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# Chelyshkov polynomials approximation for solving a class of linear and nonlinear equations arising in astrophysics

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**Abstract:** The aim of this paper is to present a numerical approach for solving linear and nonlinear differential equations arising in astrophysics, commonly known as Lane-Emden equations. The proposed method is based on the collocation method and first involves taking the truncated Chelyshkov series of the function in the equation. The computational cost is reduced due to the orthogonality of Chelyshkov polynomials, and the solution of a linear or nonlinear Lane-Emden equation is reduced to solving a system of linear or nonlinear algebraic equations. Several test examples of these types of differential equations, modeling different physical problems with initial and boundary conditions, are solved to demonstrate the reliability of the method. To demonstrate its effectiveness, absolute error tables and graphs are presented, and the numerical results are compared with other methods and exact solutions. It is observed that when the exact solution has a polynomial form, the proposed method proves to be highly accurate and effective.

**Keywords:** Chelyshkov polynomials; Lane-Emden equations; Collocation method; Initial and boundary value problems.
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## 1 Introduction

The research of many authors focuses on singular initial value problems in specific second-order ordinary differential equations. Lane-Emden differential equations are among the most widely studied in this class of ordinary differential equations. It is well known that this type of differential equation was first studied by two astrophysicists, Jonathan Homer Lane and Robert Emden. It can be observed that this type of differential equation is used in modeling various problems in physics and astrophysics [32, 9]. Over the years, various analytical and numerical methods have been developed to solve Lane-Emden differential equations. Some studies that discuss analytical solutions can be referenced in [8, 34, 12, 10, 14]. Because it is often not possible to find analytical solutions for Lane-Emden type equations, several numerical methods have been developed. Some of these numerical methods include the Bessel collocation method [36, 35], Bernstein series matrix collocation method [13], Taylor matrix-collocation method [1], B-Spline collocation method [2], Bernoulli polynomials method [15], Hermite functions collocation method [19], Hermite polynomials method [18], Chebyshev of second kind collocation method [20], Laguerre wavelet method [37], Chebyshev polynomials method [4], Laguerre polynomial approach method [11], Haar wavelet method [30],

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He's variational iteration method [22], Chebysev wavelet method [17], Tangent chord method [5], Adomian decomposition method [29, 34], least square recursive approach [24], variational iteration method [33], A modified Chebyshev $\vartheta$-weighted Crank–Nicolson method [6], the Chebyshev cardinal functions method [26], A novel operational matrix method [27], and Legendre-spectral method [23].

Chelyshkov polynomials have been successfully applied to solve various problems in numerical analysis, including two-dimensional Fredholm–Volterra integral equations [25], multi-order fractional differential equations [31], three-dimensional linear Fredholm integral equations [28], time-delay fractional optimal control [16], distributed-order fractional differential equations [21], and the Bagley-Torvik equation [7]. In this research, our main goal is to find the numerical solution of Lane-Emden type equations using the Chelyshkov collocation method. To achieve this, we consider the following form of the Lane-Emden equation.

$$u''(t) + \frac{\kappa}{t}u'(t) + g(t, u(t)) = f(t), \quad \kappa \in R, \quad 0 < t \leq 1, \tag{1.1}$$

subject to the some cases of initial and boundary conditions as mentioned below:

**Case 1:**

$$u(0) = \nu_1, \quad u'(0) = \nu_2, \quad \nu_1, \nu_2 \in R, \tag{1.2}$$

**Case 2:**

$$u(0) = \gamma_1, \quad u(1) = \gamma_2, \quad \gamma_1, \gamma_2 \in R, \tag{1.3}$$

**Case 3:**

$$u'(0) = \eta_1, \quad u'(1) = \eta_2, \quad \eta_1, \eta_2 \in R, \tag{1.4}$$

**Case 4:**

$$u'(0) = \zeta_1, \quad \zeta_2 u(1) + \zeta_3 u'(1) = \zeta_4, \quad \zeta_1, \zeta_2, \zeta_3, \zeta_4 \in R, \tag{1.5}$$

where, $f(t) \in C[0, 1]$ and $g(t, u(t))$ is a continuous real value function.

Although several numerical methods have been developed to solve Lane-Emden equations, such as the Bessel and Bernstein collocation methods, these approaches often suffer from high computational costs and limited accuracy, particularly when dealing with highly nonlinear terms. In many cases, they also struggle to provide accurate solutions for certain boundary conditions. The need for a more efficient and accurate method remains, especially for equations modeling complex physical phenomena. This study aims to address these limitations by utilizing the Chelyshkov collocation method, which offers the advantage of orthogonality, reducing computational complexity and providing more accurate solutions. The main advantages of the proposed method are: (1) it can solve the nonlinear form of Lane-Emden equation with any nonlinear term; (2) it does not require to complex operations to implement the algorithm; and (3) by converting the differential equation into a system of algebraic equations, the numerical approximation of $u(t)$ can be obtained. Results obtained from our study indicate that the Chelyshkov collocation method significantly improves the accuracy of solutions for both linear and nonlinear Lane-Emden equations. Additionally, the method reduces computational costs compared to other numerical techniques. We have tested several examples, demonstrating the method's effectiveness in handling different boundary conditions and comparing the results with other well-known methods.

The remainder of this paper is structured as follows: In Section 2, we provide a detailed introduction to Chelyshkov polynomials and their properties. Section 3, discusses the operational matrix of derivatives and its application to the problem. In Section 4, we present the implementation of the Chelyshkov collocation method for solving Lane-Emden equations. Section 5, provides several numerical results to demonstrate the effectiveness of the proposed method, and finally, Section 5, concludes the paper with a summary of the findings and future directions for research.

