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Upadhyaya integral transform: a tool for solving non-linear Volterra integral equations

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Abstract: These days, integral equations govern a wide range of physical processes, including those related to electricity, mechanics, fluid dynamics, ecology, soil moisture dynamics, shallow water wave propagation, and chemical science. As a result, creating new techniques and putting them into practice for solving integral equations becomes more crucial. In this paper, the Upadhyaya integral transform is employed for determining the analytical solutions of the non-linear Volterra integral equations (NVIEs) of the first kind. Four examples suggest the effectiveness of the Upadhyaya integral transform, particularly for NVIEs of the first kind. The calculation results suggest that the proposed method provides accurate solutions to the original problem and this method is a valuable tool for researchers and scientists working on the broader range of problems involving NVIEs of the first kind.

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1 Introduction

Integral equations have drawn the interest of many scholars [21, 23] due to their various applications in science and engineering such as compartment model of drugs delivery, population growth model and Abel's problem of mechanics. For additional applications, interested scholars can refer to [22,35,16]. Particularly, NVIEs of the second kind have numerous applications in mathematics, astrophysics, visco-elasticity, mechanics and electrical engineering [36]. Specialized methods are required for finding the accurate and reliable solutions of Volterra integral equations (VIEs). Some of the commonly used methods to solve VIEs are integral transform methods [25,26,20,4,5], Adomian method [12], Sawi decomposition method [19], Chebyshev spectral method [24], hybrid orthonormal Bernstein and block-pulse functions wavelet (HOBW) method [11], Tikhonov regularization method [28], homotopy perturbation method [27, 13], and Taylor's series method [6, 3]. A wide range of other numerical methods [32,29,37,30,10] are also available to solve VIEs. Limited analytical methods are available for the NVIEs of first kind and existing methods may not be applicable for obtaining the exact solution of NVIEs of the first kind. The intricate nature of tackling the challenges posed by

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NVIEs demands the utilization of specialized methodologies and precise techniques to yield results that are both accurate and dependable. This motivates our endeavor to discuss an efficient method the “Upadhyaya integral transform” method for solving the NVIEs of first kind. Upadhyaya integral transform is the most advanced integral transform till date of the Laplace class of integral transforms and all the existing integral transforms are the special cases of this integral transform. The two special cases of Upadhyaya integral transform namely Shehu and Sawi transforms are used for solving the problems of medical and chemical engineering [18, 17]. Duality relations of Laplace class of integral transforms play an important role for solving the vast problems of science and engineering. These relations of Laplace class of integral transforms are well documented [33,7,8,1,2,14,15,9,31]. In this paper, this transform is applied to solve the NVIEs of the first kind. We discuss how to solve the NVIEs of the first kind using this transform. This is how the remainder of the paper is structured. In Section 2, several fundamental definitions and theorems are provided for our proposed methodology. In Section 3, we introduce the Upadhyaya transform for NVIEs of the first kind. In Section 4, the effectiveness of the suggested approach is illustrated with four numerical applications, proving its accuracy and dependability. The last section (Section 5) presents a brief conclusion.

2 Basic definitions and theorems

In this informative section, we give a comprehensive explanation of the essential definitions and theorems that form the basis of the Upadhyaya transform. These crucial definitions and theorems serve as the building blocks for understanding the intricate concepts and applications of the Upadhyaya transform in various fields. By delving into these foundational principles, readers can gain a deeper insight into the underlying principles that govern the transformative power of the Upadhyaya transform. Through the detailed exploration of these fundamental concepts, readers can not only grasp the theoretical framework of the Upadhyaya transform but also appreciate its practical implications across different domains. Thus, this section aims to equip the readers with a solid understanding of the core principles and theoretical underpinnings of the Upadhyaya transform, setting the stage for a more profound exploration of its significance and impact in the realm of theory of integral transformations.

