Vol 5(1), 2024, pp:55-64
DOI:10.30511/mcs.2024.2017490.1145
ISSN:2717-2708
http://mcs.qut.ac.ir/

Fourth-kind Chebyshev Computational Approach for Integro-Differential Equations

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Abstract: This study proposes a numerical approach for solving Integro-Differential Equations (IDEs) of the Fredholm and Volterra types. The method utilizes a collocation computational approach with fourth-kind shifted Chebyshev polynomials. By employing this approach, the original IDE problem is transformed into a set of linear algebraic equations, which are subsequently solved using the matrix inversion strategy. The proposed method is applied to three numerical instances, and the obtained results are compared with existing literature solutions. The comparison demonstrates the accuracy and effectiveness of the proposed approach. The study presents the results in tables and figures to provide a clear visual representation of the findings.

Keywords: Fourth kind Chebyshev polynomials, Fredholm and Volterra integro-differential equations; Approximate solution; Matrix inversion.

2020 Mathematics Subject Classification: 65C30; 65L06; 65C03
Receive: 9 December 2023, **Accepted:** 25 March 2024

1 Introduction

Integro-differential equations (IDEs) are encountered in both engineering and science, playing a significant role in studying various phenomena. Pioneering research by Abel, Lotka, Fredholm, Malthus, Verhulst, and Volterra in Mechanics, Mathematical Biology, and Economics laid the foundation for the study of integral and IDEs. For additional information, see [14] and the references cited therein. Epidemiology and mathematical models of epidemics have found IDEs useful, especially when considering age-structured or spatial epidemics [9,17]. IDEs have garnered substantial interest among researchers due to their relevance and applicability across diverse scientific fields. The majority of IDEs cannot be solved analytically, making numerical methods crucial for obtaining approximate solutions. Various authors have contributed numerical methods to tackle IDEs. For instance, Bernstein

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polynomials were used in [12] to solve a family of linear IDEs with weakly singular kernels. The Trapezoidal rule and the Variational Iteration Method (VIM) were investigated for linear IDEs in [24], while VIM was applied to solve fourth-order IDEs in [27]. The Adomian decomposition method was employed in [11] to solve Boundary Value Problems (BVPs) for fourth-order IDEs. Solving Fredholm-Volterra IDEs saw the usage of different approaches: [10] employed a spectral method using Chebyshev polynomials as the basis function, [7] used a fixed-point iterative algorithm, [29] applied the Chebyshev polynomial approach, [3] utilized the Bernstein collocation method, [30] implemented a collocation method based on Bernstein polynomials, and [1] utilized a Bernstein collocation approach known as the projection method. In solving Volterra-Fredholm IDEs, various methods were employed: [6] used the differential transform method, [16] employed the Bernstein operational matrix approach, [18] utilized the Chebyshev collocation method, and [26] adopted the reliable iterative method. For Fredholm IDEs, methods like Adomian's decomposition approach [28], the extended minimal residual method [4], the Hermite collocation method [2], the Bernoulli matrix method [8], and the quadrature-difference method [13] were all utilized. Furthermore, a new numerical method for solving the Volterra integro-differential equation system was developed in [15]. The study in [25] and [5] examined the use of third-kind Chebyshev polynomials to solve IDEs, offering a technique termed a "computational approach using Fourth Kind Shifted Chebyshev Polynomials." This approach was motivated and inspired by the previous work and aimed to enhance the outcomes produced in [5]. [19] utilized the least squares collocation Chebyshev method to solve systems of linear fractional integral-differential equations (IDEs). [20] utilized a computational algorithm for fractional Fredholm integral-differential equations (IDEs). [21] Employed the second-kind Chebyshev collocation technique for Volterra-Fredholm fractional-order IDEs. [22] Explored the solution of Volterra-Fredholm IDEs using the Chebyshev computational approach. [23] Implemented the Homotopy Perturbation Technique for fractional Volterra and Fredholm IDEs. The general form of the class of problem considered in this work is given as:

$$\sum_{i=0}^{n}\rho_i(\tau)\xi^i(\tau) = f(\tau) + \int_0^{\tau} K(\tau,\upsilon)\xi(t)d\upsilon \tag{1.1}$$

and

$$\sum_{i=0}^{n}\rho_i(\tau)\xi^i(\tau) = f(\tau) + \int_0^{1} K(\tau,\upsilon)\xi(t)d\upsilon \tag{1.2}$$

With the initial conditions

$$\xi^r(0) = \xi_i \quad r = 0,1,2,\cdots n-1 \tag{1.3}$$

where K and $\rho_i(\tau)$, $i = 0,1,2,\cdots n$ with $\rho_i(\tau) \neq 0$ are known functions, $f(\tau)$ is a known function and $\xi^i(\tau)$ is the i^{th} derivatives unknown function $\xi(\tau)$ to be determined, and Eq. (1.1) and Eq. (1.2) are referred to as Volterra and Fredholm FIDEs respectively.