# 2 Definition of Chelyshov polynomials

In this section, we introduce Chelyshkov polynomials and their properties. Chelyshkov, in [3], presented a class of orthogonal polynomials with the following explicit formula:

$$P_{Nm}(t) = \sum_{i=0}^{N-m} c_{i,m} t^{m+i}, \qquad m = 0, 1, 2, ..., N, \tag{2.1}$$

where $c_{im} = (-1)^i \binom{N-m}{i} \binom{N+nmi+1}{N-m}$.
The orthogonality of (2.1) over the interval $[0, 1]$ with respect to the weight function $w(t) = 1$, can be expressed as follows:

$$\int_0^1 P_{Nn}(t) P_{Nm}(t) = \begin{cases} \frac{1}{n+m+1} & n = m, \\ 0 & n \neq m, \end{cases}$$

and (2.1) has Rodrigues's type form

$$P_{Nm}(t) = \frac{1}{(N-m)} \times \frac{1}{t^{m+1}} \times \frac{d^{N-m}}{dt^{N-m}} (t^{N+m+1} - (1-t)^{N-m}), \qquad m = 0, 1, ..., N.$$

The matrix representation of (2.1) is given as follows:

$$\underbrace{\begin{bmatrix} P_{N0}(t) \\ P_{N1}(t) \\ P_{N2}(t) \\ \vdots \\ P_{NN}(t) \end{bmatrix}}_{\text{P(t)}} = \underbrace{\begin{bmatrix} c_{00} & c_{10} & c_{20} & \cdots & c_{(N-1)0} & c_{N0} \\ 0 & c_{01} & c_{11} & \cdots & c_{(N-2)1} & c_{(N-1)1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_{0(N-1)} & c_{1(N-1)} \\ 0 & 0 & 0 & \cdots & 0 & c_{0N} \end{bmatrix}}_{\text{C}} \underbrace{\begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^N \end{bmatrix}}_{\text{T}} \Rightarrow P(t) = CT(t).$$

Due to the orthogonality of Chelyshkov polynomials and their ability to form a basis in $L^2(R)$, any function $u(t) \in L^2[0, 1]$ can be expressed in terms of these polynomials. On the other hand, for the function $u(t)$, we have:

$$u(t) = \sum_{j=0}^{\infty} \lambda_j P_{Nj}(t).$$

However, in practice only the first $(N + 1)$ term of Chelyshkov polynomials are considered as follow:[12]

$$u(t) = \sum_{j=0}^{N} \lambda_j P_{Nj}(t) = \lambda^t P(t), \tag{2.2}$$

where $\lambda = [\lambda_0, \lambda_1, ..., \lambda_N]^t$ and due to the orthogonality of Chelyshkov polynomials the coefficients $\lambda_j$ are given by

$$\lambda_j = (2j + 1) \int_0^1 u(t) P_{Nj}(t)\, dt, \qquad j = 0, 1, ..., n.$$

The behavior of the Chelyshkov polynomials is shown in Figure 1.

## 2.1 Error bounds

Since the truncated series of Chelyshkov polynomials appears in (2.2), this section establishes the convergence and error bounds.

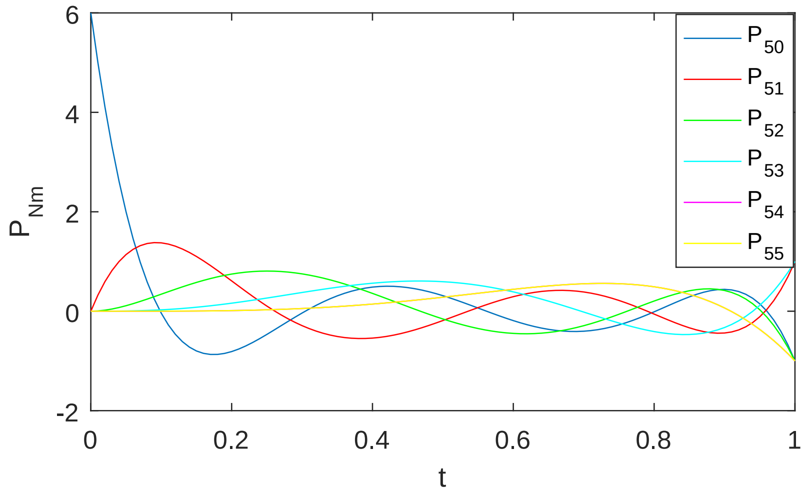


Figure 1: Chelyshkov polynomials for $N = 5$ and $m = 0, 1, 2, 3, 4, 5$.

**Theorem 2.1.** *Let $H$ be Hilbert space and $Y$ with $dim Y < \infty$ be a closed subspace of $H$ and $\{y_1, y_2, ..., y_m\}$ is any basis for $Y$. Let $u \in H$ be an arbitrary function and $u_0$ be the unique best approximation to $u$ out of $Y$. Then $\|u - u_0\| = F_u$ where $F_u = \sqrt{\frac{G(u,y_1,y_2,...,y_m)}{G(y_1,y_2,...,y_m)}}$ and $G(y_1, y_2, ..., y_m)$ is called the Gram determinant of $y_1, y_2, ..., y_m$.*

*Proof.* See [13, 14]. □

**Theorem 2.2.** *Suppose that $u \in C^{m+1}[0, 1]$ is a $m + 1$ times continuously differentiable function and let $S = span\{P_{N0}(t), ..., P_{Nm}(t)\}$. If $u_0$ is the best approximation of $u$ from $S$, then $\|u - u_0\| \leq \frac{k}{(N+1)!\sqrt{2N+3}}$ where $k = \max_{t\in[0,1]} \mid u^{N+1}(t) \mid$.*

*Proof.* See [14, 15]. □

# 3 Operational matrix of derivative

In this section, the operational matrix of derivative is derived from the vector $P(t)$, as defined in (2.2).