2.1 Definition of the Upadhyaya transform

A function$\eta(t) \in \mathcal{C}, t \geq 0$, where $\mathcal{C}$ is the collection of the sequentially continuous exponential order functions, has the Upadhyaya transform that is given by [33]

$$\mathcal{U}\{\eta(t)\} = \alpha \int_0^\infty \eta(\gamma t) e^{-\beta t} dt = \mathcal{T}(\alpha, \beta, \gamma),\ \alpha, \beta, \gamma > 0 \tag{1}$$

In Table 1, Upadhyaya transformations of some basic functions are presented in a visually comprehensible manner, offering a clear depiction of the variations and modifications that these functions undergo. By mapping out the changes brought about by Upadhyaya [33, 34], readers can gain a deeper understanding of how the core functions evolve within the context of the study. The visual representation in Table 1 not only simplifies the complexities of the transformation process but also serves as a valuable reference point for analyzing the data and drawing meaningful insights. Through this tabular format, Upadhyaya’s innovative approach to transforming core functions becomes more accessible and interpretable, enabling researchers and practitioners to delve into the nuances of these changes. As a result, Table 1 serves as a comprehensive visual guide that enhances the reader's comprehension of the transformations that occur within the core functions, shedding light on the nuances and intricacies of this process.

Table1: Upadhyaya transforms of some core functions [33, 34]

S.N.	$\eta(t) \in \mathcal{C}, t \geq 0$	$\mathcal{U}\{\eta(t)\} = \mathcal{T}(\alpha,\beta,\gamma)$
1	1	$\left(\frac{\alpha}{\beta}\right)$
2	e^{at}	$\left(\frac{\alpha}{\beta - a\gamma}\right)$
3	$t^a, a \in N$, N: set of natural numbers	$a!\left(\frac{\alpha\gamma^a}{\beta^{a+1}}\right)$
4	$t^a, a > -1, a \in R$, R: set of real numbers	$\left(\frac{\alpha\gamma^a}{\beta^{a+1}}\right)\Gamma(a+1)$
5	$sin(at)$	$\frac{a\alpha\gamma}{(\beta^2 + a^2\gamma^2)}$
6	$cos(at)$	$\frac{\alpha\beta}{(\beta^2 + a^2\gamma^2)}$
7	$sinh(at)$	$\frac{a\alpha\gamma}{(\beta^2 - a^2\gamma^2)}$
8	$cosh(at)$	$\frac{\alpha\beta}{(\beta^2 - a^2\gamma^2)}$

2.2 Inverse Upadhyaya transform

The inverse Upadhyaya transform of $\mathcal{T}(\alpha,\beta,\gamma)$, designated by$\mathcal{U}^{-1}\{\mathcal{T}(\alpha,\beta,\gamma)\}$, is another function $\eta(t)$ having the property that $\mathcal{U}\{\eta(t)\} = \mathcal{T}(\alpha,\beta,\gamma)$.

The Table 2 provides a visual representation of the Inverse Upadhyaya transforms applied to the core functions under study. As illustrated in the table, the transformations of the core functions are clearly displayed for analysis and comparison. By examining the data presented in Table 2, researchers can gain valuable insights into how the core functions are modified through the application of Inverse Upadhyaya transforms. This visual representation serves as a useful tool for researchers and analysts to better understand the effects of these transformations on the underlying data. Through the visual aid of the table, it becomes easier to discern the patterns, trends, and variations in the transformed core functions. The clarity and structure of the table facilitates a comprehensive analysis of the inverse transformations, enabling researchers to draw meaningful conclusions and make informed decisions based on the displayed information. Overall, Table 2 serves as a valuable resource for studying the effects of Inverse Upadhyaya transforms on the core functions and their visual representation aids in the interpretation of the data.

