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Qom University of Technology

On the reduced minimum modulus of multiplication operators

Hamid Rezaei [1]

Abstract: In this paper, we investigate the properties of the reduced minimum modulus in the context of Banach spaces. Given a Banach space X, we denote the algebra of bounded operators on X as $B(X)$. Our primary focus is on examining the relationship between the reduced minimum modulus of a given operator $T \in B(X)$ and its associated left and right multiplication operators, denoted by $L_T : S \mapsto TS$ and $R_T : S \mapsto ST$, respectively. By analyzing these relationships, we present a comprehensive analysis of their properties and derive novel results concerning the reduced minimum modulus of L_T and R_T.

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1 Introduction

Let X be any Banach space, $T : X \to X$ be a bounded linear operator, and $N(T)$ denote the kernel of T. The reduced minimum modulus of T (see [1, 14]) is defined by

$$\gamma(T) = \inf\Big\{\frac{\|Tx\|}{dist(x, N(T))} : x \in X\backslash N(T)\Big\},$$

with the agreement that $\gamma(T) = +\infty$ whenever $T = 0$. When X is a Hilbert space, $\gamma(T)$ changes to the following simple formula:

$$\gamma(T) = \inf\Big\{\frac{\|Tx\|}{\|x\|} : x \neq 0, x \in N(T)^{\perp}\Big\}.$$

The reduced minimum modulus is a classical notion that originates from perturbation theory. It measures the closeness of the range of operators and thus plays an essential role in studying the spectral properties of operators. For example, (see [10]) it provides more information about the Kato resolvent set, $\rho_K(T)$, which is defined as the set of all scalars λ such that $\mathrm{Ran}(\lambda - T)$ is closed and

$$Ker(\lambda - T) \subseteq \bigcap_{n=1}^{\infty} Ran(\lambda - T)^n.$$

An easy application of the Open Mapping Theorem shows that $\gamma(T) > 0$ if and only if $Ran(T)$ is closed. Moreover, $Ran(T)$ is closed if and only if T is relatively open, which means that $T : X \to Ran(T)$ is an open map [14]. Clearly, $\gamma(T) = \|T^{-1}\|^{-1}$ when T is invertible, and by the Closed Range Theorem, $\gamma(T) = \gamma(T^*)$.

[1]Corresponding author: Department of Mathematics, College of Sciences, Yasouj University, Yasouj-75914-74831, Iran, Email:rezaei@yu.ac.ir

Furthermore, if $\tilde{T} : X \backslash N(T) \to Y$ is the natural induced operator determined by $\tilde{T}(x + N(T)) = Tx$, then it is easy to see that $\gamma(\tilde{T}) = \gamma(T)$. For more information about the reduced minimum modulus, one can refer to [10, 3, 4, 7, 11, 13].

Let X be a Banach space, and let $B(X)$ denote the Banach algebra of all bounded linear operators $T : X \to X$. Let $T \in B(X)$, consider the left multiplication operator L_T defined by

$$L_T : B(X) \to B(X) \text{ by } L_T(U) = TU,$$

and the right multiplication operator R_T defined by

$$R_T : B(X) \to B(X) \text{ by } R_T(U) = UT.$$

In this paper, we explore the intricate relationships between the reduced minimum modulus of a given operator $T \in B(X)$ and its associated left and right multiplication operators, denoted by $L_T : S \mapsto TS$ and $R_T : S \mapsto ST$, respectively. We provide a comprehensive analysis of their properties and derive novel results concerning the reduced minimum modulus of L_T and R_T in terms of $\gamma(T)$.

In general, for $T \in B(X)$, we establish that $\gamma(L_T) = \gamma(R_T) \leq \gamma(T)$. Furthermore, we demonstrate that equality holds when T is restricted to the range-kernel complemented operator. We recall that a bounded linear operator $T \in B(X)$ is considered range-kernel complemented if either $N(T)$ or $\mathrm{Ran}(T)$ is a complemented subspace in X. Notably, every Hilbert space operator or finite-rank operator is range-kernel complemented.

More specifically, we show that $\gamma(L_T) = \gamma(T)$ when $N(T)$ is complemented and $\gamma(R_T) = \gamma(T)$ when $\mathrm{Ran}(T)$ is a closed complemented subspace in X. In particular, we prove that $\gamma(L_T) = \gamma(R_T) = \gamma(T)$ when X is a Hilbert space.

