_i = -s'''_i + \tfrac{2}{h}[s''_i - s''_{i-1}],
\end{aligned} \tag{3.5}$$

where $s''_i = -D_x^{-\alpha}s(x_i) + f(x_i)$. We can determine the coefficients s_i, s'_i and s'''_i, $i = 1,\ldots,n$ by solving the linear system (3.5).

3.2 Sextic C^3-spline method

This subsection formulates the sextic C^3-spline scheme to approximate the solution of the FIDE (1.1) subject to ICs (1.2). For a given positive integer n, the interval $[0,b]$ is partitioned into n equal subintervals $I_i=[x_{i-1},x_i]$, $i=1\,(1)\,n$, with the step size $h=\frac{b}{n}$. Let Π_6 be the collection of all polynomials of degree at most 6 and

$$S^{(3)}_{n,6}=\left\{s\,(x)\;:\;s\in C^3\,[0,b]\,,\,s\in\Pi_6,\text{ for }\,x\in I_i,\,i=1,\dots,n\right\}.$$

We establish a piecewise polynomial $s(x)\in S^{(3)}_{n,6}$ satisfying (1.1) and (1.2), that is,

$$s''\,(x)=-D_x^{-\alpha}s\,(x)+f\,(x)\;,\qquad s\,(0)=\gamma_1\;,s'\,(0)=\gamma_2, \tag{3.6}$$

where $s(x)$ satisfies the following conditions
(1) $s''\,(x_i)=-D_x^{-\alpha}s\,(x_i)+f(x_i)$
(2) for $x\in[0,b]\,,\,s(x)$ and its derivatives up to order 3 are continuous.

For the real numbers $s''_{i-1},s''_{i-\frac{2}{3}},s''_{i-\frac{1}{3}},s''_i,s'''_{i-1},$ with $i=1,\dots,n,$ s_0,s'_0,s'''_0, we represent the unique sextic spline $s\in S^{(3)}_{n,6}$ on I_i as

$$\begin{aligned}&s\,(x)=s_{i-1}+hs'_{i-1}\mathcal{A}\,(t)+h^2s''_{i-1}\mathcal{B}\,(t)+h^2s''_{i-\frac{2}{3}}\mathcal{C}\,(t)+h^2s''_{i-\frac{1}{3}}\mathcal{D}\,(t)+h^2s''_iE\,(t)+\\&h^3s'''_{i-1}\mathcal{F}\,(t)\,,\end{aligned} \tag{3.7}$$

in which $t=\frac{x-x_{i-1}}{h}$ and $\mathcal{E}(t),\mathcal{D}(t),\mathcal{C}(t),\mathcal{B}(t),\mathcal{A}(t)$ and $\mathcal{F}(t)$ represent the polynomials of degree at most 6. Similarly, we can calculate $\mathcal{E}(t),\mathcal{D}(t),\mathcal{C}(t),\mathcal{B}(t),\mathcal{A}(t)$ and $\mathcal{F}(t)$ for quartic C^3-splines as

$$\begin{aligned}&\mathcal{A}\,(t)=t,\\&\mathcal{B}\,(t)=\tfrac{-33}{40}t^6+\tfrac{9}{4}t^5-\tfrac{85}{48}t^4+\tfrac{1}{2}t^2,\\&\mathcal{C}\,(t)=\tfrac{27}{20}t^6-\tfrac{27}{8}t^5+\tfrac{9}{4}t^4,\\&\mathcal{D}\,(t)=\tfrac{-27}{40}t^6+\tfrac{27}{20}t^5-\tfrac{9}{16}t^4,\\&\mathcal{E}\,(t)=\tfrac{3}{20}t^6-\tfrac{9}{40}t^5+\tfrac{1}{12}t^4,\\&\mathcal{F}\,(t)=\tfrac{-3}{20}t^6+\tfrac{9}{20}t^5-\tfrac{11}{24}t^4+\tfrac{1}{6}t^3,\end{aligned} \tag{3.8}$$