Table 2: Inverse Upadhyaya transforms of some core functions [33]

S.N.	$\mathcal{T}(\alpha,\beta,\gamma)$	$\eta(t) = \mathcal{U}^{-1}\{\mathcal{T}(\alpha,\beta,\gamma)\}$
1	$\left(\frac{\alpha}{\beta}\right)$	1
2	$\left(\frac{\alpha}{\beta - a\gamma}\right)$	e^{at}

3	$\left(\frac{\alpha\gamma^{a}}{\beta^{a+1}}\right), a \in N$	$\frac{t^{a}}{a!}$
4	$\left(\frac{\alpha\gamma^{a}}{\beta^{a+1}}\right), a > -1, a \in R$	$\frac{t^{a}}{\Gamma(a+1)}$
5	$\frac{\alpha\gamma}{(\beta^2 + a^2\gamma^2)}$	$\frac{sin(at)}{a}$
6	$\frac{\alpha\beta}{(\beta^2 + a^2\gamma^2)}$	$cos(at)$
7	$\frac{\alpha\gamma}{(\beta^2 - a^2\gamma^2)}$	$\frac{sinh(at)}{a}$
8	$\frac{\alpha\beta}{(\beta^2 - a^2\gamma^2)}$	$cosh(at)$

2.3 Linearity Property of the Upadhyaya transform [33]

If $\eta_i(t) \in \mathcal{C}, t \geq 0$ and $\mathcal{U}\{\eta_i(t)\} = \mathcal{T}_i(\alpha,\beta,\gamma)$ then
$\mathcal{U}\{\sum_{i=1}^{n} a_i\eta_i(t)\} = \sum_{i=1}^{n} a_i\, \mathcal{U}\{\eta_i(t)\} = \sum_{i=1}^{n} a_i\, \mathcal{T}_i(\alpha,\beta,\gamma)$, where a_i are arbitrary constants.

2.4 Translation Property of the Upadhyaya transform [33]

If $\eta(t) \in \mathcal{C}, t \geq 0$ and $\mathcal{U}\{\eta(t)\} = \mathcal{T}(\alpha,\beta,\gamma)$then$\mathcal{U}\{e^{at}\eta(t)\} = \mathcal{T}(\alpha,\beta - a\gamma,\gamma)$, where a is arbitrary constant.

2.5 Change of Scale Property of the Upadhyaya transform [33]

If $\eta(t) \in \mathcal{C}, t \geq 0$and $\mathcal{U}\{\eta(t)\} = \mathcal{T}(\alpha,\beta,\gamma)$then$\mathcal{U}\{\eta(at)\} = \mathcal{T}\left(\frac{\alpha}{a},\frac{\beta}{a},\gamma\right)$, where a is arbitrary constant.

2.6 Convolution (Faltung) theorem of the Upadhyaya transform [33]

If $\eta_i(t) \in \mathcal{C}, t \geq 0, i = 1,2$ and $\mathcal{U}\{\eta_i(t)\} = \mathcal{T}_i(\alpha,\beta,\gamma), i = 1,2$ then
$\mathcal{U}\{\eta_1(t) * \eta_2(t)\} = \left(\frac{\gamma}{\alpha}\right)\mathcal{U}\{\eta_1(t)\}\mathcal{U}\{\eta_2(t)\} = \left(\frac{\gamma}{\alpha}\right)\mathcal{T}_1(\alpha,\beta,\gamma)\mathcal{T}_2(\alpha,\beta,\gamma)$.

3 Solution of non-linear Volterra integral equations of the first kind through the Upadhyaya integral transform

Here, we delve into the intricacies of the Upadhyaya transform, presenting a comprehensive exploration of its application in solving NVIEs of the first kind. By introducing the Upadhyaya transform, we aim to provide readers with a clear understanding of how this powerful mathematical tool can be effectively utilized to tackle complex problems within the realm of NVIEs. Through a detailed analysis of the theory behind the Upadhyaya transform and its practical implications, we illuminate the significance of this technique in the context of mathematical modeling and computational analysis. Furthermore, we elucidate the key features that make the Upadhyaya transform a valuable asset for researchers and practitioners working in the field of nonlinear integral equations. By elucidating the principles and methodologies associated with the Upadhyaya transform, we seek to equip the readers with the knowledge and insights necessary to leverage this transformative approach for addressing challenges posed by NVIEs of the first kind.The NVIEsof first kind are given by

