

# New study on Caputo-Hadamard type fractional Neutral Integro-Differential equations

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**Abstract:** In this work, we focus on the analysis of fractional-order neutral integro-differential equations using the Caputo-Hadamard fractional derivative. We employed the topological degree method (TDM) to derive results and solutions for these equations. To illustrate the applicability of our findings, we provided a concrete example demonstrating the practical implications and outcomes of our analysis.

**Keywords:** Fractional calculus; Integro-Differential equation; Topological degree method

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## 1 Introduction

Over the past few decades, fractional calculus has emerged as a significant field for addressing real-world problems in physics, chemistry, engineering, & other disciplines. Extensive literature on fractional differential equations (FDEs) provides both analytical and numerical results, Local and global existence and uniqueness theorems for a functional delay fractional differential equation with bounded delay are discussed in [2, 7, 18], existence and multiplicity of positive solutions for a class of higher-order nonlinear fractional differential equations with integral boundary conditions are in detail [19, 22, 25, 28, 29, 34] Fixed point(FP) theory (FPT) has been instrumental in proving the existence and uniqueness of solutions to FDEs. Addressing challenges in FPT, to resolve integral equations, Mahwin [30] developed topological degree theory (TDT). Subsequently, Isaia [23] applied TDT to various integral equations. Analysis of TDT in existing literature can be found in [16, 39, 40, 45, 51, 52].

Bashir Ahmad and S. Sivasundaram [8] elaborated on the nonlinear term in FIDE's within the context of boundary value problems. Mingyue Zuo and et al.[31] explored FIDE's characterized by anti-periodic boundary conditions and impulsive conditions. Agarwal et al. [3] provided insights into neutral fractional equations with a bounded delay considering its initial value problem (IVP). Based on fractional derivatives, Doa Santos and et al. [26, 37] focused on identifying existence solutions for neutral equations and integro-differential equations. Ravichandran et al. [36] provided a concise examination of fractional neutral systems incorporating infinite delay. Yan et al. [43, 44] applied neutral functions featuring state-dependent delays to fractional integro-differential equations. Zhou et al. [48-50] explored p-type fractional neutral evolution equations featuring infinite delay.

We've encountered numerous instances of research exploring RL or CF fractional derivatives within both

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analytical and numerical contexts in FDEs. Hadamard [21] pioneered novel derivative featuring logarithmic kernel term, now recognized as Hadamard (HD) fractional derivative. Utilizing this HD fractional derivative, Butzer et al. [13–15] delved into its analysis alongside HD fractional integral employing the Mellin transform method. Employing derivative of HD along integral boundary conditions, Ardjouni [5] elucidated positive solutions for FDEs. Various studies Like implicit Hadamards fractional differential equations, nonlocal multi-point boundary conditions, higher-order nonlinear fractional differential equation and inclusions with integral boundary conditions, two dimensional equations and sequential Caputo types are studied in [10, 11, 20, 27, 33, 41] Subsequently, Jarad et al. [24] explored modification of Caputo FDEs, a Caputo type Hadamard derivative (CH-derivative). In [1, 4, 6, 12] FDEs along boundary value problems via CH-derivative were elucidated.

In this study, the investigation rigorously explores the existence outcomes of neutral FIDEs featuring the CH-derivative, utilizing TDM.

$$\left. \begin{aligned} {}_H^C D^\varrho[\Upsilon(t) - g(t, \Upsilon(t), P\Upsilon(t))] &= f(t, \Upsilon(t), O\Upsilon(t), S\Upsilon(t)), \quad t \in \mathcal{J} := [0, \mathfrak{L}] \\ a\Upsilon(1) + b {}_H^C D^\rho \Upsilon(1) &= c_1^H \mathcal{I}^{\varrho_1} \Upsilon(\theta_1), \quad 1 < \theta_1 < \mathfrak{L}, \varrho_1 > 0 \\ c\Upsilon(L) + d {}_H^C D^\rho \Upsilon(L) &= c_2^H \mathcal{I}^{\varrho_2} \Upsilon(\theta_2), \quad 1 < \theta_2 < L, \varrho_2 > 0 \end{aligned} \right\} \quad (1.1)$$

here  ${}_H^C D^\varrho$  &  ${}_H^C D^\rho$  represent the CH-derivatives of orders  $\varrho$  &  $\rho$ , along  $1 < \varrho \leq 2, 0 < \rho \leq 1$ ,  ${}^H \mathcal{I}^{\varrho_i}$  stands as the definitive HD integral of fractional order  $\varrho_i, i = 1, 2$ ,  $f \in C(\mathcal{J} \times \mathcal{R} \times \mathcal{R} \times \mathcal{R}, \mathcal{R})$  are functions that remains continuous, while the operators  $P$  and  $Q$  are determined as

$$(P\Upsilon)(t) = \int_0^t k(t, s)\Upsilon(s)ds$$

$$(O\Upsilon)(t) = \int_0^t l(t, s)\Upsilon(s)ds$$

$$(S\Upsilon)(t) = \int_0^1 h(t, s)\Upsilon(s)ds$$

here  $t \in \mathcal{J}, k, l \in C(D, \mathcal{R}), D = \{(t, s) \in \mathcal{J} \times \mathcal{J} : t \geq s\}$  &  $h \in C(\mathcal{J} \times \mathcal{J}, \mathcal{R})$

Suppose  $a, b, c$ , and  $d$  are constants from  $\mathcal{R}$ , fulfilling

$$\Psi = \left( a - \frac{c_1(\log \theta_1)^{\varrho_1}}{\Gamma(\varrho_1 + 1)} \right) \left( c \log L + \frac{d(\log \mathfrak{L})^{1-\rho}}{\Gamma(2-\rho)} - \frac{c_2(\log \theta_2)^{\varrho_2+1}}{\Gamma(\varrho_2 + 2)} \right) + \frac{c_1(\log \theta_1)^{\varrho_1+1}}{\Gamma(\varrho_1 + 2)} \left( c \log \mathfrak{L} - \frac{c_2(\log \theta_2)^{\varrho_2}}{\Gamma(\varrho_2 + 1)} \right) \neq 0 \quad (1.2)$$

The structure of the article unfolds like this: Section 2 establishes fundamental results to demonstrate existence via topological degree theory. Section 3 delves into the existence of FIDEs utilizing boundary conditions employing TDM. Sections 5 and 6 present illustrations and conclusions, respectively.

## 2 Basic results

Suppose a Banach space be  $\mathfrak{X}$ , and let  $\mathcal{B}$  be the set containing all bounded subsets within  $\mathfrak{X}$ .

**Definition 2.1.** [16]  $\nu : \mathcal{B} \rightarrow \mathcal{R}_+$  is rigorously established as follows: For any  $\mathcal{B} \in \mathcal{B}$ , its Kuratowski measure of non-compactness, symbolized as  $\Upsilon(\mathcal{B})$ , is established as the smallest positive value  $d$  for which  $\mathcal{B}$  can be encompassed by a finite assortment of sets, each possessing a diameter not surpassing  $d$ .

We will provide a concise elucidation of this definition. Let's start by considering a subset  $\mathcal{B}$  contained within the space  $X$ . We define  $\mathcal{B}$  as compact if, and only if, it satisfies two conditions simultaneously: completeness, indicating that all Cauchy sequence in  $\mathcal{B}$  has a limit that lies within  $\mathcal{B}$  as dictated by  $\|\cdot\|$ , and total boundedness, wherein for any  $\epsilon > 0$ , there exists a finite covering of  $\mathcal{B}$  such that each subset in the covering contains a diameter that does not exceed  $\epsilon$ . Hence, when  $\mathcal{B}$  exhibits compactness, it inevitably exhibits total boundedness, thereby implying  $\nu(\mathcal{B}) = 0$ . This naturally positions  $\nu$  as a metric for assessing the compactness of a set. A zero value of  $\nu$  implies compactness, while a larger  $\nu$  suggests a greater deviation from compactness.

Armed with this definition, we assert the subsequent findings, whose verifications are detailed in [17].

**Proposition 2.2.** [16] *Let  $\mathcal{U}, \mathcal{U}_1, \mathcal{U}_2$  be bounded subsets on Banach space, then*

1.  $\nu(\mathcal{U}) = 0 \Leftrightarrow \overline{\mathcal{U}}$  is compact.
2.  $\nu(\zeta\mathcal{U}) = \|\zeta\|\nu(\mathcal{U}), \quad \zeta \in \mathcal{R}$
3.  $\nu(\mathcal{U}_1 + \mathcal{U}_2) \leq \nu(\mathcal{U}_1) + \nu(\mathcal{U}_2), \quad \mathcal{U}_1, \mathcal{U}_2 \in \mathcal{B}$
4.  $\mathcal{U}_1 \subset \mathcal{U}_2 \Rightarrow \nu(\mathcal{U}_1) \leq \nu(\mathcal{U}_2)$
5.  $\nu(\mathcal{U}_1 \cup \mathcal{U}_2) = \max\{\nu(\mathcal{U}_1), \nu(\mathcal{U}_2)\}$
6.  $\nu(\text{conv } \mathcal{U}) = \nu(\mathcal{U})$
7.  $\nu(\overline{\mathcal{U}}) = \nu(\mathcal{U})$

We establish  $\Xi$  as the set of continuous functions  $\Upsilon : [0, \mathcal{T}] \rightarrow \mathcal{R}$  defined over the interval  $I'$ . Furthermore, we assert that  $(\Xi, |\cdot|)$  forms a Banach space equipped with the supremum  $|\Upsilon|$ , defined as the maximum absolute value of  $\Upsilon(t)$  over the interval  $[0, \mathcal{T}]$ .