**Definition 3.1.** *$D^{(1)}_{(N+1)\times N}$ and $D^{(2)}_{N\times(N-1)}$ are the operational matrices of first and second derivative of $P(t)$ if and only if*

$$\frac{dP(t)}{dt} = D^{(1)}T^{(1)}(t), \quad \frac{d^2P(t)}{dt} = D^{(2)}T^{(2)}(t)$$

*where,* $T^{(1)}(t) = \begin{bmatrix} 1 \\ t \\ \vdots \\ t^{N-1} \end{bmatrix}$ *and* $T^{(2)}(t) = \begin{bmatrix} 1 \\ t \\ \vdots \\ t^{N-2} \end{bmatrix}$

**Theorem 3.2.** *The operational matrices $D^{(1)}$ and $D^{(2)}$ are defined by*

$$D^{(1)} = \underbrace{\begin{bmatrix} c_{00} & c_{10} & c_{20} & \cdots & c_{(N-1)0} & c_{N0} \\ 0 & c_{01} & c_{11} & \cdots & c_{(N-2)1} & c_{(N-1)1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_{0(N-1)} & c_{1(N-1)} \\ 0 & 0 & 0 & \cdots & 0 & c_{0N} \end{bmatrix}}_{\text{(N+1)}\times\text{(N+1)}} \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & N \end{bmatrix}}_{\text{(N+1)}\times\text{(N)}} \tag{3.1}$$

$$D^{(2)} = D^{(1)} \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & N \end{bmatrix}}_{\text{(N)}\times\text{(N}-1)} \tag{3.2}$$

*Proof.* With respect to the representation of Chelyshkov polynomials, we have:

$$P(t) = CT(t) = \begin{bmatrix} c_{00} + c_{10}t + c_{20}t^2 + \cdots + c_{(N-1)0}t^{N-1} + c_{N0}t^N \\ c_{01}t + c_{11}t^2 + \cdots + c_{(N-2)1}t^{N-1} + c_{(N-1)1}t^N \\ \vdots \\ c_{0(N-1)}t^{N-1} + c_{1(N-1)}t^N \\ c_{0N}t^N \end{bmatrix},$$

therefore

$$\frac{dP(t)}{dt} = \begin{bmatrix} 0c_{00} + c_{10} + 2c_{20}t + \cdots + (N-1)c_{(N-1)0}t^{N-2} + Nc_{N0}t^{N-1} \\ c_{01} + 2c_{11}t + \cdots + (N-1)c_{(N-2)1}t^{N-2} + Nc_{(N-1)1}t^{N-1} \\ \vdots \\ (N-1)c_{0(N-1)}t^{N-2} + Nc_{1(N-1)}t^{N-1} \\ Nc_{0N}t^{N-1} \end{bmatrix} =$$

$$C \begin{bmatrix} 0 \\ 1 \\ 2t \\ 3t^2 \\ \vdots \\ Nt^{N-1} \end{bmatrix} = C \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & N \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^{N-1} \end{bmatrix} = D^{(1)}T^{(1)}.$$

Similarly, it can be demonstrated that $\frac{d^2P(t)}{dt^2} = D^{(2)}T^{(2)}$. □

# 4 Solution of Lane-Emden equation

The Chelyshkov collocation method is applied to the Lane-Emden type equations in this section. In order to solve the considered type of Lane-Emden equations (1.1), we first approximate $u(t)$ as:

$$u(t) = \lambda^t CT(t), \tag{4.1}$$

and hence

$$u'(t) = \lambda^t D^{(1)} T^{(1)}(t), \tag{4.2}$$

$$u''(t) = \lambda^t D^{(2)} T^{(2)}(t). \tag{4.3}$$

Then by substituting (4.1), (4.2) and (4.3) in (1.1) we have

$$\lambda^t D^{(2)} T^{(2)}(t) + \frac{\kappa}{t}\lambda^t D^{(1)} T^{(1)}(t) + g(t, \lambda^t CT(t)) = f(t), \tag{4.4}$$

also, the initial and boundary conditions for different cases yields:

**Case 1:**

$$\lambda^t CT(0) = \nu_1, \quad \lambda^t D^{(1)} T^{(1)}(0) = \nu_2, \quad \nu_1, \nu_2 \in R, \tag{4.5}$$

**Case 2:**

$$\lambda^t CT(0) = \gamma_1, \quad \lambda^t CT(1) = \gamma_2, \quad \gamma_1, \gamma_2 \in R, \tag{4.6}$$

**Case 3:**

$$\lambda^t D^{(1)} T^{(1)}(0) = \eta_1, \quad \lambda^t D^{(1)} T^{(1)}(1) = \eta_2, \quad \eta_1, \eta_2 \in R, \tag{4.7}$$

**Case 4:**

$$\lambda^t D^{(1)} T^{(1)}(0) = \zeta_1, \quad \zeta_2 \lambda^t CT(1) + \zeta_3 \lambda^t D^{(1)} T^{(1)}(1) = \zeta_4, \quad \zeta_1, \zeta_2, \zeta_3, \zeta_4 \in R, \tag{4.8}$$

By considering equation (4.4) along with each of the initial and boundary conditions, we have $N+1$ unknown coefficients. Therefore, $N+1$ algebraic equations are needed to determine them. By replacing $t$ with the first $N-2$ roots $t_i$ of the well-known shifted Legendre polynomials (or even $N-2$ arbitrary points in $(0,1])$ as collocation points, the following algebraic equations are obtained.

$$\lambda^t D^{(2)} T^{(2)}(t_i) + \frac{\kappa}{t_i}\lambda^t D^{(1)} T^{(1)}(t_i) + g(t_i, \lambda^t CT(t_i)) = f(t_i), \quad i = 1, 2, ..., N-2. \tag{4.9}$$

Finally, to find the approximate solution of (1.1), the system of equations of (4.9), along with one of the conditions (4.5) or (4.6) or (4.7) or (4.8) must be solved.

# 5 Numerical results

Several Lane–Emden type equations are studied to provide a clear overview of the proposed method in this paper. To demonstrate the usefulness and accuracy of the method, problems with exact analytical solutions are selected initially. Subsequently, some problems without analytical solutions are studied, and the results are compared with those obtained from other numerical methods. The errors are given by $\mid u_{ex}(t) - u_{ap}(t) \mid$ . at selected points. Furthermore, the numerical results indicate that the approach is easy to implement.It is important to note that for linear equations, the proposed method can be applied using the Gaussian elimination method, while for nonlinear cases, MATLAB tools can be utilized.