Now, with the help of Definition 2.4 and setting $x=x_i$, we obtain

$$\begin{aligned}&D^{-\alpha}\,s\,(x)|_{x=x_i}=\tfrac{t^\alpha}{\Gamma(\alpha+1)}\,s_{i-1}|_{t=1}+h\,D^{-\alpha}\,(\mathcal{A}\,(t))|_{t=1}s'{}_{i-1}+h^2\,D^{-\alpha}\,(\mathcal{B}\,(t))|_{t=1}s''{}_{i-1}\\&+h^2\,D^{-\alpha}\,(\mathcal{C}\,(t))|_{t=1}s''{}_{i-\frac{2}{3}}+h^2\,D^{-\alpha}\,(\mathcal{D}\,(t))|_{t=1}s''{}_{i-\frac{1}{3}}+h^2\,D^{-\alpha}\,(\mathcal{E}\,(t))|_{t=1}s''{}_i+h^2\\&D^{-\alpha}\,(\{\,(t))|_{t=1}s'''{}_{i-1},\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}&D^{-\alpha}\,(\mathcal{A}\,(t))|_{t=1}=\tfrac{1}{\Gamma(\alpha+2)},\\&D^{-\alpha}\,(\mathcal{B}\,(t))|_{t=1}=\tfrac{-594}{\Gamma(\alpha+7)}+\tfrac{270}{\Gamma(\alpha+6)}-\tfrac{85}{2\Gamma(\alpha+5)}+\tfrac{1}{\Gamma(\alpha+3)},\\&D^{-\alpha}\,(C\,(t))|_{t=1}=\tfrac{972}{\Gamma(\alpha+7)}-\tfrac{405}{\Gamma(\alpha+6)}+\tfrac{54}{\Gamma(\alpha+5)},\\&D^{-\alpha}\,(\mathcal{D}\,(t))|_{t=1}=\tfrac{-486}{\Gamma(\alpha+7)}+\tfrac{162}{\Gamma(\alpha+6)}-\tfrac{27}{2\Gamma(\alpha+5)},\\&D^{-\alpha}\,(\mathcal{E}\,(t))|_{t=1}=\tfrac{108}{\Gamma(\alpha+7)}-\tfrac{27}{\Gamma(\alpha+6)}+\tfrac{2}{\Gamma(\alpha+5)},\\&D^{-\alpha}\,(\mathcal{F}\,(t))|_{t=1}=\tfrac{-108}{\Gamma(\alpha+7)}+\tfrac{54}{\Gamma(\alpha+6)}-\tfrac{11}{\Gamma(\alpha+5)}+\tfrac{1}{\Gamma(\alpha+4)}.\end{aligned}$$

Finally, we establish $s(x)$ to approximate the numerical solution $y(x)$ of Eqs. (1.1)-(1.2), which fulfills the following properties for $i = 1, \ldots, n$:

$$\begin{aligned}
&s_{i-\frac{2}{3}} = s_{i-1} + \tfrac{1}{3}hs'_{i-1} + \tfrac{271}{6480}h^2s''_{i-1} + \tfrac{17}{1080}h^2s''_{i-\frac{2}{3}} - \tfrac{1}{432}h^2s''_{i-\frac{1}{3}} + \tfrac{1}{3240}h^2s''_i + \\
&\tfrac{7}{3240}h^3s'''_{i-1}, \\
&s_{i-\frac{1}{3}} = s_{i-1} + \tfrac{2}{3}hs'_{i-1} + \tfrac{13}{135}h^2s''_{i-1} + \tfrac{16}{135}h^2s''_{i-\frac{2}{3}} + \tfrac{1}{135}h^2s''_{i-\frac{1}{3}} + \tfrac{2}{405}h^3s'''_{i-1}, \\
&s_i = s_{i-1} + hs'_{i-1} + \tfrac{37}{240}h^2s''_{i-1} + \tfrac{9}{40}h^2s''_{i-\frac{2}{3}} + \tfrac{9}{80}h^2s''_{i-\frac{1}{3}} + \tfrac{1}{120}h^2s''_i + \tfrac{1}{120}h^3s'''_{i-1}, \\
&s'_i = s'_{i-1} + \tfrac{13}{60}hs''_{i-1} + \tfrac{9}{40}hs''_{i-\frac{2}{3}} + \tfrac{9}{20}hs''_{i-\frac{1}{3}} + \tfrac{13}{120}hs_i'' + \tfrac{1}{60}h^2s'''_{i-1}, \\
&s'''_i = \tfrac{-13}{2}\tfrac{1}{h}s''_{i-1} + \tfrac{27}{2}\tfrac{1}{h}s''_{i-\frac{2}{3}} - \tfrac{27}{2}\tfrac{1}{h}s''_{i-\frac{1}{3}} + \tfrac{13}{2}\tfrac{1}{h}s_i'' - s'''_{i-1},
\end{aligned} \tag{3.10}$$

in which $s''_a = f(x_a) - D_x^{-\alpha}s\,(x_a)$, $a = i-1, i-\frac{2}{3}, i-\frac{1}{3}, i,\ s_0 = \gamma_1, s'_0 = \gamma_2$, and the unknown coefficients $s_{i-\frac{2}{3}}, s_{i-\frac{1}{3}}, s_i,\ i \geq 1$ may be computed after solving the system (3.10) for $i = 1, \ldots, n$.