2 Some Relevant Basic Definitions

An integro-differential equation is an equation that has an unknown function, $\xi(\tau)$, that appears under the integral sign. Standard integro-differential equations have the following form:

$$\xi(\tau) = f(\tau) + \lambda \int_{g(\tau)}^{h(\tau)} K(\tau,\upsilon)\xi(\tau)d\upsilon$$

where $K(\tau,\upsilon)$ is a function of two variables τ and υ known as the kernel or the nucleus of the integral equation, g(τ) and h(τ) are the limits of integration, λ is a constant parameter.

Definition 2.2

Chebyshev Polynomials of the Fourth Kind (CPFK): The CPFK are orthogonal polynomials with respect to the weight function $W(x) = \sqrt{\frac{1-\tau}{1+\tau}}\ \forall\ \tau \in [-1,1]$, is defined by $W_n(\tau) = \frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\left(\frac{\theta}{2}\right)}$, where $\tau = \cos\theta$ and the recurrence relation

$\varsigma_{n+1}(\tau) = 2\tau\varsigma_n(\tau) - \varsigma_{n-1}(\tau); n \geq 1$, starting with

$\varsigma_0(\tau) = 1, \varsigma_1(\tau) = 2\tau + 1$

Hence, the first few Chebyshev Polynomials of the fourth kind is given below:

$\varsigma_0(\tau) = 1, \varsigma_1(\tau) = 2\tau + 1, \varsigma_2(\tau) = 4\tau^2 + 2\tau - 1, \varsigma_3(\tau) = 8\tau^3 + 4\tau^2 - 3\tau - 1$

Definition 2.3

Shifted Chebyshev Polynomials of the Fourth Kind (SCPFK): The SCPFK is orthogonal polynomials with respect to the weight function $W^*(\tau) = \sqrt{\frac{1-\tau}{t}}\ \forall\ t \in [0,1]$, is defined by

${\varsigma^*}_n(\tau) = \varsigma_n(2\tau - 1)$ Where $\varsigma_n(\tau)$ is CPFK and the recurrence relation

${\varsigma^*}_{n+1}(\tau) = 2(2\tau - 1){\varsigma^*}_n(\tau) - {\varsigma^*}_{n-1}(\tau); n \geq 1$, starting with

${\varsigma^*}_0(\tau) = 1, {\varsigma^*}_1(\tau) = 4\tau - 1$

Hence, the first few Shifted Chebyshev Polynomials of the fourth kind is given below:

${\varsigma^*}_0(\tau) = 1, {\varsigma^*}_1(\tau) = 4\tau - 1, {\varsigma^*}_2(\tau) = 16\tau^2 - 12\tau + 1, {\varsigma^*}_3(\tau) = 64\tau^3 - 80\tau^2 + 24\tau - 1$

Definition 2.4

Absolute Error: We defined absolute error as follows in this study: Absolute Error=$|\boldsymbol{\xi}(\tau) - \xi(\tau)|$; $0 \leq \tau \leq 1$, where $\boldsymbol{\varsigma}(\tau)$ is the exact solution and $\xi(x)$ is the approximate solution.

3 Proposed method

In order to find the numerical approximation to the general class of problem considered in this study, we assumed an approximate solution by means of the fourth kind shifted Chebyshev polynomials in the form:

$$\xi(\tau) = \sum_{r=0}^{n} {\varsigma^*}_r(\tau)\, e_r \tag{3.1}$$

Where, $e_r, r = 0(1)n$ are to be found.