**Example 5.1.** *Consider the following linear and nonhomogeneous Lane–Emden equation with initial value*

$$u''(t) + \frac{8}{t}u'(t) + tu(t) = t^5 - t^4 + 44t^2 - 30t,$$

$$u(0) = u'(0) = 0, \quad t \in (0, 1],$$

*with the correct solution* $u(t) = t^4 - t^3$.

Now, by applying the technique described in Section 4 with $N = 4$, the solution of equation is approximate as follows:

$$u_{ap}(t) = \sum_{j=0}^{4} \lambda_j P_{4j}(t).$$

Then by forming (4.9) for this equation we get a $5 \times 5$ system of linear equations and by solve it the five unknowns are as below:

$$\begin{bmatrix}\lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4\end{bmatrix} = \begin{bmatrix}0 & 0 & 0 & -\frac{1}{8} & -\frac{1}{8}\end{bmatrix}$$

therefore, we obtain $u_{ap}(t) = t^4 - t^3$ which is the exact solution.

**Example 5.2.** *Consider the following singular Lane-Emden equation*

$$u''(t) + \frac{2}{t}u'(t) + u(t) = t^3 + t^2 + 12t + 6,$$

$$u(0) = u'(0) = 0, \quad t \in (0, 1],$$

*The exact solution of the problem is* $u(t) = t^3 + t^2$.

By choose $N = 3$, similar to previous example, after applying the proposed method we get a $4 \times 4$ system of linear equation and by solve it the four unknowns are as below:

$$\begin{bmatrix}\lambda_0 & \lambda_1 & \lambda_2 & \lambda_3\end{bmatrix} = \begin{bmatrix}0 & 0 & \frac{1}{6} & \frac{13}{6}\end{bmatrix}$$

therefore, we obtain $u_{ap}(t) = t^3 + t^2$ which is the exact solution.

**Example 5.3.** *Consider the nonlinear initial value problem*

$$u''(t) + \frac{2}{t}u'(t) + u^3(t) = t^6 + 6,$$

$$u(0) = u'(0) = 0, \quad t \in (0, 1],$$

*the exact solution of this equation is* $u(t) = t^2$.

In this example we let $N = 2$ and get three unknown coefficients in approximation (2.2)

$$\begin{bmatrix}\lambda_0 & \lambda_1 & \lambda_2\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}$$

then can be seen that $u_{ap}(t) = t^2$ which is the exact solution.

**Example 5.4.** *Cconsider the linear singular two-point boundary value problem*

$$u''(t) + \frac{1}{t}u'(t) + u(t) = \frac{5}{4} + \frac{t^2}{16},$$

$$u'(0) = 0, \quad u(1) = \frac{17}{16}, \quad t \in (0, 1],$$

where $u(t) = 1 + \frac{t^2}{16}$ is the exact solution of this problem. For this problem by letting $N = 2$, the coefficients of Chelyshkov series are $\begin{bmatrix}\lambda_0 & \lambda_1 & \lambda_2\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & 1 & \frac{83}{48}\end{bmatrix}$ and it can be seen that $u_{ap}(t) = \frac{1}{3}P_{20}(t) + P_{21}(t) + \frac{83}{48}P_{22}(t) = 1 + \frac{t^2}{16}$.

These four examples show that, it seems in both linear and nonlinear Lane-Emden equations, when the solution is a polynomial by applying the proposed method, the exact solution can be obtained.

**Example 5.5.** *Consider the following linear Lane-Emden equation*

$$u''(t) + \frac{2}{t}u'(t) - (4t^2 + 6)u(t) = 0,$$

$$u(0) = 1, \quad u'(0) = 0, \quad t \in (0, 1],$$

this is the isothermal gas spheres equation and has exact solution $u(t) = e^{t^2}$. The numerical solution is obtained for $N = 4, 5, 7, 10, 15$ and the absolute errors of results are presented in Table 1. Then for comparison the results of three other methods are shown in Table 2, such as, Hermite functions collocation method [19], Hermite polynomials method [18] and Chebyshev of second kind collocation method [20]. Figure 2 compare the exact solution and approximate solutions and errors functions of the approximate solutions for various N is shown in Fig. 3.

Table 1: Error analysis of Example 5.5 with Chelyshkov method for different values of $N$.

| $t$ | $N = 4$ | $N = 5$ | $N = 7$ | $N = 10$ | $N = 15$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0.1 | 1.9355e-05 | 9.4415e-06 | 1.8470e-06 | 7.6054e-08 | 1.5542e-11 |
| 0.2 | 2.1575e-03 | 1.9419e-04 | 1.4394e-05 | 6.3713e-09 | 1.3617e-11 |
| 0.3 | 7.9625e-03 | 4.2362e-04 | 2.9339e-05 | 1.8208e-09 | 1.3437e-11 |
| 0.4 | 1.6442e-02 | 3.5967e-04 | 3.7218e-05 | 7.8821e-08 | 1.3994e-11 |
| 0.5 | 2.4615e-02 | 1.3968e-04 | 4.4915e-05 | 1.8514e-07 | 1.5112e-11 |
| 0.6 | 2.8041e-02 | 7.3096e-04 | 3.6965e-05 | 2.5992e-07 | 1.6741e-11 |
| 0.7 | 2.1700e-02 | 4.0009e-03 | 1.8397e-05 | 3.3612e-07 | 1.8869e-11 |
| 0.8 | 1.3106e-03 | 1.2343e-02 | 9.9875e-05 | 4.0836e-07 | 2.2358e-11 |
| 0.9 | 3.4683e-02 | 2.7633e-02 | 6.0379e-05 | 5.1457e-07 | 2.4237e-11 |
| 1.0 | 8.2102e-02 | 4.9166e-02 | 2.5833e-04 | 9.0427e-08 | 4.5107e-11 |

Table 2: Error analysis of Example 5.5 three other methods.