4 Convergence Analysis

This section studies the convergence analysis of the problem (1.1) with homogenous conditions. For this aim, we consider that

$$u''(x) = \psi(x), \tag{4.1}$$

with ICs

$$u(0) = 0,\ u'(0) = 0, \tag{4.2}$$

has a unique solution. Therefore, there is a Green's function $H(x,s)$ for the problem, such that

$$u(x) = \int_0^x H(x,s)\,\psi(s)\,ds = H\psi(x)$$

and

$$H(x,s) = (x-s).$$

The operator $H\psi(x)$ fulfills the following conditions:

(1)

$$\lim_{h\to 0}\left(\max_{t,s\in[0,b]}\ \max_{|s-t|\leq h}\int_0^b |H(t,x) - H(s,x)|\,dx\right) = 0,$$

(2)

$$\max_{t\in[0,b]}\int_0^b |H(t,s)|\,ds < \infty,$$

which implies that $H\psi(x)$ is a bounded and compact operator [3]. In what follows, we consider a theorem as follows.

Theorem 4.1. *Suppose that $u(x)$ fulfills the FIDE problem (4). Then, we obtain*

$$D^{-\alpha}u(x) = D^{-\alpha}\int_a^x H(x,s)\,\psi(s)\,ds = \int_a^x \left(D^{-\alpha}H(x,s)\right)\psi(s)\,ds = D^{-\alpha}H\psi(x)\,.$$

Proof. By using the definition of RLFI in Eq. (4), we have

$$\begin{aligned} D^{-\alpha}u\left(x\right) &= D^{-\alpha}\int_{s=a}^{s=x} H\left(x,s\right)\psi\left(s\right)ds \\ &= \frac{1}{\Gamma\left(\alpha\right)}\int_{t=a}^{t=x}\left(x-t\right)^{\alpha-1}\left(\int_{s=a}^{s=x} H\left(x,s\right)\psi\left(s\right)ds\right)\right)dt \\ &= \int_{s=a}^{s=x}\frac{1}{\Gamma\left(\alpha\right)}\left(\int_{t=a}^{t=x}\left(x-t\right)^{\alpha-1}H\left(t,s\right)dt\right)\psi\left(s\right)ds \\ &= \int_{s=a}^{s=x}\left(D^{-\alpha}H\left(x,s\right)\right)\psi\left(s\right)ds = D^{-\alpha}H\psi\left(x\right). \end{aligned}$$

□

Theorem 4.2. *Suppose that* $s(x) \in S_{n,i}^{(3)}$, $i = 4, 6$, *and* $u(x)$ *represent the solutions of Eq.* (3.1) *and Eqs.* (1.1)–(1.2), *respectively. If* $n \geq N_0$, *then for the constants* c_k *and* c_0, *independent of* u *and* h, *we obtain*

$$\begin{aligned} &\left\|u-s\left(x\right)\right\| \leq c_k\left\|u^{(k+2)}\right\|h^k, \quad for\, u\in C^{k+2}\left[a,b\right], \quad k=1,\ldots,2, \\ &\left\|u-s\left(x\right)\right\| \leq c_0\psi\left(u'',h\right), \quad for\, u\in C^{2}\left[a,b\right], \end{aligned} \tag{4.3}$$

where

$$\psi\left(\phi,h\right)=\sup\left\{\left|\phi\left(x+h\right)-\phi(x)\right|: x, x+h\in\left[a,b\right]\right\}. \tag{4.4}$$