Thus, substituting Eq. (3.1) into Eq. (1.1) gives

$$\sum_{i=0}^{n} \rho_i(\tau) \sum_{r=0}^{n} {\varsigma^{*i}}_r(\tau)\, e_r = f(\tau) + \int_0^x k(\tau, v) \sum_{i=0}^{n} \varsigma^*(v)\, e_i dv \tag{3.2}$$

Let $p(\tau) = \sum_{i=0}^{n} \rho_i(\tau) \sum_{r=0}^{n} {\varsigma^{*i}}_r(\tau)\, e_r$ and $q(\tau) = \int_0^x k(\tau, v) \sum_{i=0}^{n} \varsigma^*(v)\, e_i dv$

Thus, Eq. (3.2) becomes

$$p(\tau) - q(\tau) = f(\tau) \tag{3.3}$$

The linear algebraic system of equations in (n+1) unknown constants $e'_i s$ is obtained by collocating Eq. (3.3) at the evenly spaced point $x_i = a + \frac{(b-a)i}{n}$, $(i = 0(1)n)$. Additional equations are obtained from Eq. (1.3), which are represented in matrix form:

$$\begin{pmatrix} H_{11} & H_{12} & H_{13} & H_{14} & \cdots & H_{1n} \\ H_{21} & H_{22} & H_{23} & H_{24} & \cdots & H_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ H_{m1} & H_{m2} & H_{m3} & H_{m4} & \cdots & H_{mn} \\ {H_{11}}^{0} & {H_{12}}^{0} & {H_{13}}^{0} & {H_{14}}^{0} & \cdots & {H_{1n}}^{0} \\ {H_{21}}^{1} & {H_{22}}^{1} & {H_{23}}^{1} & {H_{24}}^{1} & \vdots & {H_{2n}}^{0} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ {H_{m1}}^{n-1} & {H_{m2}}^{n-1} & {H_{m3}}^{n-1} & {H_{m4}}^{n-1} & & {H_{mn}}^{n-1} \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} E_{11} \\ E_{21} \\ \vdots \\ E_{mn} \\ {E_{11}}^{0} \\ {E_{22}}^{1} \\ \vdots \\ {E_{mn}}^{n-1} \end{pmatrix} \tag{3.4}$$

Where $H_i's$ and $H_i^{*}{}'s$ are the coefficients of $e_i's$ and $E_i's$ are values of $f(\tau_i)$.
The matrix inversion approach is then used to solve the system of equations in order to obtain the unknown constants.

$$\begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} & H_{13} & H_{14} & \cdots & H_{1n} \\ H_{21} & H_{22} & H_{23} & H_{24} & \cdots & H_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ H_{m1} & H_{m2} & H_{m3} & H_{m4} & \cdots & H_{mn} \\ {H_{11}}^{0} & {H_{12}}^{0} & {H_{13}}^{0} & {H_{14}}^{0} & \cdots & {H_{1n}}^{0} \\ {H_{21}}^{1} & {H_{22}}^{1} & {H_{23}}^{1} & {H_{24}}^{1} & \vdots & {H_{2n}}^{0} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ {H_{m1}}^{n-1} & {H_{m2}}^{n-1} & {H_{m3}}^{n-1} & {H_{m4}}^{n-1} & & {H_{mn}}^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} E_{11} \\ E_{21} \\ \vdots \\ E_{mn} \\ {E_{11}}^{0} \\ {E_{22}}^{1} \\ \vdots \\ {E_{mn}}^{n-1} \end{pmatrix} \tag{3.5}$$

The required approximate solution is obtained by solving Eq. (3.5) and then substituting the unknown constant values into the assumed approximate solution.

4 Numerical Examples

Example 4.1 [5]: Consider the fourth-order Volterra integro- differential equation
$\xi^{(iv)}(\tau) = -1 + \xi(\tau) + \int_0^{\tau}(\tau - \upsilon)\xi(\upsilon)d\upsilon$

Subject to the initial conditions

$\xi(0) = -1, \xi'(0) = 1, \xi''(0) = 1, \xi'''(0) = -1,$

The exact solution is $\xi(\tau) = \sin\tau - \cos\tau$
By applying the aforementioned technique to example 4.1, which is solved at n = 10, we are able to determine the following constants and the necessary approximation.
$e_0 = -0.702418604428818$, $e_1 = 0.316575052724382$, $e_2 = 0.0156642092773358$, $e_3 = -0.00341500400772488$, $e_4 = -0.0000749271943344854$, $e_5 = 0.0000107950292978990$, $e_6 = 1.50113588427696 \times 10^{-7}$, $e_7 = -1.61558826215297 \times 10^{-8}$, $e_8 = -1.63786226307026 \times 10^{-10}$, $e_9 = 1.40203579745820 \times 10^{-11}$, $e_{10} = 1.08912208183271 \times 10^{-13}$
$\varsigma^{*}(\tau) = -1 + 0.9999999996\tau + 0.5000000002\tau^2 - 0.1666666668\tau^3 - 0.04166666682\tau^4 + 0.008333354748\tau^5 + 0.001388767122\tau^6 - 0.0001980941364\tau^7 - 0.00002526207961\tau^8 + 1.142027276 \times 10^{-7}\tau^{10}$