As a consequence of our findings, we deduce that if T is a range-kernel complemented operator, then T has a closed range in X if and only if $L_T($ or $R_T)$ has a closed range in $B(X)$ (see [5, 6]). Lastly, we obtain a sufficient condition for the generalized conjugate operator $C_{T,S} : U \to TUS$ to have a closed range in $B(X)$.

2 Main results

In what follows, let X be a Banach space and $B(X)$ be the Banach algebra of bounded linear operators on X. For every $x \in X$ and $f \in X^*$, the rank one operator $x \otimes f$ is defined by $(x \otimes f)(t) = f(t)x$. Note that $x \otimes f$ is bounded and $\|x \otimes f\| = \|x\|\|f\|$.

Suppose M is a closed subspace of a Banach space X. If there exists a closed subspace N of X such that $X = M + N$ and $M \cap N = \emptyset$, then M is said to be complemented in X. In this case, we write $X = M \oplus N$. It is well known that a closed subspace of a Banach space X is complemented in X if and only if it is the range of some continuous projection in X. Additionally, if M is a closed subspace of a Banach space X such that either $\dim(M) < \infty$ or $\dim(X/M) < \infty$, then M is complemented ([12]).

Proposition 2.1. *Let $T \in B(X)$ then $\gamma(L_T) = \gamma(R_{T^*})$ and $\gamma(L_T) \leq \gamma(T)$.*

Proof. For the first part, it is enough to consider the following relations:

$$\|L_T(U)\| = \|TU\| = \|U^*T^*\| = \|R_{T^*}(U^*)\|,$$

and

$$U \in N(L_T) \Longleftrightarrow U^* \in N(R_{T^*}),$$

for every $U \in B(X)$.

For the second part, if $\gamma(L_T) = 0$ there is nothing to prove, so assume that $\gamma(L_T) > 0$. Then for $x \in X \setminus N(T)$, pick a linear functional $f \in X^*$ such that $\|f\| \leq 1$ and $f(x) = 1$. Consider the rank one operator $U = x \otimes f$, we observe that $U \in B(X) \setminus N(L_T)$ and so for $\varepsilon > 1$, there exists some $V \in N(L_T)$ such that

$$\varepsilon dist(U, N(L_T)) > \|U - V\|.$$

Hence

$$\varepsilon^{-1}\gamma(L_T) \leq \frac{\|L_T(U)\|}{\varepsilon dist(U, N(L_T))} < \frac{\|TU\|}{\|U - V\|} \leq \frac{\|Tx \otimes f\|}{\|Ux - Vx\|}$$
$$\leq \frac{\|Tx\|}{\|x - Vx\|} \leq \frac{\|Tx\|}{dist(x, N(T))}.$$

Taking the infimum on the right hand some of the above with respect to x, gives $\varepsilon^{-1}\gamma(L_T) \leq \gamma(T)$. Letting $\varepsilon \to 1$, we get $\gamma(L_T) \leq \gamma(T)$. This completes the proof. □

Theorem 2.1. *Let $T \in B(X)$ be a range-kernel complemented operator then*

(i) *If $N(T)$ is complemented in X then $\gamma(L_T) = \gamma(T)$.*

(ii) *If $R(T)$ is complemented in X then $\gamma(R_T) = \gamma(T)$.*

Proof. In the event that $\gamma(T) = 0$, Theorem 2.1 implies $\gamma(L_T) = \gamma(T) = 0$. Assuming $\gamma(T) > 0$, T possesses a closed range, and since T is a range-kernel complemented operator, either $N(T)$ or $R(T)$ is a complemented subspace in X. Initially, let us suppose that $N(T)$ is complemented. Consequently, there exists a closed subspace M satisfying $X = N(T) \oplus M$. Let P be the projection from X onto the closed subspace M, then $Ran(P) = M$, $N(P) = N(T)$, and $(I - P)$ is a projection onto $N(T)$. Observe that

$$d(Px, N(T)) = \|Px\| \quad (x \in X). \tag{2.1}$$

To see this, it is clear that $d(Px, N(T)) \leq \|Px\|$. Now for $\varepsilon > 1$, there exists some $y \in N(T)$ such that