Proof. Following [23] and employing Eq. (4.1) and Theorem 4.1, the FIDE (1.1) can be rewritten as

$$\psi\left(x\right)+\int_0^x D^{-\alpha}G\left(x,s\right)\psi\left(s\right)ds = f\left(x\right), \tag{4.5}$$

in which the operator $D^{-\alpha}G$ is compact. Therefore, the solution of Eqs. (1.1)-(1.2) is equivalent to the solution of Eq. (4.5), which can be written in operator form as:

$$(I + D^{-\alpha}G)\psi = f. \tag{4.6}$$

Since $s(x) \in S_{n,i}^{(3)}$, $i = 4, 6$, $s(x) \in C^3[0, b]$ and $s''(x) \in C^1[a, b]$, we set

$$s''\left(x\right) = \psi_n\left(x\right). \tag{4.7}$$

Then, we can notice that $\psi_n(x)$ represents a continuous piecewise polynomial that fulfills the homogeneous ICs. Now, let us introduce a linear projection P_c, which maps every continuous function into

$$S_{n,j} = \left\{s(x):\ s \in C^1[0,b],\ s\in\Pi_j\right\},\ j=4,6,$$

where $\lim_{h\to 0}\|P_c\psi-\psi\|_\infty = 0$, which implies that [31]:

$$\lim_{h\to 0}\left\|P_cD^{-\alpha}G - D^{-\alpha}G\right\|_\infty = 0,$$

for any continuous function ψ. Based on Theorem 4.1, we get

$$s''\left(x\right) = -D^{-\alpha}Gs\left(x\right)+f\left(x\right). \tag{4.8}$$

Substituting Eq. (4.7) into Eq. (4.8), taking P_c on both sides of Eq. (4.8) and knowing that $P_c\psi_n = \psi_n$, we can obtain

$$\psi_n + P_c D^{-\alpha} G\psi_n = P_c f. \tag{4.9}$$

Applying the linear projection operator P_c on both sides of (4.6) yields

$$P_c\psi + P_c D^{-\alpha} G\psi = P_c f. \tag{4.10}$$

According to Eqs. (4.9) and (4.10), we easily obtain

$$(I + P_c D^{-\alpha} G)(\psi - \psi_n) = \psi - P_c\psi. \tag{4.11}$$

Following [33], $(I + P_c D^{-\alpha} G)^{-1}$ is bounded. Therefore, it follows that

$$\psi - \psi_n = (I + P_c D^{-\alpha} G)^{-1}(\psi - P_c\psi). \tag{4.12}$$

By taking G over both sides of (4.12) and using (4.1) and (4), we obtain

$$u - s(x) = H(I + P_c D^{-\alpha} G)^{-1}(u'' - P_c u''). \tag{4.13}$$

Due to boundness of the operator G, we get

$$\|u - s(x)\| \le \|G\| \left\|(I + P_c D^{-\alpha} G)^{-1}\right\| \|u'' - P_c u''\| . \tag{4.14}$$

In view of the interpolation theory [31], we obtain

$$\begin{aligned} &\|u'' - P_c u''\| \le \eta_k \left\|u^{(k+2)}\right\| h^k, \quad for\, u \in C^{k+2}\left[a,b\right], \quad 1 \le k \le 2, \\ &\|u'' - P_c u''\| \le \eta_0 \psi\left(u'', h\right), \quad for\, u \in C^{2}\left[a,b\right]. \end{aligned} \tag{4.15}$$

Following [33], we have $\left\|(I + P_c D^{-\alpha} G)^{-1}\right\| \le \delta$, for $n \ge N_0$. Finally, we get $c_0 = \delta\eta_0 \|G\|$ and $c_k = \delta\eta_k \|G\|$, $k = 1, 2$, which finishes the proof. □

5 Numerical examples and discussion

To the best of our knowledge, the FIDE (1.1) with ICs (1.2) has not been solved yet numerically. Herein, we present three numerical examples in order to highlight the effectiveness of the proposed strategy. For this purpose, we define the L_{rms} error as:

$$L_{\text{rms}} = \left[\sum_{j=0}^{n} \frac{{e_n}^2\left(x_j\right)}{n}\right]^{\frac{1}{2}},$$

where $e_n = u(x_n) - s(x_n)$.

Example 5.1. *Let us consider the FIDE*

$$u''\left(x\right) = -D^{-\alpha} u\left(x\right) + \frac{120}{20\Gamma\left(6+\alpha\right)} x^{5+\alpha} - \frac{24}{12\Gamma\left(5+\alpha\right)} x^{4+\alpha} + x^3 - x^2,$$

$$u\left(0\right) = 0,\ u'\left(0\right) = 0.$$

The corresponding analytical solution is $u(x) = \frac{1}{20}x^5 - \frac{1}{12}x^4$.