**Definition 2.3.** [16] *If  $\mathcal{F} : \mu \rightarrow \mathcal{Y}$  exhibits boundedness, continuity and  $\mu \subset \mathcal{Y}$ , & if  $\mathcal{F}$  is  $\Upsilon$ -Lipschitz and satisfies  $\nu(\mathcal{F}(\mathcal{H})) \leq k\nu(\mathcal{H})$  for all  $\mathcal{H} \subset \mu$  and some  $k \geq 0$ , where  $k < 1$ , results in  $\mathcal{F}$  exhibiting  $\nu$ -contraction.*

**Proposition 2.4.** [16] *Suppose  $T_1, T_2, T_3, T_4 : \mu \rightarrow Y$  are  $\nu$ -Lipschitz maps with respective constants  $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4$ , then collectively, they form a  $\nu$ -Lipschitz map with the constant  $\mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3 + \mathbf{m}_4$ .*

For an  $i \in \{1, 2, 3, 4\}$ , results in ,

$$\|\mathcal{F}_{\mathcal{I}(x)} - \mathcal{F}_{\mathcal{I}(y)}\| \leq m_{\mathcal{I}}\|x - y\| \leq (\mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3 + \mathbf{m}_4)\|x - y\|$$

Therefore,  $\forall \mathcal{I}$  in the set 1, 2, 3, 4,  $\mathcal{F}_i$  exhibits  $\nu$ -Lipschitz mappings combined with  $\mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3 + \mathbf{m}_4$ .

**Proposition 2.5.** [16] *Let  $\mathcal{G} : \mu \rightarrow \mathcal{Y}$  be compact, then  $\mathcal{G}$  exhibits  $\nu$ -Lipschitz with zero constant.*

Proposition 2.2 unequivocally establishes that for every compact subset  $\mathcal{H}$ ,  $\nu(\mathcal{G}(\mathcal{H}))$  equals zero. Therefore, the conclusion naturally ensues.

**Proposition 2.6.** [16] *Let  $\mathcal{G} : \mu \rightarrow Y$  exhibits Lipschitz continuity, results as it also exhibits  $\nu$ -Lipschitz with  $k$ .*

**Definition 2.7.** [16] *Let  $\mathcal{F} : \mathcal{Y} \rightarrow \mathcal{F}$  is termed  $\nu$ -condensing when, for every set  $A \in \mathcal{B}$ ,  $\nu(\mathcal{F}(A))$  is less than or equal to  $\nu(A)$ . We represent the set  $C_\nu(\Omega)$  as the collection of all  $\nu$ -condensing mappings  $\mathcal{F} : \Omega \rightarrow X$ .*

We hereby establish D as the "degree function" utilized in TDT. [17, 23]

**Theorem 2.8.** [16]  $\mathcal{F} : \mathcal{Y} \rightarrow \mathcal{Y}$  is  $\nu$ -condensing map and

$$\mathcal{H} = \{\Upsilon \in \mathcal{Y} : \exists \zeta \in [1, \mathfrak{L}] \text{ such that } \Upsilon = \zeta \mathfrak{t}\Upsilon\}.$$

Let  $\mathcal{H}$  exhibits boundedness in  $\Xi$ , so  $\exists r > 0$  s. t.  $\mathcal{H} \subset B_r(0)$ , results in

$$D(\mathfrak{I} - \zeta \mathfrak{t}, B_r(0), 0) = 1, \quad \forall \zeta \in [1, \mathfrak{L}].$$

Therefore,  $\mathfrak{t}$  necessarily possesses at least one FP.

**Definition 2.9.** [28] Integral in HD of  $\varrho > 0$  for  $\Upsilon \in \mathfrak{L}^1(\mathcal{J})$  is rigorously defined by:

$${}^H\mathcal{I}^\varrho \Upsilon(\mathfrak{t}) = \frac{1}{\Gamma(\varrho)} \int_1^{\mathfrak{t}} \log(\mathfrak{t}/s)^{\varrho-1} \Upsilon(s) \frac{ds}{s}$$

where

$$\Gamma(\varrho) = \int_0^\infty e^{-\mathfrak{t}} \mathfrak{t}^{\varrho-1} d\mathfrak{t}, \quad \varrho > 0$$

Let  $\delta = \mathfrak{t} \frac{d}{d\mathfrak{t}}$ ,  $\varrho > 0, n = [\varrho] + 1$ .

**Definition 2.10.** [28] Derivative in HD of  $\varrho > 0$  and  $\Upsilon \in \Xi$  is emphatically asserted by:

$${}^H D^\varrho \Upsilon(\mathfrak{t}) = \delta^n ({}^H \mathcal{I}^{n-\varrho} \Upsilon(\mathfrak{t})).$$

**Definition 2.11.** [24, 28] Derivative in CH of  $\varrho > 0$  and  $\Upsilon \in \Xi$  is emphatically declared by:

$${}^H_C D^\varrho \Upsilon(\mathfrak{t}) = {}^H \mathcal{I}^{n-\varrho} \delta^n \Upsilon(\mathfrak{t})$$

We draw upon several essential lemmas which is fundamental analyzing FDEs.

**Lemma 2.12.** [24, 28] Suppose  $\varrho > 0, r > 0, n = [\varrho] + 1$  and  $a > 0$ . Then

$$1. {}^H \mathcal{I}^\varrho (\log \frac{\mathfrak{t}}{a})^{r-1} = \frac{\Gamma(r)}{\Gamma(\varrho+r)} (\log \frac{\mathfrak{t}}{a})^{e+r+1},$$

Especially, when  $a$  equals 1 and  $r$  equals 1, the aforementioned outcome simplifies to  ${}^H \mathcal{I}^\varrho(1)(\varrho) = \frac{1}{\Gamma(\varrho+1)} (\log(\mathfrak{t}))^{e+1}$

$$2. {}^H_C D^\varrho (\log \frac{\mathfrak{t}}{a})^{r-1} = \begin{cases} \frac{\Gamma(r)}{\Gamma(r-\varrho)} (\log \frac{\mathfrak{t}}{a})^{r-\varrho-1}, & r > n \\ 0, & r \in \{0, 1, \dots, n-1\}. \end{cases}$$

**Lemma 2.13.** [24, 28] Let  $n\varrho_1, \varrho_2 > 0$  and  $\Upsilon \in \Xi$ . Then,

1.  ${}^H \mathcal{I}^{\varrho_1} ({}^H \mathcal{I}^{\varrho_2} \Upsilon(\mathfrak{t})) = ({}^H \mathcal{I}^{\varrho_1+\varrho_2} \Upsilon(\mathfrak{t}))$
2.  ${}^H_C D^{\varrho_1} ({}^H \mathcal{I}^{\varrho_2} \Upsilon(\mathfrak{t})) = ({}^H \mathcal{I}^{\varrho_2-\varrho_1} \Upsilon(\mathfrak{t}))$
3.  ${}^H_C D^{\varrho_1} ({}^H \mathcal{I}^{\varrho_1} \Upsilon(\mathfrak{t})) = \Upsilon(\mathfrak{t})$ .

**Lemma 2.14.** [24, 28] Suppose  $\varrho > 0$ . Set  $n = [\varrho] + 1, \forall \Upsilon \in \Xi$  the CH-derivative of FDEs

$${}^H_C D^\varrho \Upsilon(\mathfrak{t}) = 0$$

has a solution

$$\Upsilon(\mathfrak{t}) = \sum_{i=0}^n c_i (\log \mathfrak{t})^i$$

and  $c_i, i = 0, 1, \dots, n-1$  be real constants. Furthermore, subsequent outcome ensues promptly.

$${}^H I_C^\varrho ({}^H_C D^\varrho \Upsilon(\mathfrak{t})) = \Upsilon(\mathfrak{t}) + \sum_{i=0}^{n-1} c_i (\log \mathfrak{t})^i.$$

### 3 Main Results

We will establish several hypotheses that we will firmly presume:

(A1)  $\exists$  constant  $\mathfrak{E}, \tilde{\mathfrak{E}} > 0$  and  $p \in [0, 1)$  s. t.

$$\|g(t, \Upsilon(t), P\Upsilon(t))\| \leq (\mathfrak{E} + \tilde{\mathfrak{E}}\|\Upsilon\|^p)$$

for all  $\Upsilon \in \Xi$ .

(A2)  $\exists$  constants  $\mathfrak{E}_1, \mathfrak{E}_2$  s. t.

$$\|g(t, \Upsilon_1(t), P\Upsilon_1(t)) - g(t, \Upsilon_2(t), P\Upsilon_2(t))\| \leq \mathfrak{E}_1\|\Upsilon_1 - \Upsilon_2\| + \mathfrak{E}_2\|P\Upsilon_1 - P\Upsilon_2\|$$

(A3)  $\exists$  constant  $\mathcal{F}, \check{\mathcal{F}}, \tilde{\mathcal{F}} > 0$  and  $p \in [0, 1)$  s. t.

$$\|f(t, \Upsilon(t), O\Upsilon(t), S\Upsilon(t))\| \leq (\mathcal{F} + \check{\mathcal{F}}\|\Upsilon\|^p + \tilde{\mathcal{F}}\|\Upsilon\|^p)$$

for all  $\Upsilon \in \Xi$ .