Table 1 Shows comparison of the absolute errors for example 4.1

x	Exact Solution	Approximate Solution	Absolute Error of our Method n=10	Absolute Error of [5] n=10
0.0	-1.000000000	-1.000000000000	0.000e+00	6.00e-09
0.2	-0.7813972470	-0.781397247100	1.000e-10	2.10e-09
0.4	-0.5316426517	-0.531642652000	3.000e-10	6.20e-09
0.6	-0.26069314150	-0.260693141500	0.000e+00	6.80e-09
0.8	0.020649381600	0.020649381000	1.000e-10	4.77e-09
1.0	0.301168678900	0.301168680000	0.000e+00	9.55e-07

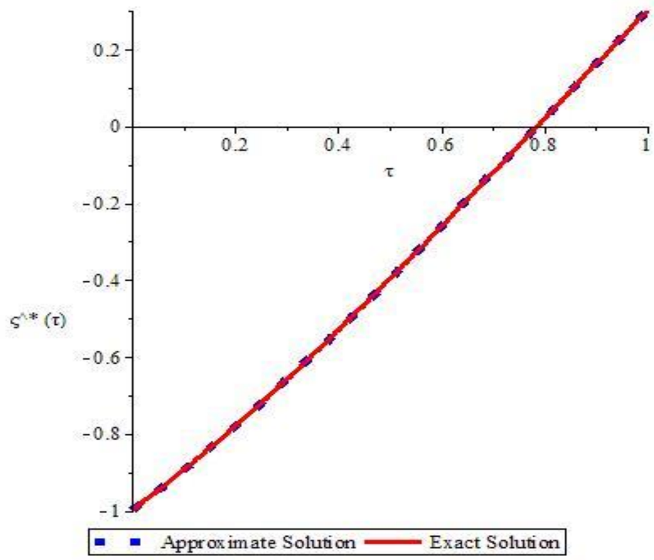

Figure 1: Shows the graphical representation of the exact and approximate solutions to the example 4.1

Example 4.2 [5]: Consider the second-order Volterra integro- differential equation
$\xi^{(ii)}(\tau) = 2 - 2\tau \sin \tau - \int_0^{\tau} (\tau - \upsilon)\xi(\upsilon)d\upsilon$

Subject to the initial conditions

$\xi(0) = 0, \xi'(0) = 0$

The exact solution is $\xi(\tau) = \tau \sin \tau$

By applying the aforementioned technique to example 4.2, which is solved at n = 10, we are able to determine the following constants and the necessary approximation.

$e_0 = 0.116149685819937$, $e_1 = 0.169864800444855$, $e_2 = 0.0500277617217244$, $e_3 = -0.00425159101715328$, $e_4 = -0.000544932540461251$, $e_5 = 0.0000210751400980353$, $e_6 = 0.00000172924157487990$, $e_7 = -4.29634817635666 \times 10^{-8}$, $e_8 = -2.59349039466450 \times 10^{-9}$, $e_9 = 4.72742209961519 \times 10^{-11}$, $e_{10} = 2.27751107903518 \times 10^{-12}$

$\varsigma^*(\tau) = 2.296965508 \times 10^{-11} - 1.053327574 \times 10^{-10}\tau + 0.9999999994\tau^2 + 1.9650563 \times 10^{-9}\tau^3 - 0.1666666845\tau^4 + 0.16080561 \times 10^{-7}\tau^5 + 0.008332879919\tau^6 + 0.00000103339049\tau^7 - 0.0001997991416\tau^8 + 0.00000104897197\tau^9 + 0.000002388143457\tau^{10}$

