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# New definition of fractional derivative included Mittag-Leffler function of conformable type

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**Abstract :** In this paper a new definitionn of fractional derivative and fractional integral in the sense of conformable derivative type is presented. This form of definition shows that it is more compatible with classical natural definition of derivative and is more convinient fractional derivative one. We will define this for $0 \leq \alpha < 1$ and $n-1 \leq \alpha < n$ and further, if $\alpha = 1$ the definition coincides with the classical definition of derivative of first order.

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## 1 Introduction

The origin of the fractional derivative goes back to Hopital's question in the late 17th century, when he asked what happens for $\frac{d^n f}{dx^n}$ if $n = \frac{1}{2}$. After that, mathematicians provided new definitions for the derivative of the arbitary order, most of which were defined in integral form. In 2014, Khalil et al.[13] presented a new definition of fractional derivative, which has many commonalities with ordinary derivative, and called it conformable fractional derivative. In recent years, fractional calculus and consequently fractional differential equations have attracted the attention of many researchers in this field[15, 4, 5, 2, 7, 9]. Fractional differential equations are very important among mathematicians due to their many physical applications [11, 12, 16, 8, 10]. Different definitions are provided for fractional derivatives and fractional integrals. The most important fractional derivatives like Riemann-Liouville, Caputo, Caputo-Fabrizio[6] and Atangana-Baleanu[3] that have been presented have an integral representation [14, 1].

1. Riemann-Liouville fractional derivative of order $\alpha \in [n-1, n)$ of $f$ is

$$^{RL}D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx.$$

2. Caputo fractional derivative of order $\alpha \in [n-1, n)$ of $f$ is

$$^{C}D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx.$$

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3. Caputo-Fabrizio fractional derivative of order $\alpha \in (0,1)$ of $f$ is

$$^{CF}D_a^\alpha(f)(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t f'(x) \exp[-\frac{(t-x)^\alpha}{1-\alpha}]dx,$$

   where $M(\alpha)$ is known to be a normalized function such that $M(0) = M(1) = 1$.

4. Atangana-Baleanu fractional derivative of order $\alpha \in (0,1)$ of $f$ in Caputo sense is

$$^{ABC}D_a^\alpha(f)(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t f'(x) E_\alpha[-\frac{(t-x)^\alpha}{1-\alpha}]dx,$$

   where $M(\alpha)$ has the same properties in the case of the Caputo-Fabrizio fractional derivative.

Most of these definitions do not apply to the elementary properties of the ordinary derivative except the property of linearity. In [13], a new definition of fractional derivative presented that applied to the basic properties of ordinary derivative. In this paper, with the use of this definition and Mittag-Leffler function, we introduce a new fractional derivative and prove the properties of the derivative, and finally, by presenting some examples, we find out the efficiency of this definition.

## 2 Definition of new fractional derivative

We start this section by defining the Mittag-Leffler function. This function appears in many physical phenomena and also occurs naturally in the solution of fractional differential equations. In fact, this function is a generalization of the exponential function, while the gamma function is a generalization of the factorial function.

**Definition 2.1.** *The one parameter Mittag-Leffler function $E_\alpha$ is defined as*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \alpha \in \mathbb{R}, z \in \mathbb{C}$$

In the next definition, we present a new fractional derivative using the Mittag-Leffler function. In the new definition, we will use the first two terms of the expansion of the Mittag-Leffler function, that is, $E_\alpha(\varepsilon t^{-\alpha}) = 1 + \frac{\varepsilon t^{-\alpha}}{\Gamma(\alpha+1)} + O(t^{-2\alpha})$.

**Definition 2.2.** *Suppose function $f$ is given that $f : [0, \infty) \longrightarrow \mathbb{R}$, then the new fractional derivative of order $\alpha$ in the sense of conformable fractional derivative is defined*

$$D^\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(tE_\alpha(\varepsilon t^{-\alpha})) - f(t)}{\varepsilon}, \tag{2.1}$$

*for $t > 0$ and $\alpha \in (0,1)$. In this case we say that $f$ is $\alpha$-differentiable.*

We suppose $f$ is $\alpha$-differentiable in $(0,t)$, and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t)$$

Now by the use of (2.1), we obtain the following important theorem.