Table 1 reports the L_{rms} errors of the numerical solution and the CPU running time (in seconds) of the quartic and sextic C^3-spline methods. We verify that the numerical and analytical solutions are close to each other.

Table 1: The L_{rms} errors of the approximate solution and the computational time of Example 5.1.

Quartic C^3-spline	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 0.95$	CPU time (s)
$n = 16$	$2.7639\ E-7$	$3.3236E-4$	$3.7863E-4$	0.078125
$n = 32$	$1.6936\ E-8$	$3.3573\ E-4$	$3.8662\ E-4$	0.171875
$n = 64$	$1.0480E-9$	$3.3698\ E-4$	$3.9013\ E-4$	0.453125
$n = 128$	$8.1587E-11$	$7.8840E-5$	$8.9344E-5$	1.281250
Sextic C^3-spline				
$n = 16$	$2.8093\ E-7$	$2.3549\ E-4$	$2.3958\ E-4$	0.328125
$n = 32$	$3.4697\ E-8$	$2.3522\ E-4$	$2.4178\ E-4$	0.625125
$n = 64$	$4.3129\ E-9$	$2.3468\ E-4$	$2.4251\ E-4$	1.375526
$n = 128$	$5.3766\ E-10$	$2.3427\ E-4$	$2.4280\ E-4$	4.515630

Example 5.2. *We consider the following FIDE*

$$u''(x) = -D^{-\alpha}u(x) + \frac{40320}{56\Gamma(9+\alpha)}x^{8+\alpha} - \frac{720}{30\Gamma(7+\alpha)}x^{6+\alpha} + x^6 - x^4,$$

$$u(0) = 0,\ u'(0) = 0.$$

The analytical solution of this example is $u(x) = \frac{1}{30}x^6 - \frac{1}{56}x^8$.

Table 2 lists the L_{rms} error and the CPU running time (in seconds) of the quartic and sextic C^3-spline methods. One can verify that the numerical solutions are in excellent agreement with the analytical ones.

Table 2: The L_{rms} errors of the approximate solutions and the computational time of Example 5.2.

Quartic C^3-spline	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 0.95$	CPU time (s)
$n = 16$	$3.8826\ E-7$	$7.8555\ E-5$	$8.642\ E-5$	0.093751
$n = 32$	$2.2345\ E-8$	$7.8762\ E-5$	$8.8150\ E-5$	0.187512
$n = 64$	$1.3358\ E-9$	$7.8839\ E-5$	$8.8975\ E-5$	0.453125
$n = 128$	$8.1587\ E-11$	$7.8840\ E-5$	$8.9344\ E-5$	1.328131
Sextic C^3-spline				
$n = 16$	$1.6015\ E-7$	$5.7470\ E-5$	$5.6370\ E-5$	0.32812
$n = 32$	$1.9720\ E-8$	$5.6844\ E-5$	$5.6637\ E-5$	0.65625
$n = 64$	$2.4453\ E-9$	$5.6395\ E-5$	$5.6634\ E-5$	1.35938
$n = 128$	$3.0441\ E-10$	$5.6126\ E-5$	$5.6599\ E-5$	4.56250

Example 5.3. *Finally, we consider the FIDE*

$$u''(x) = -D^{-\alpha}u(x) + t^{\alpha+1}E_{2,\alpha+2}(-x^2) - \sin(x)$$

$$u(0) = 0,\ u'(0) = 1.$$

The analytical solution is $u(x) = \sin(x)$.

Table 3 reports the L_{rms} errors of the quartic and sextic C^3-spline methods. W notice that the patterns of the numerical solutions are in excellent agreement with the analytical ones.

Table 3: The L_{rms} errors of the approximate solutions of Example 5.3.

Quartic C^3-spline	$\alpha = 0.5$	$\alpha = 0.95$
$n = 16$	$5.0325E{-4}$	$7.6420E{-4}$
$n = 32$	$3.1702E0{-5}$	$8.8150E{-5}$
Sextic C^3-spline		
$n = 16$	$6.0420E{-5}$	$9.6350E{-5}$
$n = 32$	$1.2844E{-6}$	$3.2341E{-6}$

6 Conclusions

This paper developed numerical schemes based on C^3- splines for the computation of approximate solutions of FIDEs. The uniqueness and existence of the solution were investigated. Additionally, the convergence analysis of the proposed strategy was discussed in detail. Three numerical examples were given to clarify the effectiveness and accuracy of the proposed strategy. Following the results, future study will address the solution of two-dimensional FIDEs.

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