Table 2 Shows comparison of the absolute errors for example 4.2

x	Exact Solution	Approximate Solution	Absolute Error of our Method n=10	Absolute Error of [5] n=10
0.0	0.00000000000000	0.00000000002297	2.297e-11	1.13e-10
0.2	0.03973386616000	0.03973386615000	2.000e-11	2.56e-07
0.4	0.15576733690000	0.15576733680000	1.000e-10	2.22e-07
0.6	0.33878548400000	0.33878548380000	4.000e-10	1.68e-07
0.8	0.57388487270000	0.57388487230000	5.000e-10	5.38e-07
1.0	0.84147098480000	0.84147098420000	6.000e-10	9.55e-07

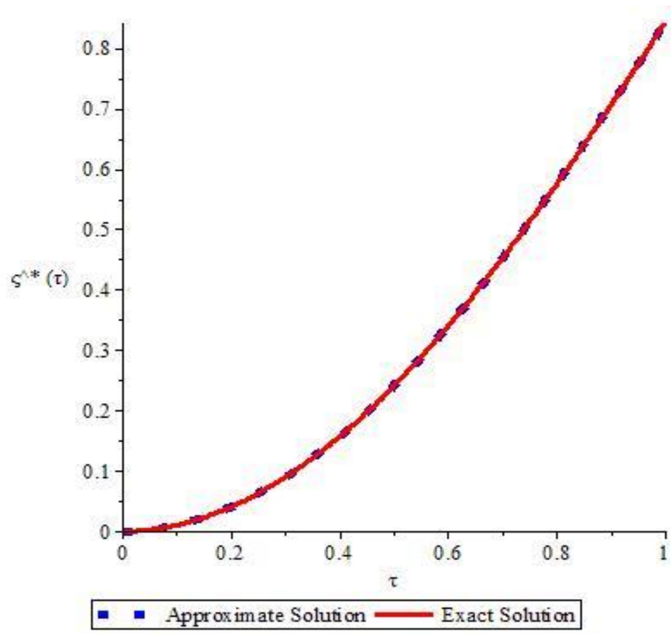

Figure 2: Shows the graphical representation of the exact and approximate solutions to the example 4.2

Example 4.3 [5]: Consider the second-order Fredholm integro- differential equation

$\xi^{(ii)}(\tau) = e^{\tau} - \frac{4}{3}\tau + \int_0^1 \tau\upsilon\xi(\upsilon)d\upsilon$

Subject to the initial conditions

$\xi(0) = 1, \xi'(0) = 2$

The exact solution is $\xi(\tau) = \tau + e^{\tau}$

By applying the aforementioned technique to example 4.3, which is solved at n = 9, we are able to determine the following constants and the necessary approximation.

$e_0 = 1.57819182754921$, $e_1 = 0.622591480137829$, $e_2 = 0.0482432944405358$, $e_3 = 0.00408933394967816$, $e_4 = 0.000258160701943420$, $e_5 = 0.0000129936485438083$, $e_6 = 5.43944662926427 \times 10^{-7}$, $e_7 = 1.94942116615859 \times 10^{-8}$, $e_8 = 6.11360058064281 \times 10^{-10}$, $e_9 = 1.74654873896785 \times 10^{-11}$

$\varsigma^*(\tau) = 1 + 2\tau + 0.5\tau^2 + 0.1666667083\tau^3 + 0.04166629763\tau^4 + 0.008334857410\tau^5 + 0.001385295380\tau^6 + 0.0002034838614\tau^7 + 0.00002060758368\tau^8 + 0.000004578472726\tau^9$

Table 3 Shows comparison of the absolute errors for example 4.3

x	Exact Solution	Approximate Solution	Absolute Error of our Method n=9	Absolute Error of [5] n=9
0.0	1.00000000000000	1.00000000000000	0.000e+00	4.79e-06
0.2	1.42140275800000	1.42140275900000	1.600e-10	5.03e-06
0.4	1.89182469800000	1.89182469700000	5.000e-11	6.74e-06
0.6	2.42211880000000	2.42211880000000	2.000e-10	7.91e-06
0.8	3.02554092800000	3.02554093000000	8.000e-10	7.58e-06
1.0	3.71828182800000	3.71828182800000	6.000e-10	1.11e-05