**Theorem 2.3.** *If function $f : [0, \infty) \longrightarrow \mathbb{R}$ is $\alpha$-differentiable at $t_0 > 0$, $0 < \alpha \leq 1$, then $f$ is continuous at $t_0$.*

*Proof.* Using $f(tE_\alpha(\varepsilon t_0^{-\alpha})) - f(t_0) = \frac{f(tE_\alpha(\varepsilon t_0^{-\alpha}))-f(t_0)}{\varepsilon}\varepsilon$. So we have

$$\lim_{\varepsilon\to 0}[f(tE_\alpha(\varepsilon t_0^{-\alpha})) - f(t_0)] = \lim_{\varepsilon\to 0} \frac{f(tE_\alpha(\varepsilon t_0^{-\alpha})) - f(t_0)}{\varepsilon} \lim_{\varepsilon\to 0} \varepsilon.$$

Put $h = \frac{\varepsilon t^{1-\alpha}}{\Gamma(\alpha+1)}$, then we have

$$\lim_{h\to 0}[f(t_0 + h) - f(t_0)] = f^{(\alpha)}(t_0) \times 0,$$

then we obtain

$$\lim_{h\to 0} f(t_0 + h) = f(t_0),$$

hence, $f$ will be comtinuous at $t_0$. □

Now we can show that definition (2.1) satisfies all the properties in the following theorem.

**Theorem 2.4.** *Let $f, g$ be $\alpha$-differentiable functions at a point $t > 0$ and $0 < \alpha \leq 1$, then we have*

1. $D^\alpha(af + bg) = aD^\alpha(f) + bD^\alpha(g)$, *for all $a, b \in \mathbb{R}$*
2. $D^\alpha(t^p) = \frac{pt^{p-\alpha}}{\Gamma(\alpha+1)}$, *for all $p \in \mathbb{R}$*
3. $D^\alpha(c) = 0$, *for all constant function $f(t) = c$*
4. $D^\alpha(fg) = fD^\alpha(g) + gD^\alpha(f)$
5. $D^\alpha(\frac{f}{g}) = \frac{gD^\alpha(f)-fD^\alpha(g)}{g^2}$

*Proof.* Here we prove relation (4) and the others are so easy to stablish.

$$\begin{aligned}
D^\alpha(fg)(t) &= \lim_{\varepsilon\to 0} \frac{f(tE_\alpha(\varepsilon t^{-\alpha}))g(tE_\alpha(\varepsilon t^{-\alpha})) - f(t)g(t)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{f(tE_\alpha(\varepsilon t^{-\alpha}))g(tE_\alpha(\varepsilon t^{-\alpha})) - f(t)g(tE_\alpha(\varepsilon t^{-\alpha})) + f(t)g(tE_\alpha(\varepsilon t^{-\alpha})) - f(t)g(t)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \Big(\frac{f(tE_\alpha(\varepsilon t^{-\alpha})) - f(t)}{\varepsilon} \times g(tE_\alpha(\varepsilon t^{-\alpha}))\Big) + f(t) \lim_{\varepsilon\to 0} \frac{g(tE_\alpha(\varepsilon t^{-\alpha})) - g(t)}{\varepsilon} \\
&= D^\alpha(f)(t) \lim_{\varepsilon\to 0} g(tE_\alpha(\varepsilon t^{-\alpha}) + f(t)D^\alpha(g)(t),
\end{aligned}$$

now by the use of (2.3) we have $\lim_{\varepsilon\to 0} g(tE_\alpha(\varepsilon t^{-\alpha}) = g(t)$, so proof completed. □

By the use of definition (2.1) we have the following important lemma.

**Lemma 2.5.** *Let $f$ be $\alpha$-differentiable and also differentiable function at a point $t > 0$ and $0 < \alpha \leq 1$, then*

$$D^\alpha(f)(t) = \frac{t^{1-\alpha}}{\Gamma(\alpha + 1)} f'(t). \tag{2.2}$$

*Proof.* In defenition (2.1) let $h = \frac{\varepsilon t^{1-\alpha}}{\Gamma(\alpha+1)}$, then $\varepsilon = t^{\alpha-1}\Gamma(\alpha + 1)h$, this completes the proof. □

Now by using of (2.2) we give new fractional derivative of some important functions.

**Proposition 2.6.** *Suppose $a \in \mathbb{R}$, then*

1. $D^\alpha(e^{at}) = \frac{at^{1-\alpha}}{\Gamma(\alpha+1)}e^{at}$

2. $D^{\alpha}(\sin at) = \frac{at^{1-\alpha}}{\Gamma(\alpha+1)} \cos at$
3. $D^{\alpha}(\cos at) = -\frac{at^{1-\alpha}}{\Gamma(\alpha+1)} \sin at$

**Remark 2.7.** *Noice that function $f$ could not be differentiable at a point but it could be $\alpha$-differentiable. for example $f(t) = \sqrt{t}$, then $D^{(\frac{1}{2})}f(0) = \frac{1}{2\Gamma(\frac{3}{2})} \neq 0$, but $f'(0)$ does not exist. This case is not true for the known classical fractional derivatives.*

Now for the case $n < \alpha \leq n+1$ we have the following definition.