$$\varepsilon d(Px, N(T)) \geq \|Px - y\|.$$

Simultaneously, $Py = 0$ and

$$\|Px - y\| \geq \|P(Px - y)\| = \|P^2x\| = \|Px\|$$

Hence $\varepsilon d(Px, N(T)) \geq \|Px\|$. Taking $\varepsilon \to 1$, $d(Px, N(T)) \geq \|Px\|$ and (1) is proved.
Now assume that $U \in B(X)\setminus N(L_T)$ hence $(I - P)U \in N(L_T)$ and so

$$L_T((I - P)U) = T(I - P)U = 0.$$

The above observations imply that

$$\frac{\|TU\|}{dist(U, N(L_T))} \geq \frac{\|TU\|}{\|U - (I - P)U\|} \geq \frac{\|TU\|}{\|PU\|}. \tag{2.2}$$

On the other hand, by considering 2.1, we have

$$\|TU\| \geq \|T(Ux)\| = \|T(PUx + (I - P)Ux)\|$$
$$= \|TPUx\| \geq \gamma(T)d(PUx, N(T)) = \gamma(T)\|PUx\|$$

for every $x \in ball(X)$. It implies that $\|TU\| \geq \gamma(T)\|PU\|$ and considering 2.2

$$\gamma(T) \leq \frac{\|TU\|}{\|PU\|} \leq \frac{\|TU\|}{dist(U, N(L_T))}.$$

Taking the infimum on the right hand side of the above inequality with respect to U, we obtain $\gamma(T) \leq \gamma(L_T)$. Therefore $\gamma(L_T) = \gamma(T)$ and the proof of part (i) is complemented.

Now assume that $R(T)$ is complemented in X. In this case $Y = R(T) \oplus M$ for some closed subspace M in Y. Since

$$M^* = \big(Y/R(T)\big)^* = R(T)^{\perp} = N(T^*),$$

we observe that $N(T^*)$ is complemented in Y^* and so by part (i), $\gamma(T^*) \leq \gamma(L_{T^*})$. Hence

$$\gamma(T) = \gamma(T^*) = \gamma(L_{T^*}) = \gamma(R_T).$$

Now the proof of part (ii) is also completed. □

Since every closed subspace in a Hilbert space is complemented, the following corollary is immediate:

Corollary 2.2. *Let H be a Hilbert space and $T \in B(X)$ then $\gamma(T) = \gamma(L_T) = \gamma(R_T)$*

The following corollary is also immediate from Theorem 2.1

Corollary 2.3. *Let $T \in B(X)$ be a range-kernel complemented operator then the following statements are equivalent:*

(1) *T has a closed range in X.*

(2) *L_T has a closed range in $B(X)$.*

(3) *R_T has a closed range in $B(X)$.*

Recall that Corollary 2.3 have been established directly in ([6], Theorem 3) for Hilbert spaces, and for unbounded operators on Banach spaces ([5], Theorem 2.3 and Lemma 2.4).

It is possible to extend Corollary 2.3 to the generalized conjugate operators. Recall that for any operator $T, S \in B(X)$, the generalized conjugate operator $C_{T,S}$ on $B(X)$ is defined by $C_{T,S}(U) = TUS$. It is straightforward to verify that $C_{T,S}$ is bounded and $\|C_{T,S}\| \leq \|T\|\|S\|$. The next corollary provides a sufficient condition for $C_{T,S}$ to have a closed range. Before stating it, we need the following simple observation:

Theorem 2.4. *Assume that T, S have closed ranges in X and $N(T) \subseteq Ran(S)$ then TS has closed range in X.*

Proof. Applying the open mapping principle, the induced operator $\tilde{T} : X/N(T) \to Ran(T)$ is a surjective isomorphism. Since $N(T) \subseteq Ran(S)$, the space $Ran(S)/N(T)$ is a closed subspace of $X/N(T)$. Since $\tilde{T}$ is an isomorphism, the image $Ran(TS) = T(Ran(S)) = \tilde{T}\Big(Ran(S)/N(T)\Big)$ is closed in X. This completes the proof. □

Corollary 2.5. *Let T, S in $B(X)$ and L_T, R_S are closed range such that $N(L_T) \subseteq Ran(R_S)$ then the generalized conjugate operator $C_{T,S}$ has closed range.*

We end the paper by the following question:
Question: Does for every bounded linear operator $T : X \to X$ on a Banach space X, $\gamma(L_T) = \gamma(T)$?

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