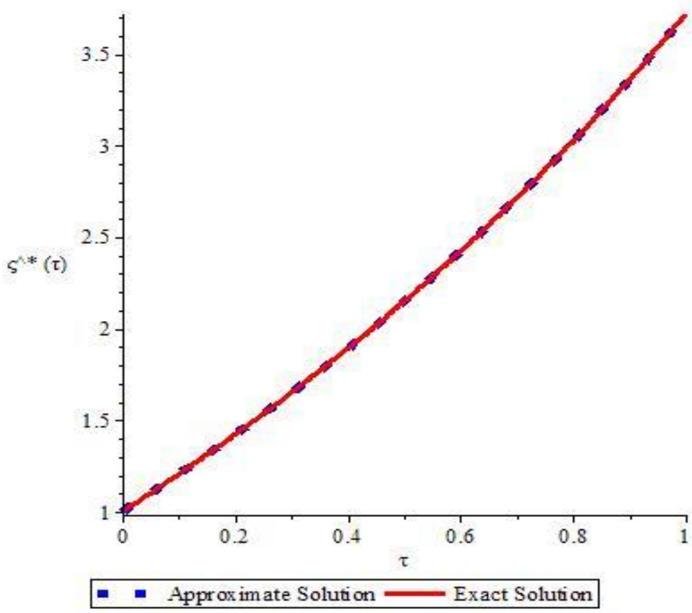

Figure 3: Shows the graphical representation of the exact and approximate solutions to the example 4.3

5 Conclusion

In this work, the successful application of the collocation method to IDEs enabled the researchers to obtain numerical solutions using fourth-kind shifted Chebyshev polynomials. In a related study [5], the collocation method and Chebyshev third-kind polynomials were utilized with n = 5 and 10 to solve three numerical examples considered in this research. The results presented in the table demonstrate that the proposed strategy outperforms their approach in terms of performance. Additionally, Figures 1–3 exhibit excellent agreement between the approximations and exact solutions achieved through the suggested method. Based on these findings, we highly recommend employing the suggested approach when dealing with other integro-differential equations.

Acknowledgement: Authors thank those who contributed to write this article and give some valuable comments.

Funding Statement: The authors received no specific funding for this study.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

References

[1] N.I. Acar, A. Daşcıoglu, A projection method for linear Fredholm-Volterra integro-differential equations, Journal of Taibah University for Science, 13(1) 2019, 644-650.

[2] N. Akgonullu, N. sahin, M. Sezer, A. Hermite collocation method for the approximation solutions of higher-order linear Fredholm integro-differential equations, Numerical Methods for Partial Differential Equations, 27(6) 2011, 1707-1721.

[3] A. Akyuz-DaGcJoLlu, N. Acar, C. Guler, Bernstein collocation method for solving nonlinear Fredholm-Volterra integro differential equations in the most general form, Journal of Applied Mathematics, 2014, 1-8.

[4] E. Aruchunan, J. Sulaiman, Numerical solution of second order linear Fredholm integro-differential equations using generalized minimal residual method, American Journal of Applied Sciences,7(6) 2010, 780-783.

[5] A.M. Ayinde, A.A James, A.A. Ishaq, T. Oyedepo, A new numerical approach using Chebyshev third kind polynomial for solving integro-differential equations of higher order, Gazi University Journal of Science, Part A, 9(3) 2022, 259-266.

[6] S.H. Behiry, S.I. Mohamed, Solving high-order nonlinear Volterra-Fredholm integro-differential equations by differential transform method, Natural Science, 4(8) 2012, 581-587.

[7] M.I. Berenguer, D. Gamez, A.J.L. Opez Linares, Fixed-point iterative algorithm for the linear Fredholm-Volterra integro-differential equation, Journal of Computational and Applied Mathematics, Article ID 370894, 2012, 1-12.

[8] A. Bhrawy, E. Tohidi, F. Soleymani, A new Bernoulli matrix method for solving high-order linear and nonlinear Fredholm integro-differential equations with piecewise intervals, Appl. Math. Comput. 219 (2) 2012, 482–497.

[9] F. Brauer, V.D.Pauline, W.Jianhong, Mathematical Epidemiology, Lecture Notes in Mathematics, 1945 2008, 205–22.

[10] H.M. El-Hawary, S.T. El-Sheshtawy, Spectral method for solving the general form linear Fredholm–Volterra integro differential equations based on Chebyshev polynomials, J Mod Met Numer Math, 1 2010, 1-11.

[11] I. Hashim, Adomian decomposition method for solving BVPs for fourth-order integro-differential equations, Journal of Computer and Applied Mathematics, 193 2006, 658-664.

[12] O.R. Isık, M. Sezer, Z. Guney, Bernstein series solution of a class of linear integro-differential equations with weakly singular kernel, Applied Mathematics and Computation, 217(16) 2011, 7009-7020.