| $t$ | [19] | [18] | [20] |
|---|---|---|---|
| 0 | 0 | 0.285000e-18 | 0 |
| 0.1 | 1.78e-08 | 0.903323e-11 | $1.27e-10$ |
| 0.2 | 2.09e-08 | 0.111835e-11 | $1.44e-10$ |
| 0.5 | 2.62e-08 | 0.149890e-11 | $1.78e-10$ |
| 0.7 | 3.27e-08 | 0.192421e-1 | $1.98e-10$ |
| 0.8 | 3.79e-08 | 0.224083e-11 | $2.87e-10$ |
| 0.9 | 5.48e-08 | 0.266033e-11 | $3.11e-10$ |
| 1.0 | 2.51e-09 | 0.322054e-11 | $3.64e-10$ |

**Example 5.6.** *The following nonlinear Lane–Emden equation is used in the theory of stellar structure and describes the temperature distribution of a spherical gas cloud. It also describes the variation of density as a function of the radial distance for a polytrope.*

$$u''(t) + \frac{2}{t}u'(t) + u^m(t) = 0,$$

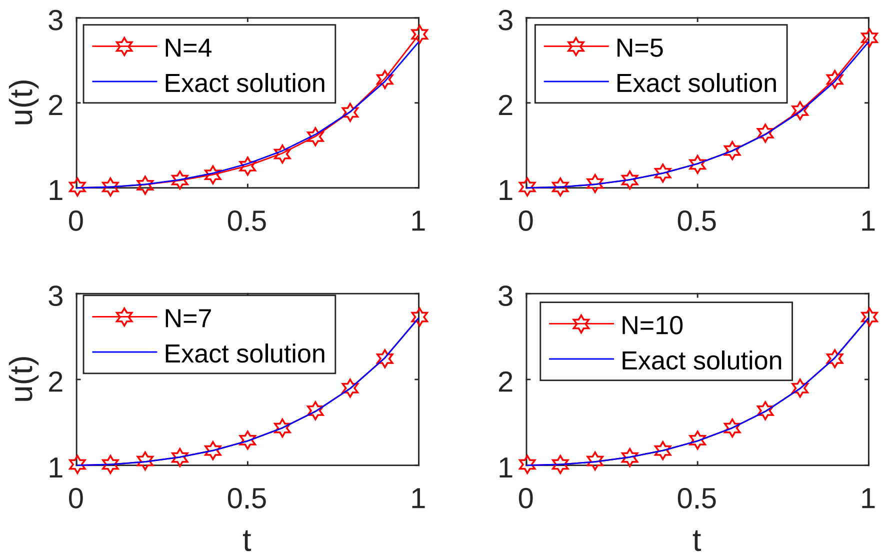


Figure 2: Comparison of the Chelyshkov method in four cases and exact solution of Example 5.5.

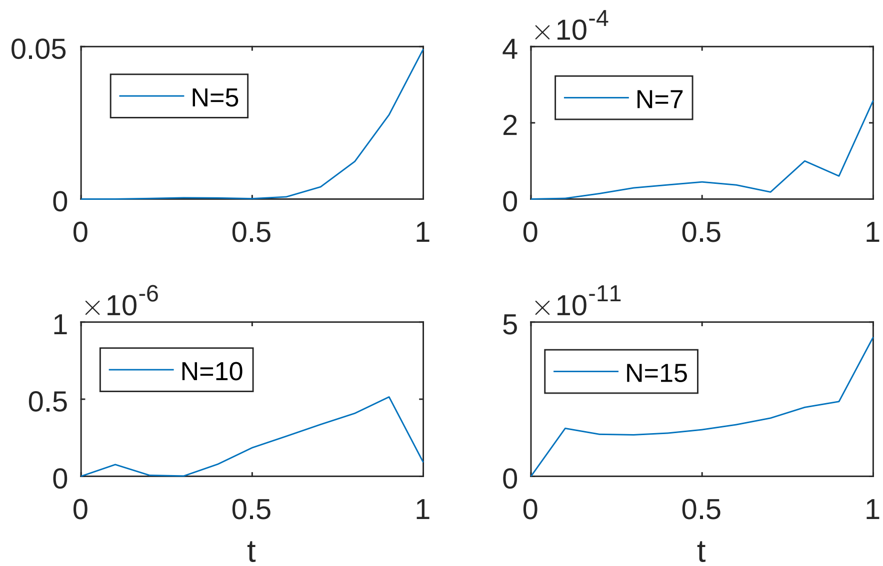


Figure 3: Plot of absolute errors at N =5,7,10,15 for Example 5.5.

$$u(1)=\frac{\sqrt{3}}{2},\qquad u'(0)=0,\quad t\in(0,1],$$

Exact solutions exist only for $m = 0, 1, 5$. As we know for $m = 0, 1$ this is a linear equation so only for $m = 5$ is considered and the exact solution in this case is $u(t) = \sqrt{\frac{3}{3+t^2}}$.
Table 3, presents the absolute error of the Chelyshkov method for various choices of $N$. These results show that the error decreases as $N$ increases. The results obtained from [9, 29, 22] where, respectively are the Taylor wavelet method (using $8^{th}$ order polynomial), the improved decomposition method (using $12^{th}$ order polynomial) and He's variational iteration method (VIM) (using $12^{th}$ order polynomial) are presented in Table 4. It seems that the Chelyshkov method has higher accuracy than other methods. Due to the fact that in Chelyshkov method a lower degree polynomial is used, we might say that this method is more effective than others. In order to compare the Chelyshkov method with the exact solution, the resulting graph of this type of Lane-Emden equation is shown in Fig. 4 and Fig. 5 shows the absolute errors of approximate solutions.