(A4)  $\exists$  constants  $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$  s. t.

$$\begin{aligned} \|f(t, \Upsilon_1(t), O\Upsilon_1(t), S\Upsilon_1(t)) - f(t, \Upsilon_2(t), O\Upsilon_2(t), S\Upsilon_2(t))\| &\leq \mathcal{F}_1\|\Upsilon_1 - \Upsilon_2\| + \mathcal{F}_2\|O\Upsilon_1 - O\Upsilon_2\| \\ &\quad + \mathcal{F}_3\|S\Upsilon_1 - S\Upsilon_2\| \end{aligned}$$

(A5) Let  $k_{\max} = \sup_{t \in \mathcal{J}} \int_0^t \|k(t, s)\| ds$ ,  $l_{\max} = \sup_{t \in \mathcal{J}} \int_0^t \|l(t, s)\| ds$  &  $h_{\max} = \sup_{t \in \mathcal{J}} \int_0^1 \|h(t, s)\| ds$ .

**Lemma 3.1.** [16] When dealing with  $V$  over  $\mathcal{J}$ , FODE as follows:

$$\left. \begin{aligned} {}_H^C D^\varrho[\Upsilon(t) - W(t)] &= V(t), \quad t \in \mathcal{J} := [0, \mathfrak{L}] \\ a\Upsilon(1) + b {}_H^C D^\rho \Upsilon(1) &= c_1^H \mathcal{I}^{\varrho_1} \Upsilon(\theta_1), \quad 1 < \theta_1 < \mathfrak{L}, \varrho_1 > 0 \\ c\Upsilon(\mathfrak{L}) + d {}_H^C D^\rho \Upsilon(\mathfrak{L}) &= c_2^H \mathcal{I}^{\varrho_2} \Upsilon(\theta_2), \quad 1 < \theta_2 < \mathfrak{L}, \varrho_2 > 0 \end{aligned} \right\} \quad (3.1)$$

Then  $\exists$  only one solution, referred to as

$$\begin{aligned} \Upsilon(t) &= {}^H \mathcal{I}^\varrho V(t) + W(t) + K_1(t)({}^H \mathcal{I}^{\varrho_1 + \varrho} V(\theta_1) + W(\theta_1)) + K_2(t)(c_2({}^H \mathcal{I}^{\varrho_2 + \varrho} V(\theta_2) + W(\theta_2))) \\ &\quad - c({}^H \mathcal{I}^\varrho V(L) + W(L)) + d({}^H \mathcal{I}^{\varrho - \rho} V(L) + W(L)) \end{aligned} \quad (3.2)$$

where,

- $K_1(t) = c_1(\Psi_1 - \Psi_2 t), \quad K_2(t) = c_1 \Psi_3 + \Psi_4 t$
- $\Psi_1 = \frac{1}{\Psi} \left( c \log \mathfrak{L} + \frac{d(\log \mathfrak{L})^{1-\rho}}{\Gamma(2-\rho)} - \frac{c_2(\log \theta_2)^{\varrho_2+1}}{\Gamma(\varrho_2+2)} \right)$
- $\Psi_2 = \frac{1}{\Psi} \left( c \log \mathfrak{L} - \frac{c_2(\log \theta_2)^{\varrho_2}}{\Gamma(\varrho_2+1)} \right)$
- $\Psi_3 = \frac{1}{\Psi} \left( \frac{c_1(\log \theta_1)^{\varrho_1+1}}{\Gamma(\varrho_1+2)} \right)$
- $\Psi_4 = \frac{1}{\Psi} \left( a - \frac{c_1(\log \theta_1)^{\varrho_1}}{\Gamma(\varrho_1+1)} \right)$

$\mathfrak{E} \Psi$  is given by (1.2)

*Proof.* From Lemma 2.14, equation (3.1) becomes,

$$\Upsilon(t) = {}^H \mathcal{I}^\varrho V(t) + W(t) + k_0 + k_1 \log(t), \quad k_0, k_1 \in \mathcal{R}$$

Using boundary conditions, we have,

$$\begin{aligned} {}^H \mathcal{I}^{\varrho_i} \Upsilon(\theta_i) &= {}^H \mathcal{I}^{\varrho_i + \varrho} V(\theta_i) + W(\theta_i) + k_0 \frac{(\log \theta_i)^{\varrho_i}}{\Gamma(\varrho_i + 1)} + k_1 \frac{(\log \theta_i)^{\varrho_i + 1}}{\Gamma(\varrho_i + 2)}, \quad i = 1, 2, \dots \\ {}^C_H D^\rho \Upsilon(\mathfrak{L}) &= {}^H \mathcal{I}^{\varrho - \rho} V(\mathfrak{L}) + W(\mathfrak{L}) + k_1 \frac{\Gamma(2)(\log \mathfrak{L})^{1-\rho}}{\Gamma(2 - \rho)} \end{aligned}$$

Solving for  $k_0, k_1$  we get the solutions:

$$k_0 = c_1 \Psi_1 {}^H \mathcal{I}^{\varrho + \varrho_1} \Upsilon(\theta_1) + \Psi_3 (c_2 {}^H \mathcal{I}^{\varrho + \varrho_2} \Upsilon(\theta_2) - (c({}^H \mathcal{I}^\varrho V(\mathfrak{L}) + W(\mathfrak{L})) + d({}^H \mathcal{I}^{\varrho - \rho} V(\mathfrak{L}) + W(\mathfrak{L}))))$$

&

$$\begin{aligned} k_1 &= c_1 \Psi_4 ({}^H \mathcal{I}^{\varrho + \varrho_2} V(\theta_2) + W(\theta_2)) - (c({}^H \mathcal{I}^\varrho V(\mathfrak{L}) + W(\mathfrak{L})) + d({}^H \mathcal{I}^{\varrho - \rho} V(\mathfrak{L}) + W(\mathfrak{L}))) \\ &\quad - c_1 \Psi_2 ({}^H \mathcal{I}^\varrho V(\theta_1) + W(\theta_1)) \end{aligned}$$

Substituting for  $k_0$  and  $k_1$  we get (3.2). □

Considering the issue outlined in (1.1), leveraging lemma 3.1 above yields definitive solution,

$$\begin{aligned} \Upsilon(t) &= {}^H \mathcal{I}^\varrho f_\Upsilon(t) + g_\Upsilon(t) + K_1(t) ({}^H \mathcal{I}^{\varrho_1 + \varrho} f_\Upsilon(\theta_1) + g_\Upsilon(\theta_1)) + K_2(t) (c_2 ({}^H \mathcal{I}^{\varrho_2 + \varrho} f_\Upsilon(\theta_2) + g_\Upsilon(\theta_2)) \\ &\quad - c({}^H \mathcal{I}^\varrho f_\Upsilon(\mathfrak{L}) + g_\Upsilon(\mathfrak{L})) + d({}^H \mathcal{I}^{\varrho - \rho} f_\Upsilon(\mathfrak{L}) + g_\Upsilon(\mathfrak{L}))) \end{aligned} \quad (3.3)$$

To condense, we signify  $g(t, \Upsilon(t), P\Upsilon(t))$  by  $g_\Upsilon$ ,  $f(t, \Upsilon(t), O\Upsilon(t), S\Upsilon(t))$  by  $f_\Upsilon$  and

$$\begin{aligned} K_1(t) &= c_1(\Psi_1 - \Psi_2 t), \quad K_2(t) = c_1 \Psi_3 + \Psi_4 t \\ \Psi_1 &= \frac{1}{\Psi} \left( c \log \mathfrak{L} + \frac{d(\log \mathfrak{L})^{1-\rho}}{\Gamma(2-\rho)} - \frac{c_2(\log \theta_2)^{\varrho_2+1}}{\Gamma(\varrho_2+2)} \right), \quad \Psi_2 = \frac{1}{\Psi} \left( c \log \mathfrak{L} - \frac{c_2(\log \theta_2)^{\varrho_2}}{\Gamma(\varrho_2+1)} \right), \\ \Psi_3 &= \frac{1}{\Psi} \left( \frac{c_1(\log \theta_1)^{\varrho_1+1}}{\Gamma(\varrho_1+2)} \right), \quad \Psi_4 = \frac{1}{\Psi} \left( a - \frac{c_1(\log \theta_1)^{\varrho_1}}{\Gamma(\varrho_1+1)} \right) \end{aligned}$$

and  $\Psi$  is determined through the expression provided in (1.2).

We will explicitly establish the following trio of operators:

1. Define  $T_1 : \Xi \rightarrow \Xi$  as  $T_1 \Upsilon(t) = {}^H \mathcal{I}^\varrho f_\Upsilon(t) + g_\Upsilon(t)$ .
2. Define  $T_2 : \Xi \rightarrow \Xi$  as  $T_2 \Upsilon(t) = K_1(t) ({}^H \mathcal{I}^{\varrho_1 + \varrho} f_\Upsilon(\theta_1) + g_\Upsilon(\theta_1)) + K_2(t) (c_2 ({}^H \mathcal{I}^{\varrho_2 + \varrho} f_\Upsilon(\theta_2) + g_\Upsilon(\theta_2)))$ .
3. Define  $T_3 : \Xi \rightarrow \Xi$  as  $T_3 \Upsilon(t) = K_2(t) (c({}^H \mathcal{I}^\varrho f_\Upsilon(\mathfrak{L}) + g_\Upsilon(\mathfrak{L})) + d({}^H \mathcal{I}^{\varrho - \rho} f_\Upsilon(\mathfrak{L}) + g_\Upsilon(\mathfrak{L})))$ .