**Definition 2.8.** *Suppose function $f$ is given that $f : [0, \infty) \longrightarrow \mathbb{R}$, and $n$-differentiable at $t$, and $\alpha \in (n, n+1]$ then the conformable type fractional derivative of order $\alpha$ is defined by the relation*

$$D^{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f^{(\lceil\alpha\rceil-1)}(tE_{\alpha}(\varepsilon t^{(\lceil\alpha\rceil-1)})) - f^{(\lceil\alpha\rceil-1)}(t)}{\varepsilon}, \tag{2.3}$$

*such that $\lceil\alpha\rceil$ is the smallest integer greater than or equal to $\alpha$.*

**Remark 2.9.** *Now by the use of definition (2.3), we can easily show that*

$$D^{\alpha}(f)(t) = \frac{t^{(\lceil\alpha\rceil-\alpha)}}{\Gamma(\lceil\alpha\rceil+1)} f^{(\lceil\alpha\rceil)}(t),$$

*where $\alpha \in (n, n+1]$, and $f$ is $(n+1)$-differentiable at $t > 0$.*

**Theorem 2.10.** *(Rolle's Theorem) Let $f : [a, b] \to \mathbb{R}$, with $a > 0$, is a function that satisfies*

1. *$f$ is continuous on $[a, b]$,*
2. *$f$ is $\alpha$-differentiable for $0 < \alpha < 1$,*
3. *$f(a) = f(b)$.*

*Then, there exists $c \in (a, b)$, such that $D^{\alpha}(f)(c) = 0$*

*Proof.* It is easy to investigate. □

**Theorem 2.11.** *(Mean value Theorem) Let $f : [a, b] \to \mathbb{R}$, with $a > 0$, is a function that satisfies*

1. *$f$ is continuous on $[a, b]$,*
2. *$f$ is $\alpha$-differentiable for $0 < \alpha < 1$,*

*then, there exists $c \in (a, b)$, such that $D^{\alpha}(f)(c) = \frac{f(b)-f(a)}{\frac{\Gamma(\alpha+1)}{\alpha}b^{\alpha} - \frac{\Gamma(\alpha+1)}{\alpha}a^{\alpha}}$*

*Proof.* Let $g(x)$ be a function as follows

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{\frac{\Gamma(\alpha+1)}{\alpha}b^{\alpha} - \frac{\Gamma(\alpha+1)}{\alpha}a^{\alpha}} (\frac{\Gamma(\alpha+1)}{\alpha}x^{\alpha} - \frac{\Gamma(\alpha+1)}{\alpha}a^{\alpha}),$$

this function satisfies in (2.10) conditions. So there exist $c \in (a, b)$, such that $D^{\alpha}(g)(c) = 0$. Since $g(a) = g(b)$, we have

$$f(b) - f(a) = \frac{\Gamma(\alpha+1)}{\alpha}b^{\alpha} - \frac{\Gamma(\alpha+1)}{\alpha}a^{\alpha},$$

given that we know there exist $c \in (a, b)$, such that $D^{\alpha}(g)(c) = 0$ and by using $D^{\alpha}(\frac{\Gamma(\alpha+1)}{\alpha}x^{\alpha}) = 1$, we have $D^{\alpha}(g)(c) = D^{\alpha}(f)(c)$. The completes the proof. □

## 3 New fractional integral

If the function $f$ be $\alpha$-differentiable in $(a, b)$ we define the $\alpha$-fractional integral of $f$ for $a \geq 0$ and $a < t < b$.

**Definition 3.1.** *New $\alpha$-fractional integral of $\alpha$-differentiable function $f$ define as follows*

$$I_a^\alpha(f)(t) = \int_a^t \frac{\Gamma(\alpha+1)}{x^{1-\alpha}} f(x)dx, \tag{3.1}$$

*that $\alpha \in (0, 1)$.*

One of the most imortant results of this definition is the following.