Table 3: Error analysis of Example 5.6 with Chelyshkov method for different values of $N$.

| t | $N = 4$ | $N = 5$ | $N = 6$ | $N = 8$ | $N = 9$ | $N = 10$ |
|---|---|---|---|---|---|---|
| 0 | 2.2681e-03 | 6.2361e-05 | 6.5971e-05 | 7.0414e-07 | 4.8082e-08 | 3.2460e-09 |
| 0.1 | 2.2488e-03 | 6.1924e-05 | 6.5540e-05 | 6.9188e-07 | 5.0419e-08 | 3.4910e-09 |
| 0.2 | 2.2327e-03 | 5.9498e-05 | 6.2208e-05 | 6.8419e-07 | 4.5484e-08 | 3.1952e-09 |
| 0.3 | 2.2413e-03 | 5.6242e-05 | 5.7933e-05 | 6.5469e-07 | 4.4119e-08 | 3.0417e-09 |
| 0.4 | 2.2507e-03 | 5.2457e-05 | 5.4755e-05 | 6.4005e-07 | 3.5309e-08 | 2.8261e-09 |
| 0.5 | 2.2092e-03 | 4.6288e-05 | 5.1764e-05 | 6.3071e-07 | 2.3014e-08 | 2.5935e-09 |
| 0.6 | 2.0570e-03 | 3.5695e-05 | 4.6202e-05 | 5.6230e-07 | 1.9116e-08 | 2.3404e-09 |
| 0.7 | 1.7476e-03 | 2.1169e-05 | 3.6162e-05 | 4.1849e-07 | 1.9883e-08 | 2.0884e-09 |
| 0.8 | 1.2676e-03 | 6.9450e-06 | 2.2655e-05 | 2.6345e-07 | 1.8957e-08 | 1.8167e-09 |
| 0.9 | 6.5249e-04 | 8.5558e-07 | 9.4760e-06 | 1.4065e-07 | 1.8850e-08 | 1.5884e-09 |
| 1.0 | 1.0569e-16 | 5.0175e-17 | 1.2992e-15 | 2.4126e-16 | 1.3344e-16 | 3.1385e-16 |

Table 4: Absolute errors of other methods for Example 5.6.

| t | [9] | [22] | [29] |
|---|---|---|---|
| 0 | 6.52e-06 | 6.32e-03 | 3.20e-03 |
| 0.1 | 6.46e-06 | 6.27e-03 | 3.10e-03 |
| 0.2 | 6.30e-06 | 6.12e-03 | 2.90e-03 |
| 0.3 | 6.05e-06 | 5.87e-03 | 2.60e-03 |
| 0.4 | 5.70e-06 | 5.53e-03 | 2.20e-03 |
| 0.5 | 5.30e-06 | 5.09e-03 | 1.80e-03 |
| 0.6 | 4.84e-06 | 4.53e-03 | 1.40e-03 |
| 0.7 | 4.33e-06 | 3.82e-03 | 9.84e-04 |
| 0.8 | 3.86e-06 | 2.88e-03 | 6.07e-04 |
| 0.9 | 3.24e-06 | 1.64e-03 | 2.77e-04 |
| 1.0 | 1.45e-13 | 0 | 3.49e-08 |

**Example 5.7.** *Consider the following nonlinear equation*

$$u''(t) + \frac{2}{t}u'(t) + 4(2e^{u(t)} + e^{\frac{u(t)}{2}}) = 0,$$

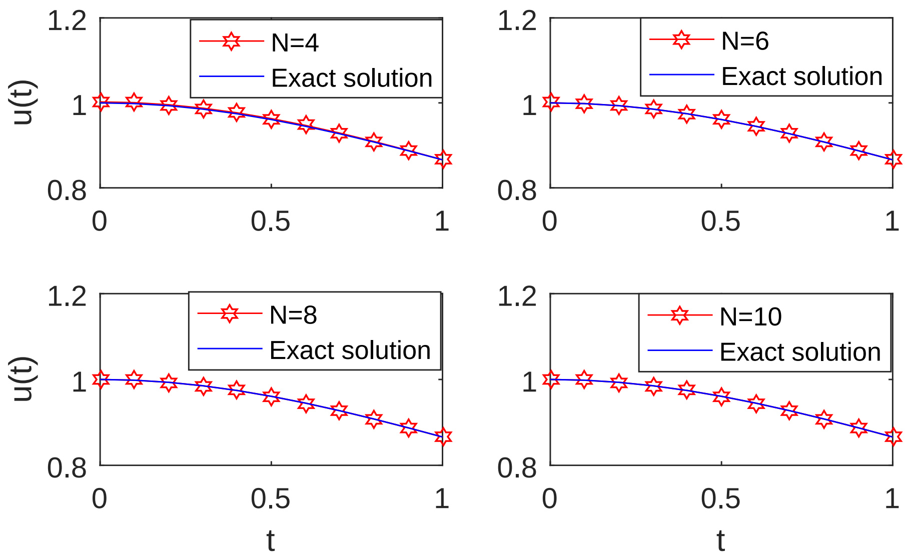


Figure 4: Comparison of the Chelyshkov method in four cases and exact solution of Example 5.6.

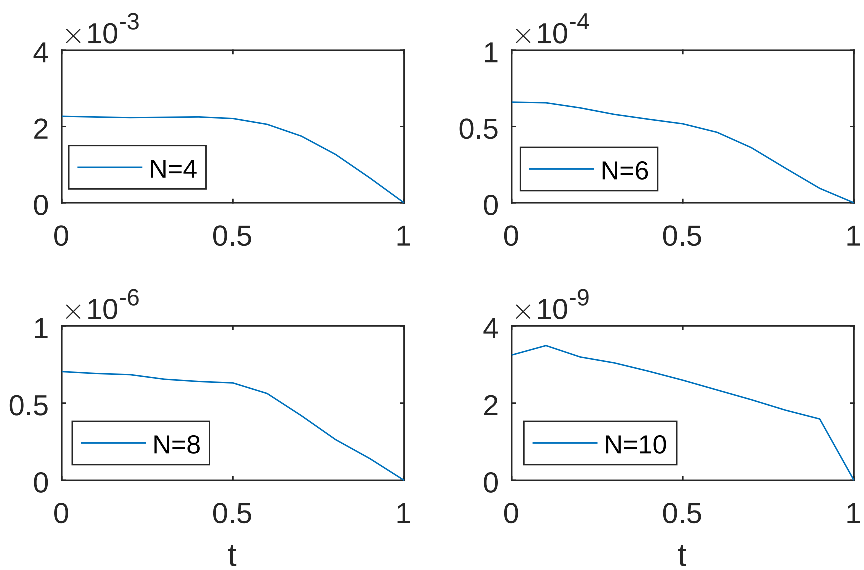


Figure 5: Absolute errors at N =4,6,8,10 for Example 5.6.

$$u(0) = 0, \qquad u'(0) = 0, \quad t \in (0, 1],$$

so that $u(t) = -2\ln(1+t^2)$ is exact solution for this problem. Absolute error of the Chelyshkov method for four choices of $N$. is presented in Table 5. Figure 10 shows the comparison between the exact solution and Chelyshkov method.