Suppose  $T : \Xi \rightarrow \Xi$  as  $T = T_1 + T_2 + T_3$ .

**Theorem 3.2.** [11, 16] Suppose  $T_1$  is Lipschitz with Lipschitz constant  $\frac{(\log \mathfrak{L})^\varrho}{\Gamma(\varrho + 1)} (\mathcal{F}_1 + \mathcal{F}_2 l_{\max} + \mathcal{F}_3 h_{\max}) + \mathfrak{E}_1 + \mathfrak{E}_2 k_{\max}$ . It also satisfies the following growth relation:

$$\|T_1 \Upsilon(t)\| \leq \frac{(\log \mathfrak{L})^\varrho}{\Gamma(\varrho + 1)} (\mathcal{F} + \check{\mathcal{F}} \|\Upsilon\|^p + \check{\mathcal{F}} \|\Upsilon\|^p) + \mathfrak{E} + \check{\mathfrak{E}} \|\Upsilon\|^p$$

*Proof.* Let  $\Upsilon_1, \Upsilon_2 \in \Xi$ . We have,

$$\begin{aligned} \|T_1\Upsilon_1(t) - T_1\Upsilon_2(t)\| &\leq \|{}^H\mathcal{I}^\varrho f_{\Upsilon_1}(t) - {}^H\mathcal{I}^\varrho f_{\Upsilon_2}(t)\| + \|g_{\Upsilon_1}(t) - g_{\Upsilon_2}(t)\| \\ &\leq {}^H\mathcal{I}^\varrho \|f_{\Upsilon_1} - f_{\Upsilon_2}\|(t) + (\mathfrak{E}_1 + \mathfrak{E}_2 k_{\max}) \|\Upsilon_1 - \Upsilon_2\| \\ &\leq [{}^H\mathcal{I}^\varrho(1)(\mathfrak{L})(\mathcal{F}_1 + \mathcal{F}_2 l_{\max} + \mathcal{F}_3 h_{\max}) + \mathfrak{E}_1 + \mathfrak{E}_2 k_{\max}] \|\Upsilon_1 - \Upsilon_2\| \\ &= \left[ \frac{(\log L)^\varrho}{\Gamma(\varrho + 1)} (\mathcal{F}_1 + \mathcal{F}_2 l_{\max} + \mathcal{F}_3 h_{\max}) + \mathfrak{E}_1 + \mathfrak{E}_2 k_{\max} \right] \|\Upsilon_1 - \Upsilon_2\| \end{aligned}$$

Hence

$$\|T_1\Upsilon_1(t) - T_1\Upsilon_2(t)\| \leq \left[ \frac{(\log L)^\varrho}{\Gamma(\varrho + 1)} (\mathcal{F}_1 + \mathcal{F}_2 l_{\max} + \mathcal{F}_3 h_{\max}) + \mathfrak{E}_1 + \mathfrak{E}_2 k_{\max} \right] \|\Upsilon_1 - \Upsilon_2\|$$

Therefore,  $T_1$  exhibits Lipschitz continuity, with a constant specified by:

$$\left[ \frac{(\log L)^\varrho}{\Gamma(\varrho + 1)} (\mathcal{F}_1 + \mathcal{F}_2 l_{\max} + \mathcal{F}_3 h_{\max}) + \mathfrak{E}_1 + \mathfrak{E}_2 k_{\max} \right]$$

For the growth relation of  $t \in \mathcal{J}$ ,

$$\begin{aligned} \|T_1\Upsilon(t)\| &\leq \|{}^H\mathcal{I}^\varrho f_\Upsilon(t) + g_\Upsilon(t)\| \\ &\leq {}^H\mathcal{I}^\varrho \|f_\Upsilon(t)\| + \|g_\Upsilon(t)\| \\ &\leq {}^H\mathcal{I}^\varrho (\mathcal{F} + \check{\mathcal{F}}\|\Upsilon\|^p + \tilde{\mathcal{F}}\|\Upsilon\|^p) + \mathfrak{E} + \tilde{\mathfrak{E}}\|\Upsilon\|^p \\ &= \frac{(\log L)^\varrho}{\Gamma(\varrho + 1)} (\mathcal{F} + \check{\mathcal{F}}\|\Upsilon\|^p + \tilde{\mathcal{F}}\|\Upsilon\|^p) + \mathfrak{E} + \tilde{\mathfrak{E}}\|\Upsilon\|^p \end{aligned}$$

Therefore, we possess

$$\|T_1\Upsilon(t)\| \leq \left[ \frac{(\log \mathfrak{L})^\varrho}{\Gamma(\varrho + 1)} (\mathcal{F} + \check{\mathcal{F}}\|\Upsilon\|^p + \tilde{\mathcal{F}}\|\Upsilon\|^p) \right] + \mathfrak{E} + \tilde{\mathfrak{E}}\|\Upsilon\|^p.$$

□

**Theorem 3.3.** Suppose  $T_2$  be continuous and satisfies the growth relation:

$$\|T_2\Upsilon(t)\| \leq A_{T_2} (\mathcal{F} + \check{\mathcal{F}}\|\Upsilon\|^p + \tilde{\mathcal{F}}\|\Upsilon\|^p) + B_{T_2} (\mathfrak{E} + \tilde{\mathfrak{E}}\|\Upsilon\|^p)$$

where,

$$\begin{aligned} A_{T_2} &= \|c_1\| \|\Psi_1\| + \|c_2\| \|\Psi_2\| \mathfrak{L} \frac{(\log \theta_1)^{\varrho + \varrho_1}}{\Gamma(\varrho_1 + \varrho + 1)} + (\|c_1\| \|c_2\| \|\Psi_3\| + \|\Psi_4\| \mathfrak{L}) \frac{(\log \theta_2)^{\varrho + \varrho_2}}{\Gamma(\varrho_2 + \varrho + 1)} \\ B_{T_2} &= (\|c_1\| \|\Psi_1\| + \|c_2\| \|\Psi_2\| \mathfrak{L}) + (\|c_1\| \|c_2\| \|\Psi_3\| + \|\Psi_4\| \mathfrak{L}). \end{aligned}$$

*Proof.* Take a series  $\Upsilon_n$  within  $\Xi$  that decisively tends toward  $\Upsilon \in \Xi$ . Given the continuity of  $f_\Upsilon$  and  $g_\Upsilon$ , we have  $f_{\Upsilon_n} \rightarrow f_\Upsilon$  and  $g_{\Upsilon_n} \rightarrow g_\Upsilon$ . Therefore, applying the Lebesgue dominated convergence theorem, we conclude that  $T_2$  exhibits continuity.

We possess,

$$\begin{aligned} \|T_2\Upsilon(t)\| &= \|K_1(t)[{}^H\mathcal{I}^{\varrho_1 + \varrho} f_\Upsilon(\theta_1) + g_\Upsilon(\theta_1)] + K_2(t)(c_2[{}^H\mathcal{I}^{\varrho_2 + \varrho} g_\Upsilon(\theta_2) + g_\Upsilon(\theta_2)])\| \\ &\leq \|K_1(t)\| [{}^H\mathcal{I}^{\varrho_1 + \varrho} \|f_\Upsilon(\theta_1)\| + \|g_\Upsilon(\theta_1)\|] + \|K_2(t)c_2\| [{}^H\mathcal{I}^{\varrho_2 + \varrho} \|g_\Upsilon(\theta_2)\| + \|g_\Upsilon(\theta_2)\|] \\ &\leq A_{T_2} (\mathcal{F} + \check{\mathcal{F}}\|\Upsilon\|^p + \tilde{\mathcal{F}}\|\Upsilon\|^p) + B_{T_2} (\mathfrak{E} + \tilde{\mathfrak{E}}\|\Upsilon\|^p) \end{aligned}$$

Consequently, we obtain the following:

$$\|T_2\Upsilon(t)\| \leq A_{T_2}(\mathcal{F} + \check{\mathcal{F}}\|\Upsilon\|^p + \tilde{\mathcal{F}}\|\Upsilon\|^p) + B_{T_2}(\mathfrak{E} + \tilde{\mathfrak{E}}\|\Upsilon\|^p).$$

□

**Theorem 3.4.** *Assume  $T_3$  exhibits continuity and adheres to specified growth condition:*

$$\|T_3\Upsilon(t)\| \leq A_{T_3}(\mathcal{F} + \check{\mathcal{F}}\|\Upsilon\|^p + \tilde{\mathcal{F}}\|\Upsilon\|^p) + B_{T_3}(\mathfrak{E} + \tilde{\mathfrak{E}}\|\Upsilon\|^p),$$

here

$$\begin{aligned} A_{T_3} &= (\|c_1\|\|\Psi_3\| + \|\Psi_4\|) \left[ \frac{\|c\|(\log \mathfrak{L})^e}{\Gamma(\varrho + 1)} + \frac{\|d\|(\log \mathfrak{L})^{e-\rho}}{\Gamma(\varrho - \rho + 1)} \right] \\ B_{T_3} &= (\|c_1\|\|\Psi_3\| + \|\Psi_4\|\|\mathfrak{L}\|) [\|c\| + \|d\|] \end{aligned}$$

*Proof.* Take a series  $\Upsilon_n$  within  $\Xi$  that decisively tends toward  $\Upsilon \in \Xi$ . Given the continuity of  $f_\Upsilon$  and  $g_\Upsilon$ , we have  $f_{\Upsilon_n} \rightarrow f_\Upsilon$  and  $g_{\Upsilon_n} \rightarrow g_\Upsilon$ . Therefore, applying the Lebesgue dominated convergence theorem, we conclude that  $T_3$  exhibits continuity.