$$\mathcal{J}(t) = \int_0^t \mathcal{K}(t,x)\,\mathcal{F}\big(\eta(x)\big)dx \tag{2}$$

where $\eta(t)$, $\mathcal{K}(t,x)$ and $\mathcal{J}(t)$ are respectively the unknown function, kernel of integral equation and known function. In equation (2), $\mathcal{F}\big(\eta(t)\big)$ is a non-linear function of $\eta(t)$ such as $log\big(\eta(t)\big)$,$\eta^2(t)$, $cos\big(\eta(t)\big)$, $e^{\eta(t)}$, $sin\big(\eta(t)\big)$, $\eta^3(t)$ etc.

Here it is assumed that the kernel $\mathcal{K}(t,x)$ of the equation (2) is a convolution kernel. Using this assumption, the equation (2) gives:

$$\mathcal{J}(t) = \int_0^t \mathcal{K}(t-x)\,\mathcal{F}\big(\eta(x)\big)dx \tag{3}$$

To solve the NVIEs of the first kind which are given by (3), we use the transformation $\mathcal{F}\big(\eta(t)\big) = \hbar(t)$ and convert (3) into a linear Volterra integral equation of the first kind as:

$$\mathcal{J}(t) = \int_0^t \mathcal{K}(t-x)\,\hbar(x)dx \tag{4}$$

where $\hbar(t) = \mathcal{F}\big(\eta(t)\big)$ or $\eta(t) = \mathcal{F}^{-1}\big(\hbar(t)\big)$ (5)

Taking the Upadhyaya transform of (4) provides

$$\mathcal{U}\{\mathcal{J}(t)\} = \mathcal{U}\left\{\int_0^t \mathcal{K}(t-x)\,\hbar(x)dx\right\} \tag{6}$$

Using the convolution property of the Upadhyaya transform, (6) reduces to

$$\mathcal{U}\{\mathcal{J}(t)\} = \mathcal{U}\{\mathcal{K}(t) * \hbar(t)\} \tag{7}$$

Applying the convolution property of the Upadhyaya transform in (7), we have

$$\mathcal{U}\{\mathcal{J}(t)\} = \left(\frac{\gamma}{\alpha}\right)\mathcal{U}\{\mathcal{K}(t)\}\mathcal{U}\{\hbar(t)\}$$

$$\Rightarrow \mathcal{U}\{\hbar(t)\} = \left(\frac{\alpha}{\gamma}\right)\left[\frac{\mathcal{U}\{\mathcal{J}(t)\}}{\mathcal{U}\{\mathcal{K}(t)\}}\right] \tag{8}$$

Use of the inverse Upadhyaya transform in (8) provides

$$\hbar(t) = \mathcal{U}^{-1}\left\{\left(\frac{\alpha}{\gamma}\right)\left[\frac{\mathcal{U}\{\mathcal{J}(t)\}}{\mathcal{U}\{\mathcal{K}(t)\}}\right]\right\} \tag{9}$$

(9) Gives the expression of $\hbar(t)$ that can be used further in (5) to find the required expression of $\eta(t)$.

4 Numerical Applications

This section provides four examples that highlight how well the Upadhyaya transform works to solve the NVIEs of the first kind.

Application 1 Consider the following NVIE of the first kind

$$\frac{1}{3}e^{4t} - \frac{1}{3}e^{t} = \int_0^t e^{(t-x)}\,\eta^2(x)dx \tag{10}$$

Consider the transform $\hbar(t) = \eta^2(t)$ or $\eta(t) = \pm[\hbar(t)]^{1/2}$ (11)

Using (11) in the (10), we get

$$\frac{1}{3}e^{4t} - \frac{1}{3}e^{t} = \int_0^t e^{(t-x)}\,\hbar(x)dx \tag{12}$$

Upadhyaya transform of both sides of (12) gives

$$\mathcal{U}\left\{\frac{1}{3}e^{4t} - \frac{1}{3}e^{t}\right\} = \mathcal{U}\left\{\int_0^t e^{(t-x)}\,\hbar(x)dx\right\} \tag{13}$$