Table 5: Error analysis of Example 5.7 with Chelyshkov method for different values of $N$.

| t | $N = 5$ | $N = 7$ | $N = 10$ | $N = 11$ |
|---|---|---|---|---|
| 0 | 4.5348e-28 | 1.1081e-27 | 1.4536e-28 | 5.0985e-26 |
| 0.1 | 2.1441e-07 | 2.3294e-06 | 1.2976e-07 | 4.3802e-09 |
| 0.2 | 3.9474e-05 | 1.7774e-05 | 3.9620e-09 | 8.7340e-09 |
| 0.3 | 8.4892e-05 | 3.4126e-05 | 6.5712e-09 | 7.8492e-09 |
| 0.4 | 4.2792e-05 | 3.9051e-05 | 3.2370e-08 | 1.5458e-08 |
| 0.5 | 4.7413e-05 | 4.0523e-05 | 1.4141e-07 | 1.2347e-07 |
| 0.6 | 1.3460e-04 | 2.4632e-05 | 1.2159e-06 | 1.1886e-06 |
| 0.7 | 1.1635e-03 | 3.5046e-05 | 7.9527e-06 | 7.9364e-06 |
| 0.8 | 3.6288e-03 | 1.3302e-04 | 3.9263e-05 | 3.9220e-05 |
| 0.9 | 7.7121e-03 | 2.3122e-04 | 1.5091e-04 | 1.5346e-04 |
| 1.0 | 1.2636e-02 | 4.0870e-04 | 4.8349e-04 | 5.0051e-04 |

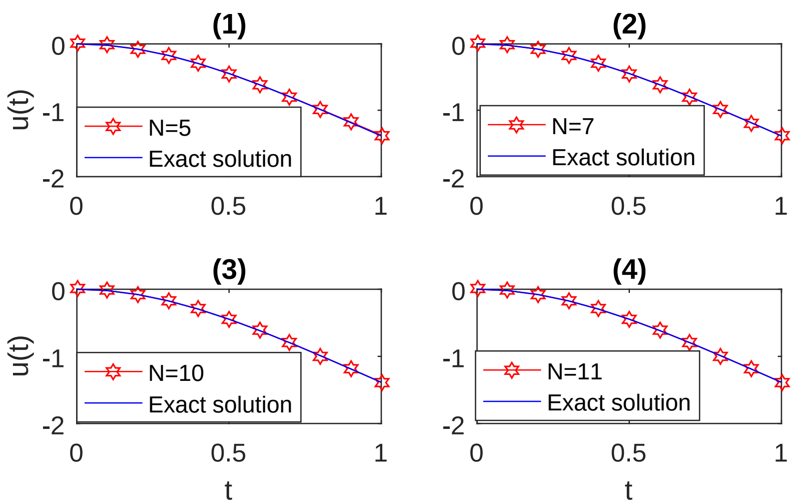


Figure 6: Comparison of the Chelyshkov method in four cases and exact solution of Example 5.7.

**Example 5.8.** *Consider the following nonlinear singular equation with boundary value condition*

$$u''(t) + \frac{2}{t}u'(t) - \frac{nu(t)}{u(t)+k} = 0,$$

$$u'(0) = 0, \qquad 5u(1) + u'(1) = 5, \quad t \in (0,1],$$

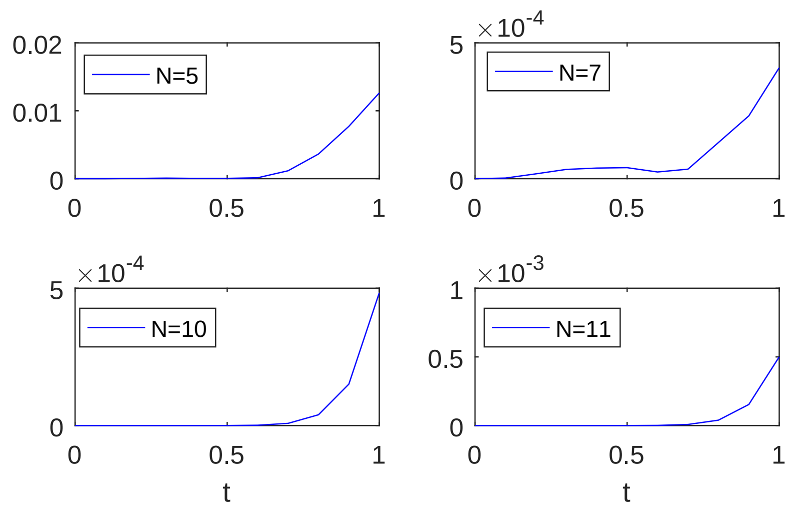


Figure 7: Absolute errors at N =5,7,10,11 for Example 5.7.

where $n = 0.76129$ and $k = 0.03119$. This problem arises in the context of oxygen tension in a cell with Michaelis-Menten oxygen uptake kinetics and is also used in modeling heat conduction through a solid. This problem does not have an exact analytical solution, so the numerical results of the Chelyshkov method are compared with those of several other numerical methods. Table 6 presents the numerical results obtained using the Chelyshkov method for $N = 5, 6, 7$. Then for compression the numerical results of four other methods are shown. Taylor wavelet method [9] with $7^{th}$ order polynomial, He's variational iteration method [22] with $6^{th}$ order polynomial, B-Spline method [2] with $n = 40$ collocation points and Bernoulli polynomials method [15] with $14^{th}$ order polynomial. The results presented show that all methods correspond to up to 6 decimal places.
Figure 8 illustrates the numerical results of the five method presented in the Table 6. This figure shows that all the obtained numerical results are coincide with each other.