[13] C. Jalius, Z. Abdul Majid, Numerical solution of second-Order Fredholm integro-differential equations with boundary conditions by Quadrature-Difference method, Hindawi Journal of Applied Mathematics, 2017, Article ID 2645097(2017), 1-5.

[14] V. Lakshmikantham, M.R.M. Rao, Theory of integro-differential equations, Breach Science Publishers, 1995.

[15] R.J. Loh, C. Phang, A new numerical scheme for solving system of Volterra integro-differential equation, Alexandria Enginerring Journal, 57(2) 2018, 1117-1124.

[16] K. Maleknejad, B. Basirat, E. Hashemizadeh, A Bernstein operational matrix approach for solving a system of high order linear Volterra–Fredholm integro-differential equations, Mathematical Computational Modell, 55(3) 2012, 1363-1372.

[17] J. Medlock, Integro-differential-Equation Models for Infectious Disease, Yale University, 2005, 1-31.

[18] V.N. Mishra, H.R. Marasi, H. Shabanian, M.N. Sahlan, Solution of Volterra-Fredholm integro-differential equations using Chebyshev collocation method, Global Journal Technology and Optimization, (1) 2017, 1-4.

[19] T. Oyedepo, O.A. Taiwo, A.J. Adewale, A.A. Ishaq, A.M. Ayinde, Numerical solution of system of linear fractional integro-differential equations by least squares collocation Chebyshev technique, Mathematics and Computational Sciences, 3(2) 2022, 10-21.

[20] T. Oyedepo, C.Y. Ishola, A.A. Ishaq, O.L. Ahmed, A.M. Ayinde, Computational algorithm for fractional Fredholm integro-differential equations, Kathmandu University Journal of Science, Engineering and Technology, 17(1) 2023, 1-7.

[21] T. Oyedepo, A.A. Ayoade, I.K. Otaide, I.A.M. Ayinde, Second kind Chebyshev collocation technique for Volterra-Fredholm fractional order integro-differential equations, Journal of Natural Sciences and Mathematical Research, 8 (2) 2022, 103-110.

[22] T. Oyedepo, A.A. Ayoade, M.O. Oluwayemi, R. Pandurangan, Solution of Volterra-Fredholm Integro-Differential Equations Using the Chebyshev Computational Approach, International Conference on Science, Engineering and Business for Sustainable Development Goals (SEB-SDG), Omu-Aran, Nigeria, 1 2023, 1-6.

[23] T. Oyedepo, M.O. Oluwayemi, M.A. Ayinde, J.A. Osilagun, L.O. Ahmed, Homotopy Perturbation Technique for Fractional Volterra and Fredholm Integro Differential Equations, International Conference on Science, Engineering and Business for Sustainable Development Goals (SEB-SDG), Omu-Aran, Nigeria, 1 2023, 1-6.

[24] R. Saadati, B. Raftari, H. Adibi, S.M. Vaezpour, S. Shakeri, A comparison between the Variational Iteration method and Trapezoidal rule for solving linear integro-differential equations, World Applied Sciences Journal, 4(3) 2008, 321-325.

[25] M.R.A. Sakran, Numerical solutions of integral and integro -differential equations using Chebyshev polynomial of the third kind, Applied Mathematics and Computation, 5 2019, 66 -82.

[26] M.Y. Samaher, Reliable Iterative Method for solving Volterra-Fredholm Integro differential equations, Al-Qadisiyah Journal of Pure Science, 26(2) 2021, 1-11.

[27] N.H. Sweilam, Fourth order integro-differential equations using variational iteration method, Computer Mathematics Applications, 54 2007, 1086-1091.

[28] A.R. Vahidi, E. Babolian, G. AsadiCordshooli, Z. Azimzadeh, Numerical solution of Fredholm integro-differential equation by Adomian's decomposition method, International Journal of Mathematical Analysis, 3 2009, 1769-1773.

[29] G. Yuksel, M. Gulsu, M. Sezer, A Chebyshev polynomial approach for high-order linear Fredholm-Volterra integro-differential equations, Gazi University Journal of Science 25(2) 2012, 393-401.

[30] S. Yuzbası, A collocation method based on Bernstein polynomials to solve nonlinear Fredholm-Volterra integro-differential equations, Applied. Mathematics Computation, 273 2016, 142-154.