We possess,

$$\begin{aligned} \|T_3\Upsilon(t)\| &= \|K_2(t)(c[{}^H\mathcal{I}^e f_\Upsilon(\mathfrak{L}) + g_\Upsilon(\mathfrak{L})] + d[{}^H\mathcal{I}^{e-\rho} f_\Upsilon(\mathfrak{L}) + g_\Upsilon(\mathfrak{L})])\| \\ &\leq \|K_1(t)\| \left( \|c\|{}^H\mathcal{I}^e \|f_\Upsilon(\mathfrak{L})\| + \|c\|\|g_\Upsilon(\mathfrak{L})\| + \|d\|{}^H\mathcal{I}^{e-\rho} \|f_\Upsilon(\mathfrak{L})\| + \|d\|\|g_\Upsilon(\mathfrak{L})\| \right) \\ &\leq A_{T_3}(\mathcal{F} + \check{\mathcal{F}}\|\Upsilon\|^p + \tilde{\mathcal{F}}\|\Upsilon\|^p) + B_{T_3}(\mathfrak{E} + \tilde{\mathfrak{E}}\|\Upsilon\|^p) \end{aligned}$$

Consequently, we obtain the following:

$$\|T_3\Upsilon(t)\| \leq A_{T_3}(\mathcal{F} + \check{\mathcal{F}}\|\Upsilon\|^p + \tilde{\mathcal{F}}\|\Upsilon\|^p) + B_{T_3}(\mathfrak{E} + \tilde{\mathfrak{E}}\|\Upsilon\|^p).$$

□

**Theorem 3.5.** *Suppose  $T_2$  happens to be compact, then it follows that  $T_2$  is Lipschitz with a zero constant.*

*Proof.* Consider  $\vartheta$  contained within  $B(r)$ . In order to establish  $T_2$  as a compact map, it's necessary to demonstrate the relative compactness of  $T_2(\vartheta)$ . This means proving that the closure of  $T_2(\vartheta)$ , denoted  $\overline{T_2(\vartheta)}$ , is compact within  $\Xi$ . With reference to Theorem 3.3, this holds true for  $\Upsilon \in \vartheta$ ,

$$\|T_2\Upsilon(t)\| \leq A_{T_2}(\mathcal{F} + \check{\mathcal{F}}r^p + \tilde{\mathcal{F}}r^p) + B_{T_2}(\mathfrak{E} + \tilde{\mathfrak{E}}r^p)$$

Therefore,  $T_2(\vartheta)$  exhibits uniform boundedness.

Subsequently, considering any  $\Upsilon \in \Xi$ , we obtain:

$$\begin{aligned} \|T_2\Upsilon'(t)\| &= \|K_1'(t)({}^H\mathcal{I}^{e_1+e} f_\Upsilon(\theta_1) + g_\Upsilon(\theta_1)) + K_2'(t)(c_2({}^H\mathcal{I}^{e_2+e} f_\Upsilon(\theta_2) + g_\Upsilon(\theta_2)))\| \\ &\leq \|K_1'(t)\| [{}^H\mathcal{I}^{e_1+e} \|f_\Upsilon(\theta_1)\| + \|g_\Upsilon(\theta_1)\|] + \|K_2'(t)c_2\| [{}^H\mathcal{I}^{e_2+e} \|f_\Upsilon(\theta_2)\| + \|g_\Upsilon(\theta_2)\|] \\ &\leq \left( \|c_1\|\|\Psi_2\| \frac{(\log \theta_1)^{e_1+e}}{\Gamma(\varrho_1 + \varrho + 1)} + \|\Psi_4\|\|c_2\| \frac{(\log \theta_2)^{e_2+e}}{\Gamma(\varrho_2 + \varrho + 1)} \right) (\mathcal{F} + \check{\mathcal{F}}\|\Upsilon\|^p + \tilde{\mathcal{F}}\|\Upsilon\|^p) \\ &\quad + (\|c_1\|\|\Psi_2\| + \|\Psi_4\|\|c_2\|)(\mathfrak{E} + \tilde{\mathfrak{E}}\|\Upsilon\|^p) \\ &\leq \overline{\mu}_1(\mathcal{F} + \check{\mathcal{F}}\|\Upsilon\|^p + \tilde{\mathcal{F}}\|\Upsilon\|^p) + \overline{\mu}_2(\mathfrak{E} + \tilde{\mathfrak{E}}\|\Upsilon\|^p) \end{aligned}$$

At this juncture, considering  $t_1$  and  $t_2 \in \mathcal{J}$ , we observe,”

$$\|T_2\Upsilon(t_2) - T_2\Upsilon(t_1)\| \leq \int_{t_1}^{t_2} \|T_2\Upsilon'(t)\| dt \leq [\bar{\mu}_1(\mathcal{F} + \check{\mathcal{F}}\|\Upsilon\|^p + \tilde{\mathcal{F}}\|\Upsilon\|^p) + \bar{\mu}_2(\mathfrak{E} + \tilde{\mathfrak{E}}\|\Upsilon\|^p)](t_2 - t_1)$$

Thus, as  $t_2 \rightarrow t_1$ ,  $\|T_2\Upsilon(t_2) - T_2\Upsilon(t_1)\| \rightarrow 0 \Rightarrow T_2$  exhibits equicontinuity. By Arzelà - Ascoli's theorem,  $T_2$  demonstrates compactness.

Therefore, as stipulated in Proposition 2.5,  $T_2$  exhibits Lipschitz continuity with a vanishing constant.  $\square$

**Theorem 3.6.** *If  $T_3$  happens to be compact, then it necessarily implies that  $T_3$  exhibits a Lipschitz constant of zero.*

*Proof.* In order to establish  $T_3$  as a compact map, it is imperative to demonstrate the relative compactness of  $T_3(\vartheta)$ , where  $\vartheta$  is a bounded subset contained in  $B(r)$ .

That is to say, the set  $\overline{T_3(\vartheta)}$  exhibits compactness within  $\Xi$ . As deduced by Theorem 3.4, for  $\Upsilon \in \vartheta$ ,

$$\|T_3\Upsilon(t)\| \leq A_{T_3}(\mathcal{F} + \check{\mathcal{F}}r^p + \tilde{\mathcal{F}}r^p) + B_{T_3}(\mathfrak{E} + \tilde{\mathfrak{E}}r^p)$$

Therefore,  $T_3(\vartheta)$  exhibits uniform boundedness.

At this juncture, when considering any  $\Upsilon$  belonging to  $\Xi$ , it is imperative to note that

$$\begin{aligned} \|T_3\Upsilon'(t)\| &\leq \|K_2'(t)\| \left( \|c\|[\|{}^H\mathcal{I}^\varrho f_\Upsilon(\mathfrak{L})\| + \|g_\Upsilon(\mathfrak{L})\|] + \|d\|[\|{}^H\mathcal{I}^\varrho f_\Upsilon(\mathfrak{L})\| + \|g_\Upsilon(L)\|] \right) \\ &\leq \|\Psi_4\| \left( \|c\| \frac{(\log L)^\varrho}{\Gamma(\varrho + 1)} + \|d\| \frac{(\log L)^{\varrho - \rho}}{\Gamma(\varrho - \rho + 1)} \right) (\mathcal{F} + \check{\mathcal{F}}\|\Upsilon\|^p + \tilde{\mathcal{F}}\|\Upsilon\|^p) \\ &\quad + \|\Psi_4\| (\|c\| + \|d\|) (\mathfrak{E} + \tilde{\mathfrak{E}}\|\Upsilon\|^p) \\ &\leq \bar{\mu}_3(\mathcal{F} + \check{\mathcal{F}}\|\Upsilon\|^p + \tilde{\mathcal{F}}\|\Upsilon\|^p) + \bar{\mu}_4(\mathfrak{E} + \tilde{\mathfrak{E}}\|\Upsilon\|^p) \end{aligned}$$

At this juncture, considering  $t_1$  and  $t_2$  within the interval  $\mathcal{J}$ , it is evident that

$$\|T_3\Upsilon(t_2) - T_3\Upsilon(t_1)\| \leq \int_{t_1}^{t_2} \|T_3\Upsilon'(t)\| dt \leq [\bar{\mu}_3(\mathcal{F} + \check{\mathcal{F}}\|\Upsilon\|^p + \tilde{\mathcal{F}}\|\Upsilon\|^p) + \bar{\mu}_4(\mathfrak{E} + \tilde{\mathfrak{E}}\|\Upsilon\|^p)](t_2 - t_1)$$

Thus, as  $t_2 \rightarrow t_1$ ,  $\|T_3\Upsilon(t_2) - T_3\Upsilon(t_1)\| \rightarrow 0 \Rightarrow T_3$  exhibits equicontinuity. According to the Arzelà-Ascoli theorem,  $T_3$  possesses the property of compactness.