Using the linearity property and the convolution definition of the Upadhyaya transform in the (13), we get

$$\frac{1}{3}\mathcal{U}\{e^{4t}\} - \frac{1}{3}\mathcal{U}\{e^{t}\} = \mathcal{U}\{e^{t} * \hbar(t)\} \tag{14}$$

Applying the convolution property of the Upadhyaya transform in (14), we have

$$\frac{1}{3}\mathcal{U}\{e^{4t}\}-\frac{1}{3}\mathcal{U}\{e^{t}\}=\left(\frac{\gamma}{\alpha}\right)\mathcal{U}\{e^{t}\}\mathcal{U}\{\hbar(t)\}$$
$$\Rightarrow\frac{1}{3}\left(\frac{\alpha}{\beta-4\gamma}\right)-\frac{1}{3}\left(\frac{\alpha}{\beta-\gamma}\right)=\left(\frac{\gamma}{\alpha}\right)\left(\frac{\alpha}{\beta-\gamma}\right)\mathcal{U}\{\hbar(t)\}$$
$$\Rightarrow\mathcal{U}\{\hbar(t)\}=\left(\frac{\alpha}{\beta-4\gamma}\right) \quad (15)$$
Use of the inverse Upadhyaya transform in (15) provides
$$\hbar(t)=\mathcal{U}^{-1}\left\{\frac{\alpha}{\beta-4\gamma}\right\}=e^{4t} \quad (16)$$
Finally the use of (16) in (11) gives the exact solution of the (10) as:
$$\eta(t)=\pm e^{2t} \quad (17)$$
Application 2 Consider the following NVIE of the first kind
$$\frac{1}{5}e^{6t}-\frac{1}{5}e^{t}=\int_0^t e^{(t-x)}\,e^{2\eta(x)}dx \quad (18)$$
Let $\hbar(t)=e^{2\eta(t)}$ or $\eta(t)=\frac{1}{2}\left[log\left(\hbar(t)\right)\right]$ (19)
Using (19) in (18), we get
$$\frac{1}{5}e^{6t}-\frac{1}{5}e^{t}=\int_0^t e^{(t-x)}\,\hbar(x)dx \quad (20)$$
Use of the Upadhyaya transform in (20) suggests
$$\mathcal{U}\left\{\frac{1}{5}e^{6t}-\frac{1}{5}e^{t}\right\}=\mathcal{U}\left\{\int_0^t e^{(t-x)}\,\hbar(x)dx\right\} \quad (21)$$
Using the linearity property and the convolution definition of the Upadhyaya transform in (21), we get
$$\frac{1}{5}\mathcal{U}\{e^{6t}\}-\frac{1}{5}\mathcal{U}\{e^{t}\}=\mathcal{U}\{e^{t}*\hbar(t)\} \quad (22)$$
Applying the convolution property of the Upadhyaya transform in (22), we obtain
$$\frac{1}{5}\mathcal{U}\{e^{6t}\}-\frac{1}{5}\mathcal{U}\{e^{t}\}=\left(\frac{\gamma}{\alpha}\right)\mathcal{U}\{e^{t}\}\mathcal{U}\{\hbar(t)\}$$
$$\Rightarrow\frac{1}{5}\left(\frac{\alpha}{\beta-6\gamma}\right)-\frac{1}{5}\left(\frac{\alpha}{\beta-\gamma}\right)=\left(\frac{\gamma}{\alpha}\right)\left(\frac{\alpha}{\beta-\gamma}\right)\mathcal{U}\{\hbar(t)\}$$
$$\Rightarrow\mathcal{U}\{\hbar(t)\}=\left(\frac{\alpha}{\beta-6\gamma}\right) \quad (23)$$
Use of the inverse Upadhyaya transform in (23) gives
$$\hbar(t)=\mathcal{U}^{-1}\left\{\frac{\alpha}{\beta-6\gamma}\right\}=e^{6t} \quad (24)$$
Lastly using (24) in (19) gives the exact solution of (18) as:
$$\eta(t)=3t \quad (25)$$
Application 3 Consider the following NVIE of the first kind
$$\frac{1}{2}t^2+\frac{1}{2}t^3+\frac{1}{4}t^4+\frac{1}{20}t^5=\int_0^t(t-x)\,\eta^3(x)dx \quad (26)$$
Applying the transformation
$$\hbar(t)=\eta^3(t) \text{ or } \eta(t)=[\hbar(t)]^{1/3} \quad (27)$$
in (26) gives
$$\frac{1}{2}t^2+\frac{1}{2}t^3+\frac{1}{4}t^4+\frac{1}{20}t^5=\int_0^t(t-x)\,\hbar(x)dx \quad (28)$$
Use of the Upadhyaya transform in (28) suggests
$$\mathcal{U}\left\{\frac{1}{2}t^2+\frac{1}{2}t^3+\frac{1}{4}t^4+\frac{1}{20}t^5\right\}=\mathcal{U}\left\{\int_0^t(t-x)\,\hbar(x)dx\right\} \quad (29)$$
Using the linearity property and the convolution of the Upadhyaya transform in the (29), we get
$$\frac{1}{2}\mathcal{U}\{t^2\}+\frac{1}{2}\mathcal{U}\{t^3\}+\frac{1}{4}\mathcal{U}\{t^4\}+\frac{1}{20}\mathcal{U}\{t^5\}=\mathcal{U}\{t*\hbar(t)\} \quad (30)$$
Applying the convolution property of the Upadhyaya transform in (30), we have
$$\frac{1}{2}\mathcal{U}\{t^2\}+\frac{1}{2}\mathcal{U}\{t^3\}+\frac{1}{4}\mathcal{U}\{t^4\}+\frac{1}{20}\mathcal{U}\{t^5\}=\left(\frac{\gamma}{\alpha}\right)\mathcal{U}\{t\}\mathcal{U}\{\hbar(t)\}$$
$$\Rightarrow\frac{1}{2}\left[2\left(\frac{\alpha\gamma^2}{\beta^3}\right)\right]+\frac{1}{2}\left[6\left(\frac{\alpha\gamma^3}{\beta^4}\right)\right]+\frac{1}{4}\left[24\left(\frac{\alpha\gamma^4}{\beta^5}\right)\right]+\frac{1}{20}\left[120\left(\frac{\alpha\gamma^5}{\beta^6}\right)\right]=\left(\frac{\gamma}{\alpha}\right)\left(\frac{\alpha\gamma}{\beta^2}\right)\mathcal{U}\{\hbar(t)\}$$