**Theorem 3.2.** *If $f$ is a continuous function in the domain of $I^\alpha$, for $t \geq a$*

$$D^\alpha(I^\alpha(f))(t) = f(t). \tag{3.2}$$

*Proof.* By the difinition of $f$, $I_a^\alpha(f)(t)$ is differentiable, so by the use of (2.5) we have

$$\begin{aligned} D^\alpha(I^\alpha(f))(t) &= \frac{t^{1-\alpha}}{\Gamma(\alpha+1)} \frac{d}{dt} I_a^\alpha(f)(t) = \frac{t^{1-\alpha}}{\Gamma(\alpha+1)} \frac{d}{dt} \int_a^t \frac{\Gamma(\alpha+1)}{x^{1-\alpha}} f(x)dx \\ &= \frac{t^{1-\alpha}}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+1)}{t^{1-\alpha}} f(t) = f(t). \end{aligned}$$

This complets the proof. □

## 4 Illustrative examples

In this section, we will solve some fractional differential equations including the new fractional derivative of order $\alpha$.

**Example 4.1.** *We consider the following equation*

$$D^\alpha y + y = 0, \quad y(0) = 1, \quad 0 < \alpha \leq 1. \tag{4.1}$$

*Substituting* (2.2) *in the equation, then*

$$\frac{x^{1-\alpha}}{\Gamma(\alpha+1)} y' + y = 0, \tag{4.2}$$

*one can easily show that the exact solution of this* (4.2) *is*

$$y(x) = \exp(-\Gamma(\alpha+1)\frac{x^\alpha}{\alpha}). \tag{4.3}$$

*This solution applies to the initial condition and the equation* (4.1)*, so*

$$D^\alpha(\exp(-\Gamma(\alpha+1)\frac{x^\alpha}{\alpha})) + \exp(-\Gamma(\alpha+1)\frac{x^\alpha}{\alpha}) = -\exp(-\Gamma(\alpha+1)\frac{x^\alpha}{\alpha})) + \exp(-\Gamma(\alpha+1)\frac{x^\alpha}{\alpha}) = 0,$$

*also $y(0) = \exp(-\Gamma(\alpha+1)\frac{0^\alpha}{\alpha}) = 1$. The plot of the solutions of the equation* (4.1) *for different values of $\alpha$ is shown in the figure (1).*

*Figure (2) shows the solution of the differential equation* (4.1) *based on the conformable fractional derivative, the new fractional derivative and the ordinary derivative. Clearly, the solution of the equation obtained by defining the new derivative is closer to the exact solution of the equation based on the ordinary derivative.*

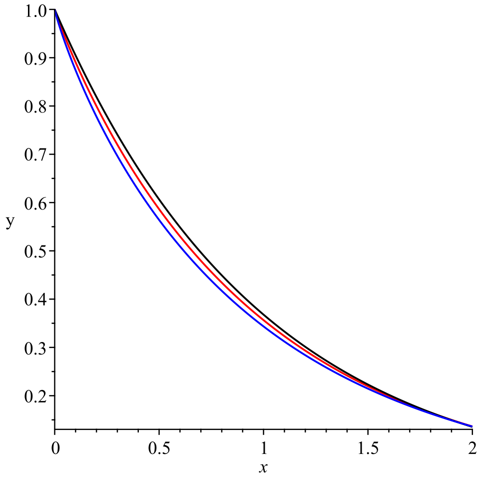

Figure 1: Plots of $y(t)$ for example (4.1) for different values of $\alpha = 0.9$(blue), $\alpha = 0.95$(red) and exact solution(black).

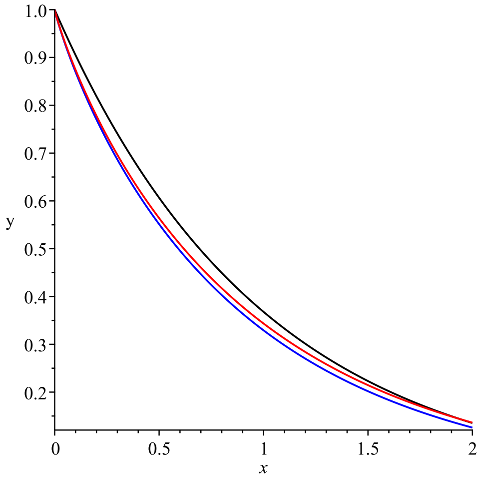

Figure 2: Plots of $y(t)$ for example (4.1) for conformable derivative(blue), new definition(red) and ordinary derivative(black).