Thus, according to Proposition 2.5,  $T_3$  possesses a Lipschitz property with a constant of zero.  $\square$

Since  $T = T_1 + T_2 + T_3$  and  $T_1$  is Lipschitz with Lipschitz constant  $\frac{(\log L)^\varrho}{\Gamma(\varrho + 1)}(\mathcal{F}_1 + \mathcal{F}_2 l_{\max} + \mathcal{F}_3 h_{\max}) + \mathfrak{E}_1 + \mathfrak{E}_2 k_{\max}$  and  $T_2, T_3$  are Lipschitz with Lipschitz constant 0, As a result,  $T$  demonstrates Lipschitz behavior, showcasing a Lipschitz constant  $\frac{(\log L)^\varrho}{\Gamma(\varrho + 1)}(\mathcal{F}_1 + \mathcal{F}_2 l_{\max} + \mathcal{F}_3 h_{\max}) + \mathfrak{E}_1 + \mathfrak{E}_2 k_{\max}$ .

If we assume that  $\frac{(\log \mathfrak{L})^\varrho}{\Gamma(\varrho + 1)}(\mathcal{F}_1 + \mathcal{F}_2 l_{\max} + \mathcal{F}_3 h_{\max}) + \mathfrak{E}_1 + \mathfrak{E}_2 k_{\max} < 1$ , Subsequently, as indicated by Definition 2.7, it can be deduced that  $T$  exhibits  $\nu$ - condensation.

Let's demonstrate the existence theorem by applying the principles of topological degree theory.

**Theorem 3.7 (Existence).** *The FODE (1.1) guarantees the existence of at least one solution provided that*

$$\left( \frac{(\log L)^\varrho}{\Gamma(\varrho + 1)} + A_{T_2} + A_{T_3} \right) + (1 + B_{T_2} + B_{T_3}) < 1.$$

*Proof.* Consider the set

$$\mathcal{H} = \{\Upsilon \in \Xi : \exists \zeta \in [0, 1] \text{ s. t. } \zeta T\Upsilon = \Upsilon\}$$

We assert that  $\mathcal{H}$  exhibits boundedness.

Suppose  $\Upsilon \in \mathcal{H}$ . Then  $\exists \zeta \in [0, 1]$  s. t.  $\zeta T\Upsilon = \Upsilon$ .

$$\text{i.e., } \Upsilon = \zeta(T_1(\Upsilon) + T_2(\Upsilon) + T_3(\Upsilon))$$

Applying  $\|\cdot\|$  to both sides yields

$$\begin{aligned} \|\Upsilon\| &= \zeta\|T_1(\Upsilon)\| + \|T_2(\Upsilon)\| + \|T_3(\Upsilon)\| \\ &\leq \left(\frac{(\log \mathfrak{L})^e}{\Gamma(\varrho+1)} + A_{T_2} + A_{T_3}\right)(\mathcal{F} + \check{\mathcal{F}}\|\Upsilon\|^p + \tilde{\mathcal{F}}\|\Upsilon\|^p) \\ &\quad + (1 + B_{T_2} + B_{T_3})(\mathfrak{E} + \check{\mathfrak{E}}\|\Upsilon\|^p) \end{aligned}$$

In the absence of boundedness in  $\mathcal{H}$ , when dividing both sides by  $\|\Upsilon\|$ , the following is obtained:

$$1 \leq \left(\frac{(\log L)^e}{\Gamma(\varrho+1)} + A_{T_2} + A_{T_3}\right) \frac{(\mathcal{F} + \check{\mathcal{F}}\|\Upsilon\|^p + \tilde{\mathcal{F}}\|\Upsilon\|^p)}{\|\Upsilon\|} + (1 + B_{T_2} + B_{T_3}) \frac{(\mathfrak{E} + \check{\mathfrak{E}}\|\Upsilon\|^p)}{\|\Upsilon\|}$$

As  $\|\Upsilon\|$  tends towards infinity, given that  $p$  lies in the range  $[0, 1)$ , the r.h.s of the previous inequality rapidly converges to zero. Consequently, we arrive at the contradiction  $1 \leq 0$ , establishing the boundedness of  $H$ . By invoking Theorem 2.8, we assert that  $T$  undoubtedly harbors a fixed point, thus unequivocally proving the existence of at least one solution to problem (1.1).  $\square$

At present, we demonstrate uniqueness pertaining to discussed problem.

**Theorem 3.8 (Uniqueness).** *Assume FODE (1.1) admits a solitary solution under the condition that*

$$\left(\frac{(\log L)^e}{\Gamma(\varrho+1)} + A_{T_2} + A_{T_3}\right)(\mathcal{F}_1 + \mathcal{F}_2 l_{\max} + \mathcal{F}_3 h_{\max}) + (1 + B_{T_2} + B_{T_3})(\mathfrak{E}_1 + \mathfrak{E}_2 k_{\max}) < 1$$

*Proof.* Consider arbitrary elements  $\Upsilon_1, \Upsilon_2 \in \Xi$ . Let  $t$  be any element in  $\mathcal{J}$ ,

$$\begin{aligned} \|T\Upsilon_1(t) - T\Upsilon_2(t)\| &= \|(T_1\Upsilon_1(t) - T_1\Upsilon_2(t)) + (T_2\Upsilon_1(t) - T_2\Upsilon_2(t)) + (T_3\Upsilon_1(t) - T_3\Upsilon_2(t))\| \\ &\leq \|(T_1\Upsilon_1 - T_1\Upsilon_2)\| + \|(T_2\Upsilon_1 - T_2\Upsilon_2)\| + \|(T_3\Upsilon_1 - T_3\Upsilon_2)\| \end{aligned}$$

Taking supremum over all  $t \in \mathcal{J}$ ,

$$\|T\Upsilon_1 - T\Upsilon_2\| \leq \|(T_1\Upsilon_1 - T_1\Upsilon_2)\| + \|(T_2\Upsilon_1 - T_2\Upsilon_2)\| + \|(T_3\Upsilon_1 - T_3\Upsilon_2)\|$$

Based on Theorem 3.1, we show that

$$\|T_1\Upsilon_1(t) - T_1\Upsilon_2(t)\| \leq \left[\frac{(\log \mathfrak{L})^e}{\Gamma(\varrho+1)}(\mathcal{F}_1 + \mathcal{F}_2 l_{\max} + \mathcal{F}_3 h_{\max}) + \mathfrak{E}_1 + \mathfrak{E}_2 k_{\max}\right] \|\Upsilon_1 - \Upsilon_2\|$$

We have from definition of  $T_2$ :

$$\begin{aligned} \|T_2\Upsilon_1(t) - T_2\Upsilon_2(t)\| &\leq \|K_1(t)\| \left[{}^H\mathcal{I}^{\varrho_1+\varrho} \|f_{\Upsilon_1}(\theta_1) - f_{\Upsilon_2}(\theta_1)\| + \|g_{\Upsilon_1}(\theta_1) - g_{\Upsilon_2}(\theta_1)\|\right] \\ &\quad + \|K_2(t)\| c_2 \left[{}^H\mathcal{I}^{\varrho_2+\varrho} \|f_{\Upsilon_1}(\theta_1) - f_{\Upsilon_2}(\theta_1)\| + \|g_{\Upsilon_1}(\theta_2) - g_{\Upsilon_2}(\theta_2)\|\right] \\ &\leq \left(\|K_1(t)\| {}^H\mathcal{I}^{\varrho_1+\varrho}(1)(\varrho_1 + \varrho) + \|K_2(t)\| c_2 {}^H\mathcal{I}^{\varrho_2+\varrho}(1)(\varrho_2 + \varrho)\right) \\ &\quad \times (\mathcal{F}_1 + \mathcal{F}_2 l_{\max} + \mathcal{F}_3 h_{\max}) \|\Upsilon_1 - \Upsilon_2\| \\ &\quad + (\|K_1(t)\| + \|K_2(t)\| c_2) (\mathfrak{E}_1 + \mathfrak{E}_2 k_{\max}) \|\Upsilon_1 - \Upsilon_2\| \\ &= [A_{T_2}(\mathcal{F}_1 + \mathcal{F}_2 l_{\max} + \mathcal{F}_3 h_{\max}) + B_{T_2}(\mathfrak{E}_1 + \mathfrak{E}_2 k_{\max})] \|\Upsilon_1 - \Upsilon_2\| \end{aligned}$$

Thus,

$$\|T_2\Upsilon_1 - T_2\Upsilon_2\| \leq [A_{T_2}(\mathcal{F}_1 + \mathcal{F}_2l_{\max} + \mathcal{F}_3h_{\max}) + B_{T_2}(\mathfrak{E}_1 + \mathfrak{E}_2k_{\max})]\|\Upsilon_1 - \Upsilon_2\|$$