$$\Rightarrow \mathcal{U}\{\hbar(t)\} = \left(\frac{\alpha}{\beta}\right) + \left[3\left(\frac{\alpha\gamma}{\beta^2}\right)\right] + \left[6\left(\frac{\alpha\gamma^2}{\beta^3}\right)\right] + \left[6\left(\frac{\alpha\gamma^3}{\beta^4}\right)\right] \quad (31)$$

Use of the inverse Upadhyaya transform in the (31) gives

$$\hbar(t) = \mathcal{U}^{-1}\left\{\left(\frac{\alpha}{\beta}\right) + \left[3\left(\frac{\alpha\gamma}{\beta^2}\right)\right] + \left[6\left(\frac{\alpha\gamma^2}{\beta^3}\right)\right] + \left[6\left(\frac{\alpha\gamma^3}{\beta^4}\right)\right]\right\}$$

$$\Rightarrow \hbar(t) = \mathcal{U}^{-1}\left\{\left(\frac{\alpha}{\beta}\right)\right\} + 3\mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma}{\beta^2}\right)\right\} + 6\mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^2}{\beta^3}\right)\right\} + 6\mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^3}{\beta^4}\right)\right\}$$

$$\Rightarrow \hbar(t) = 1 + 3t + 3t^2 + t^3 = (1+t)^3 \quad (32)$$

Substituting (32) in (27) yields the exact solution of (26) as

$$\eta(t) = (1+t) \quad (33)$$

Application 4 Consider the following the NVIE of the first kind

$$\frac{1}{2}t^2 + \frac{1}{3}t^3 + \frac{1}{12}t^4 = \int_0^t (t-x)\,\eta^2(x)dx \quad (34)$$