**Example 4.2.** *Consider the equation*

$$D^{\alpha}y+y=\sqrt{x}(1+\frac{2}{\Gamma(2.5)}x).$$

*We investigate it for* $\alpha=\frac{1}{2}$ *and plots of solutions for some different values of* $\alpha$

$$\frac{x^{1-\frac{1}{2}}}{\Gamma(\frac{1}{2}+1)}y'+y=\sqrt{x}(1+\frac{2}{\Gamma(2.5)}x), \tag{4.4}$$

*or in simple form*

$$y'+\frac{\Gamma(1.5)}{\sqrt{x}}y=(1+\frac{2}{\Gamma(2.5)}x)\Gamma(1.5), \tag{4.5}$$

*and for similar* $\alpha$*, this is a linear first order equation that one can solve with analytical, approximate or numerical methods.*

**Example 4.3.** *Consider the equation following equation for* $\alpha=\frac{1}{2}$

$$D^{\alpha}y=\frac{y\sqrt{x}+x^{\frac{3}{2}}}{2y-3x},$$

*plugging* (2.2) *in the equation, then we have*

$$\frac{x^{1-\frac{1}{2}}}{\Gamma(\frac{1}{2}+1)}y'=\frac{y\sqrt{x}+x^{\frac{3}{2}}}{2y-3x}, \tag{4.6}$$

*and so we will have the following equivalent equation that is homogeneous equation and can solve easily.*

$$y'=\frac{y+x}{2y-3x}\Gamma(1.5), \tag{4.7}$$

# 5 Results and discussion

In this paper, by using the definition of the Mittag-Leffler function, we could provide a new definition of the fractional derivative that applies to the basic properties of the ordinary derivative. Then we stated and proved two important rules of the derivative, Roll's theorem and mean value theorem. Finally, with the help of the new definition of the fractional derivative, we presented and solved some fractional differential equations including this derivative, show their graphs, and observed its effectiveness.

# References

[1] Abdeljawad, T., Baleanub, D. (2017). Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel. Journal of Nonlinear Sciences and Applications (JNSA), 10(3).

[2] Atangana, A. (2016). On the new fractional derivative and application to nonlinear Fisher's reaction–diffusion equation. Applied Mathematics and computation, 273, 948-956.

[3] Atangana, A., Baleanu, D. (2016). New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. Thermal Science, 20(2), 763.

[4] Baleanu, D., Jajarmi, A., Sajjadi, S. S., Mozyrska, D. (2019). A new fractional model and optimal control of a tumor-immune surveillance with non-singular derivative operator. Chaos: An Interdisciplinary Journal of Nonlinear Science, 29(8).

[5] Baleanu, D., Sadat Sajjadi, S., Jajarmi, A., Asad, J. H. (2019). New features of the fractional Euler-Lagrange equations for a physical system within non-singular derivative operator. The European Physical Journal Plus, 134, 1-10.

[6] Caputo, M., Fabrizio, M. (2015). A new definition of fractional derivative without singular kernel. Progress in Fractional Differentiation Applications, 1(2), 73-85.

[7] Hussein, M. A. (2022). Analysis of fractional differential equations with Antagana-Baleanu fractional operator. Mathematics and Computational Sciences, 3(3), 29-39.

[8] Demirci, E., Unal, A. (2011). A fractional order SEIR model with density dependent death rate. Hacettepe journal of mathematics and statistics, 40(2), 287-295.

[9] Evirgen, F., Uçar, E., Uçar, S., Özdemir, N. (2023). Modelling influenza a disease dynamics under Caputo-Fabrizio fractional derivative with distinct contact rates. Mathematical Modelling and Numerical Simulation with Applications, 3(1), 58-73.

[10] Hattaf, K., Yousfi, N. (2009). Mathematical model of the influenza A (H1N1) infection. Advanced Studies in Biology, 1(8), 383-390.

[11] Jahanshahi, M., Danaei, R. (2023). Analytical-Numerical Solution for a Third Order Space-time Conformable Fractional PDE with Mixed Derivative by Spectral and Asymptotic Methods. Math. Anal, 20, 81-93., 20(1) 2023, 81-93.

[12] Danaei, R., Jahanshahi, M. (2023). Investigation and solving of initial-boundary value problem including fourth order PDE by contour integral and asymptotic methods. Journal of Mathematical Modeling, 11(2), 245-255.

[13] Khalil, R., Al Horani, M., Yousef, A., Sababheh, M. (2014). A new definition of fractional derivative. Journal of computational and applied mathematics, 264, 65-70.

[14] Owolabi, K. M., Atangana, A. (2019). Numerical methods for fractional differentiation (Vol. 54). Springer Singapore.

[15] Saad, K. M., Atangana, A., Baleanu, D. (2018). New fractional derivatives with non-singular kernel applied to the Burgers equation. Chaos: An Interdisciplinary Journal of Nonlinear Science, 28(6).

[16] Zayir, M. Y., Kamil Jassim, H. (2022). A Fractional Variational Iteration Approach for Solving Time-Fractional Navier-Stokes Equations. Mathematics and Computational Sciences, 3(2), 41-47.