We have from definition of  $T_3$ :

$$\begin{aligned} \|T_3\Upsilon_1(t) - T_3\Upsilon_2(t)\| &\leq \|K_2(t)\| \left( \|c\| [{}^H\mathcal{I}^\varrho \|f_{\Upsilon_1}(\mathfrak{L}) - f_{\Upsilon_2}(\mathfrak{L})\| + \|g_{\Upsilon_1}(\mathfrak{L}) - g_{\Upsilon_2}(\mathfrak{L})\|] \right. \\ &\quad \left. + \|d\| [{}^H\mathcal{I}^{\varrho-\rho} \|f_{\Upsilon_1}(\mathfrak{L}) - f_{\Upsilon_2}(\mathfrak{L})\| + \|g_{\Upsilon_1}(\mathfrak{L}) - g_{\Upsilon_2}(\mathfrak{L})\|] \right) \\ &\leq \|K_2(t)\| \left( \|c\| [{}^H\mathcal{I}^\varrho(1)(\varrho) + \|d\|c_2[{}^H\mathcal{I}^{\varrho-\rho}(1)(\varrho - \rho)]] \right) \\ &\quad \times (\mathcal{F}_1 + \mathcal{F}_2l_{\max} + \mathcal{F}_3h_{\max})\|\Upsilon_1 - \Upsilon_2\| + \|K_2(t)(\|c\| + \|d\|)(\mathfrak{E}_1 + \mathfrak{E}_2k_{\max})\|\Upsilon_1 - \Upsilon_2\| \\ &= [A_{T_3}(\mathcal{F}_1 + \mathcal{F}_2l_{\max} + \mathcal{F}_3h_{\max}) + B_{T_3}(\mathfrak{E}_1 + \mathfrak{E}_2k_{\max})]\|\Upsilon_1 - \Upsilon_2\| \end{aligned}$$

Thus,

$$\|T_3\Upsilon_1 - T_3\Upsilon_2\| \leq [A_{T_3}(\mathcal{F}_1 + \mathcal{F}_2l_{\max} + \mathcal{F}_3h_{\max}) + B_{T_3}(\mathfrak{E}_1 + \mathfrak{E}_2k_{\max})]\|\Upsilon_1 - \Upsilon_2\|$$

Hence, we have,

$$\begin{aligned} \|T\Upsilon_1 - T\Upsilon_2\| &\leq |(T_1\Upsilon_1 - T_1\Upsilon_2)| + |(T_2\Upsilon_1 - T_2\Upsilon_2)| + |(T_3\Upsilon_1 - T_3\Upsilon_2)| \\ &\leq \left[ \left( \frac{(\log L)^\varrho}{\Gamma(\varrho + 1)} + A_{T_2} + A_{T_3} \right) (\mathcal{F}_1 + \mathcal{F}_2l_{\max} + \mathcal{F}_3h_{\max}) \right. \\ &\quad \left. + (1 + B_{T_2} + B_{T_3})(\mathfrak{E}_1 + \mathfrak{E}_2k_{\max}) \right] \|\Upsilon_1 - \Upsilon_2\| \end{aligned}$$

Since  $\left( \frac{(\log L)^\varrho}{\Gamma(\varrho + 1)} + A_{T_2} + A_{T_3} \right) (\mathcal{F}_1 + \mathcal{F}_2l_{\max} + \mathcal{F}_3h_{\max}) + (1 + B_{T_2} + B_{T_3})(\mathfrak{E}_1 + \mathfrak{E}_2k_{\max}) < 1$ , According to Banach Contraction Theorem,  $\exists$  only one fixed point for  $T$ . Consequently, the FODE (1.1) possesses solitary solution.  $\square$

## 4 4.1 Application

Allow FIDEs of neutral type with boundary value conditions

$$\left. \begin{aligned} {}^C_H D^{\frac{3}{2}}[\Upsilon(t) - \frac{1}{16} \int_0^1 e^{2s-2} \Upsilon(s) ds] &= \frac{1}{12\pi} \sin(2\pi\Upsilon) + \frac{1}{30} \Upsilon + t^2 + 2 + \frac{1}{8} \int_0^1 e^{2s-2} \Upsilon(s) ds + \frac{1}{8} \int_0^1 e^s \Upsilon(s) ds, \\ \Upsilon(0) + 3({}^C_H D^{\frac{1}{3}} \Upsilon(0)) &= \int_0^{\frac{1}{4}} \frac{(\frac{1}{4}-s)^{\frac{-1}{2}}}{\Gamma(\frac{1}{2})} \Upsilon(s) ds, \quad t \in [0, 1] \\ \Upsilon(1) + 2({}^C_H D^{\frac{1}{3}} \Upsilon(1)) &= 3 \int_0^{\frac{3}{4}} \frac{(\frac{3}{4}-s)^{\frac{-1}{2}}}{\Gamma(\frac{1}{2})} \Upsilon(s) ds, \end{aligned} \right\} \quad (4.1)$$

here  $g(t, \Upsilon(t), P\Upsilon(t)) = \frac{1}{16} \int_0^1 e^{2s-2} \Upsilon(s) ds$ ,  
 $f(t, \Upsilon(t), O\Upsilon(t), S\Upsilon(t)) = \frac{1}{12\pi} \sin(2\pi\Upsilon) + \frac{1}{30} \Upsilon + t^2 + 2 + \frac{1}{8} \int_0^1 e^{2s-2} \Upsilon(s) ds + \frac{1}{8} \int_0^1 e^s \Upsilon(s) ds$ .  
 Here  $\varrho = \frac{3}{2}, \rho = \frac{1}{3}, a = 1, b = 3, c = 1, d = 2, c_1 = 1, c_1 = 3, \varrho_1 = \frac{1}{4}$ ,

$$\varrho_2 = \frac{3}{4}, \theta_1 = \frac{1}{4}, \theta_2 = \frac{3}{4}, \mathfrak{L} = e \ \&$$

$$\begin{aligned} A_{T_2} &= \|c_1\|\|\Psi_1\| + \|c_2\|\|\Psi_2\|\mathfrak{L} \frac{(\log \theta_1)^{\varrho_1 + \varrho_2}}{\Gamma(\varrho_1 + \varrho_2 + 1)} + (\|c_1\|c_2\|\|\Psi_3\| + \|\Psi_4\|\mathfrak{L}) \frac{(\log \theta_2)^{\varrho_1 + \varrho_2}}{\Gamma(\varrho_2 + \varrho_2 + 1)} \\ B_{T_2} &= (\|c_1\|\|\Psi_1\| + \|c_2\|\|\Psi_2\|\mathfrak{L}) + (\|c_1\|c_2\|\|\Psi_3\| + \|\Psi_4\|\mathfrak{L}) \\ A_{T_3} &= (\|c_1\|\|\Psi_3\| + \|\Psi_4\|\mathfrak{L}) \left[ \frac{\|c\|(\log \mathfrak{L})^\varrho}{\Gamma(\varrho + 1)} + \frac{\|d\|(\log \mathfrak{L})^{\varrho - \rho}}{\Gamma(\varrho - \rho + 1)} \right] \\ B_{T_3} &= (\|c_1\|\|\Psi_3\| + \|\Psi_4\|\mathfrak{L})[\|c\| + \|d\|]. \end{aligned}$$

Using the above parameters in  $A_{T_2}, B_{T_2}, A_{T_3}$  and  $B_{T_3}$ , we get

$$\left( \frac{(\log L)^\varrho}{\Gamma(\varrho + 1)} + A_{T_2} + A_{T_3} \right) + (1 + B_{T_2} + B_{T_3}) = 8.0427$$

In order to demonstrate Theorem 3.6, we opt for

$$\begin{aligned} f(t, \Upsilon, O(\Upsilon), S(\Upsilon)) &= \frac{1}{12\pi} \sin(2\pi\Upsilon) + \frac{1}{30} \Upsilon + t^2 + 2 + \frac{1}{8} \int_0^1 e^{2s-2} \Upsilon(s) ds + \frac{1}{8} \int_0^1 e^s \Upsilon(s) ds \\ g(t, \Upsilon, P(\Upsilon)) &= \frac{1}{16} \int_0^1 e^{2s-2} \Upsilon(s) ds \end{aligned}$$

in (1.1) and then

$$\begin{aligned} |f(t, \Upsilon_1, O(\Upsilon_1), S(\Upsilon_1)) - f(t, \Upsilon_2, O(\Upsilon_2), S(\Upsilon_2))| &= \frac{1}{12\pi} |\sin(2\pi\Upsilon_1) - \sin(2\pi\Upsilon_2)| + \frac{1}{30} |\Upsilon_1 - \Upsilon_2| \\ &\quad + 0.054 |\Upsilon_1 - \Upsilon_2| + 0.125 |\Upsilon_1 - \Upsilon_2| \\ &= 0.212 |\Upsilon_1 - \Upsilon_2| \end{aligned}$$

&

$$|g(t, \Upsilon_1, P(\Upsilon_1)) - g(t, \Upsilon_2, P(\Upsilon_2))| = 0.027 |\Upsilon_1 - \Upsilon_2|$$

Hence the condition (A2) and (A4) holds with  $U_1 = 0.212, U_2 = 0.027$ , where  $U_1 = \mathcal{F}_1 + \mathcal{F}_2 l_{\max} + \mathcal{F}_3 h_{\max}, U_2 = \mathfrak{E}_1 + \mathfrak{E}_2 k_{\max}$ .