Consider the transform

$$\hbar(t) = \eta^2(t) \text{ or } \eta(t) = \pm[\hbar(t)]^{1/2} \quad (35)$$

which converts (34) into

$$\frac{1}{2}t^2 + \frac{1}{3}t^3 + \frac{1}{12}t^4 = \int_0^t (t-x)\,\hbar(x)dx \quad (36)$$

Use of the Upadhyaya transform in (36) provides

$$\mathcal{U}\left\{\frac{1}{2}t^2 + \frac{1}{3}t^3 + \frac{1}{12}t^4\right\} = \mathcal{U}\left\{\int_0^t (t-x)\,\hbar(x)dx\right\} \quad (37)$$

Using the linearity property and convolution definition of the Upadhyaya transform in (37), we get

$$\frac{1}{2}\mathcal{U}\{t^2\} + \frac{1}{3}\mathcal{U}\{t^3\} + \frac{1}{12}\mathcal{U}\{t^4\} = \mathcal{U}\{t * \hbar(t)\} \quad (38)$$

Applying the convolution property of the Upadhyaya transform to (38), we have

$$\frac{1}{2}\mathcal{U}\{t^2\} + \frac{1}{3}\mathcal{U}\{t^3\} + \frac{1}{12}\mathcal{U}\{t^4\} = \left(\frac{\gamma}{\alpha}\right)\mathcal{U}\{t\}\mathcal{U}\{\hbar(t)\}$$

$$\Rightarrow \frac{1}{2}\left[2\left(\frac{\alpha\gamma^2}{\beta^3}\right)\right] + \frac{1}{3}\left[6\left(\frac{\alpha\gamma^3}{\beta^4}\right)\right] + \frac{1}{12}\left[24\left(\frac{\alpha\gamma^4}{\beta^5}\right)\right] = \left(\frac{\gamma}{\alpha}\right)\left(\frac{\alpha\gamma}{\beta^2}\right)\mathcal{U}\{\hbar(t)\}$$

$$\Rightarrow \mathcal{U}\{\hbar(t)\} = \left(\frac{\alpha}{\beta}\right) + \left[2\left(\frac{\alpha\gamma}{\beta^2}\right)\right] + \left[2\left(\frac{\alpha\gamma^2}{\beta^3}\right)\right] \quad (39)$$

Use of the inverse Upadhyaya transform in (39) suggests

$$\hbar(t) = \mathcal{U}^{-1}\left\{\left(\frac{\alpha}{\beta}\right) + \left[2\left(\frac{\alpha\gamma}{\beta^2}\right)\right] + \left[2\left(\frac{\alpha\gamma^2}{\beta^3}\right)\right]\right\}$$

$$\Rightarrow \hbar(t) = \mathcal{U}^{-1}\left\{\left(\frac{\alpha}{\beta}\right)\right\} + 2\mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma}{\beta^2}\right)\right\} + 2\mathcal{U}^{-1}\left\{\left(\frac{\alpha\gamma^2}{\beta^3}\right)\right\}$$

$$\Rightarrow \hbar(t) = 1 + 2t + t^2 = (1+t)^2 \quad (40)$$

Back substituting (40) in (35) gives the exact solution of (34) as:

$$\eta(t) = \pm(1+t) \quad (41)$$

5 Conclusion

The authors successfully showed in this paper how the Upadhyaya transform can be used as a potential tool for finding the solutions of the first kind of NVIEs. The accurate analytical solutions of the NVIEs of the first class are obtained using this integral transform approach. The results of this paper suggest that the present method “Upadhyaya integral transform” has two main advantages over the other available methods. Firstly, it is free from the complicated computations for obtaining the exact analytical solution of NVIEs of the first kind and secondly, there is no necessity of using large computer memory and time for solving the NVIEs of the first kind.

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