To calculate  $t_f$  from given data is

$$t_f = \frac{U_1 (\log \mathfrak{L})^\varrho}{\Gamma(\varrho + 1)} = 0.1595$$

If not, in the case of every  $t \in \mathcal{J}$  and  $\Upsilon$  in  $\mathcal{R}$ , we obtain

$$\begin{aligned} |f(t, \Upsilon, O(\Upsilon), S(\Upsilon))| &= \left| \frac{1}{12\pi} \sin(2\pi\Upsilon) + \frac{1}{30} \Upsilon + t^2 + 2 + \frac{1}{8} \int_0^1 e^{2s-2} \Upsilon(s) ds + \frac{1}{8} \int_0^1 e^s \Upsilon(s) ds \right| \\ &= 3 + 0.087 |\Upsilon| + 0.125 |\Upsilon| \end{aligned}$$

&

$$|g(t, \Upsilon, P(\Upsilon))| = 0.027 |\Upsilon|.$$

Hence the condition (A1) and (A3) holds with  $\mathcal{F} = 3, \check{\mathcal{F}} = 0.087, \tilde{\mathcal{F}} = 0.125, \mathfrak{E} = 0, \tilde{\mathfrak{E}} = 0.027$ .

Using Theorem 3.6,

$$\mathcal{H} = \{ \Upsilon \in \Xi : \exists \zeta \in [0, 1] \text{ s. t. } \zeta T \Upsilon = \Upsilon \}$$

comprises set of solutions; subsequently,

$$\begin{aligned} \|\Upsilon\| &= \|\zeta(T_1(\Upsilon) + T_2(\Upsilon) + T_3(\Upsilon))\| \\ &\leq \left(\frac{(\log L)^e}{\Gamma(\varrho + 1)} + A_{T_2} + A_{T_3}\right)(\mathcal{F} + \check{\mathcal{F}}\|\Upsilon\|^p + \tilde{\mathcal{F}}\|\Upsilon\|^p) \\ &\quad + (1 + B_{T_2} + B_{T_3})(\mathfrak{E} + \check{\mathfrak{E}}\|\Upsilon\|^p) \end{aligned}$$

Thus,

$$\|\Upsilon\| \leq \frac{\left(\frac{(\log L)^e}{\Gamma(\varrho + 1)} + A_{T_2} + A_{T_3}\right)\mathcal{F} + (1 + B_{T_2} + B_{T_3})\mathfrak{E}}{1 - \left(\frac{(\log L)^e}{\Gamma(\varrho + 1)} + A_{T_2} + A_{T_3}\right)(\check{\mathcal{F}} + \tilde{\mathcal{F}}) - (1 + B_{T_2} + B_{T_3})\check{\mathfrak{E}}} = 53.6861$$

According to Theorem 3.6, there exists at least one solution  $\Upsilon$  in  $C(\mathcal{J}, \mathcal{R} \times \mathcal{R} \times \mathcal{R}, \mathcal{R})$  for the system (1.1) with values taken from (4.1).

Also,  $\left(\frac{(\log L)^e}{\Gamma(\varrho + 1)} + A_{T_2} + A_{T_3}\right)U_1 + (1 + B_{T_2} + B_{T_3})U_2 = 0.8184 < 1$ . Therefore, according to Theorem 3.7, the system (1.1) with values from (4.1) possesses a solitary solution.

#### 4.2 Application

The FIDE with B.V.C's is:

$$\begin{cases} {}^C_H D^{\frac{3}{2}}\zeta(q) = \frac{1}{12\pi} \sin(2\pi\zeta) + \frac{1}{30}\zeta + q^2 + 2 + \frac{1}{8} \int_0^1 e^{2s-2}\zeta(s)ds, & q \in [0, 1] \\ \zeta(0) + 3({}^C_H D^{\frac{1}{3}}\zeta(0)) = \int_0^{\frac{1}{4}} \frac{(\frac{1}{4}-s)^{\frac{-1}{2}}}{\Gamma(\frac{3}{2})}\zeta(s)ds, \\ \zeta(1) + 2({}^C_H D^{\frac{1}{3}}\zeta(1)) = 5 \int_0^{\frac{3}{4}} \frac{(\frac{3}{4}-s)^{\frac{-1}{2}}}{\Gamma(\frac{3}{2})}\zeta(s)ds, \end{cases} \quad (4.2)$$

where  $g(\zeta) = \frac{1}{8} \int_0^1 e^{2s-2}\zeta(s)ds$ . Here  $\nu = \frac{3}{2}, \gamma = 13, a, c = 1, b = 3, d = 2, c_1 = 5, \nu_1 = \frac{1}{4}, \nu_2 = \frac{3}{4}, \zeta_1 = \frac{1}{4}, \zeta_2 = \frac{3}{4}, L = e$  and

$$\begin{aligned} C_{T_2} &= |c_1||\chi_1| + |c_2||\chi_2|L \frac{(\log \zeta_1)^{\nu+\nu_1}}{\Gamma(\nu_1 + \nu + 1)} + (|c_1||c_2||\chi_3| + |\chi_4|L) \frac{(\log \zeta_2)^{\nu+\nu_2}}{\Gamma(\nu_2 + \nu + 1)}, \\ C_{T_3} &= (|c_1||\chi_3| + |\chi_4|) \left[ \frac{|c|(\log L)^\nu}{\Gamma(\nu + 1)} + \frac{|d|(\log L)^{\nu-\gamma}}{\Gamma(\nu - \gamma + 1)} \right]. \end{aligned}$$

By the above parameters in  $C_{T_2}$  and  $C_{T_3}$ , we get

$$\left(\frac{(\log L)^\nu}{\Gamma(\nu + 1)} + C_{T_2} + C_{T_3}\right) = 3.3428$$

To Prove Thm 3.6, we take

$$f(q, \zeta, g(\zeta)) = \frac{1}{12\pi} \sin(2\pi\zeta) + \frac{1}{30}\zeta + q^2 + 2 - \frac{1}{8} \int_0^1 e^{2s-2}\zeta(s)ds$$

in (1.1) and then

$$\begin{aligned} |f(q, \zeta_1, g(\zeta_1)) - f(q, \zeta_2, g(\zeta_2))| &= \frac{1}{12\pi} |\sin(2\pi\zeta_1) - \sin(2\pi\zeta_2)| + \frac{1}{30} |\zeta_1 - \zeta_2| + 0.054 |\zeta_1 - \zeta_2| \\ &= 0.087 |\zeta_1 - \zeta_2|. \end{aligned}$$

Hence the condition (A1) holds with  $P = 0.087$ , where  $P = \alpha + \beta k_{\max} + \chi h_{\max}$ . To calculate  $q_f$  from given data is

$$q_f = \frac{P(\log L)^\nu}{\Gamma(\nu + 1)} = 0.0755.$$

Let  $q \in J, \zeta \in \mathcal{B}$ ,

$$\begin{aligned} |f(q, \zeta, g(\zeta))| &= \left| \frac{1}{12\pi} \sin(2\pi\zeta) + \frac{1}{30}\zeta + q^2 + 2 - \frac{1}{8} \int_0^1 e^{2s-2} \zeta(s) ds \right| \\ &= \frac{1}{30} |\zeta| + 3 + 0.054 |\zeta|. \end{aligned}$$

Hence the condition (A2) holds with  $\mathcal{W}_1 = 3, \mathcal{W}_2 = \frac{1}{30}, \mathcal{W}_3 = 0.054, \mathcal{W}_4 = 0$ . By Theorem 3.6,

$$\mathcal{H} = \{\zeta \in \Psi : \exists \lambda \in [0, 1] \text{ s. t. } \lambda T\zeta = \zeta\}$$

is solution set,

$$\begin{aligned} \|\zeta\| &= \|\lambda(T_1(\zeta) + T_2(\zeta) + T_3(\zeta))\| \\ &\leq \left( \frac{(\log L)^\nu}{\Gamma(\nu + 1)} + C_{T_2} + C_{T_3} \right) (\mathcal{W}_1 + \mathcal{W}_2 \|\zeta\|^p + \mathcal{W}_3 \|\zeta\|^p + \mathcal{W}_4 \|\zeta\|^p) \end{aligned}$$

Thus,

$$\|\zeta\| \leq \frac{\left( \frac{(\log L)^\nu}{\Gamma(\nu + 1)} + C_{T_2} + C_{T_3} \right) A_1}{1 - \left( \frac{(\log L)^\nu}{\Gamma(\nu + 1)} + C_{T_2} + C_{T_3} \right) (A_2 + A_3 + A_4)} = 13.6109$$

by Theorem 3.6 the system (1.1) with values from (4.2) has a solution. Hence,  $\left[ \frac{(\log L)^\nu}{\Gamma(\nu + 1)} + C_{T_2} + C_{T_3} \right] P = 0.2827 < 1$ . By Theorem 3.7 the system (1.1) with values from (4.1) has a unique solution.

## 5 Conclusion

In summary, your work seems to explore the theoretical framework for solving neutral FIDEs with specific boundary conditions and employs topological methods to establish key properties of the solutions. This kind of research is crucial for advancing the understanding of complex differential equations and their applications. For analytical solutions, we shall verify theorem conditions which can save time and prevent efforts on intractable problems. Using simplification techniques such as separation of variables, integration factors, and transformations, and exploring symbolic computation to find exact solutions when possible. Also, investigating the special cases or constraints might make an analytical solution feasible. For approximate solutions, we shall use numerical methods like Euler's, perform error analysis, and leverage computational simulation tools. Adaptive techniques and comparative studies can help choose the best approach. Thus, Analytical solutions offer exact expressions and insights into system behavior, serving as benchmarks for verification, whereas numerical solutions play a crucial role in solving complex problems, provide flexible approximations, and enable dynamic exploration through simulations.

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