**Example 5.9.** *Consider the following nonlinear equation*

$$u''(t) + \frac{2}{t}u'(t) + \sinh(u(t)) = 0,$$

$$u(0) = 1, \qquad u'(0) = 0, \quad t \in (0, 1],$$

this problem has no exact solution. To solve it, the Chelyshkov method was applied for various choices of $N$, and the results are shown in Table 7. Table 8 presents the results of three other methods for comparison. The series solutions using the Chelyshkov and Adomian decomposition methods [34] are given by (5.1) and (5.2), respectively. The graphs of these two polynomials are shown in Fig. 9.

$$\begin{aligned} u(t) \quad &\cong \quad \frac{2138319383671}{9007199254740992}t^7 - \frac{377906498529}{281474976710656}t^6 + \frac{622012086111}{4503599627370496}t^5 \\ &+ \quad \frac{33937960239315}{2251799813685248}t^4 + \frac{26441746155}{4503599627370496}t^3 - \frac{220526848203165}{1125899906842624}t^2 + 1. \end{aligned} \tag{5.1}$$

Table 6: Comparison of the results of Chelyshkov method with other numerical methods for example 5.8

| | Chelyshkov method | | | Four other methods | | | |
|---|---|---|---|---|---|---|---|
| $t$ | $N=5$ | $N=6$ | $N=7$ | [9] | [22] | [2] | [15] |
| 0 | 0.82848333 | 0.82848324 | 0.82848328 | 0.8284835 | 0.8284832 | 0.8284832 | 0.8284832 |
| 0.1 | 0.82970613 | 0.82970604 | 0.82970608 | 0.8297063 | 0.8297060 | 0.8297060 | 0.8297060 |
| 0.2 | 0.83337477 | 0.83337468 | 0.83337472 | 0.8333750 | 0.8333747 | 0.8333746 | 0.8333747 |
| 0.3 | 0.83948994 | 0.83948986 | 0.83948990 | 0.8394901 | 0.8394898 | 0.8394898 | 0.8394899 |
| 0.4 | 0.84805282 | 0.84805273 | 0.84805278 | 0.8480530 | 0.8480527 | 0.8480527 | 0.8480527 |
| 0.5 | 0.85906498 | 0.85906487 | 0.85906492 | 0.8590651 | 0.8590649 | 0.8590648 | 0.8590649 |
| 0.6 | 0.87252841 | 0.87252826 | 0.87252831 | 0.8725285 | 0.8725282 | 0.8725282 | 0.8725283 |
| 0.7 | 0.88844543 | 0.88844525 | 0.88844530 | 0.8884455 | 0.8884452 | 0.8884452 | 0.8884453 |
| 0.8 | 0.90681869 | 0.90681849 | 0.90681854 | 0.9068188 | 0.9068185 | 0.9068185 | 0.9068185 |
| 0.9 | 0.92765111 | 0.92765094 | 0.92765098 | 0.9276512 | 0.9276509 | 0.9276509 | 0.9276509 |
| 1.0 | 0.95094588 | 0.95094576 | 0.95094579 | 0.9509459 | 0.9509457 | 0.9509457 | 0.9509457 |

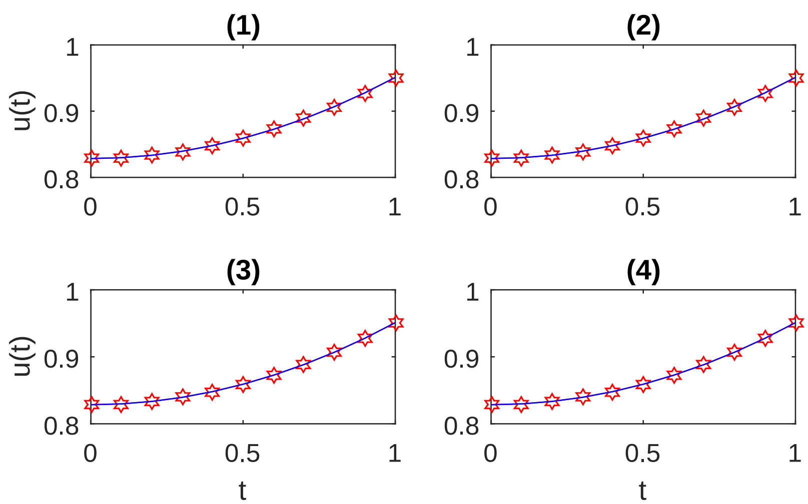


Figure 8: (1) to (4) respectively are comparison between [9],[22],[2],[15] with proposed method for Example 5.8.

$$\begin{aligned} u(t) &\cong 1-\frac{(e^2-1)}{12e}t^2+\frac{1}{480}\frac{(e^4-1)}{e^2}t^4-\frac{1}{30240}\frac{(2e^6+3e^2-3e^4-2)}{e^3}t^6 \\ &+ \frac{1}{26127360}\frac{(61e^8-104e^6+104e^2-61)}{e^4}t^8. \end{aligned} \tag{5.2}$$

**Example 5.10.** *Consider the following nonlinear equation*

$$u''(t)+\frac{2}{t}u'(t)+\sin(u(t))=0,$$

Table 7: Numerical results of Chelyshkov method for different values of $N$ for Example 5.9.

| t | $N = 5$ | $N = 7$ | $N = 9$ | $N = 11$ |
|---|---|---|---|---|
| 0 | 1.0 | 1.0 | 1.0 | 1.0 |
| 0.1 | 0.99804384931 | 0.99804284121 | 0.99804284149 | 0.99804284149 |
| 0.2 | 0.99219127537 | 0.99218943515 | 0.99218943498 | 0.99218943498 |
| 0.3 | 0.98249507831 | 0.98249359944 | 0.98249359937 | 0.98249359937 |
| 0.4 | 0.96904420261 | 0.96904375968 | 0.96904375822 | 0.96904375822 |
| 0.5 | 0.95196061278 | 0.95196109739 | 0.95196109363 | 0.95196109363 |
| 0.6 | 0.93139616894 | 0.93139709018 | 0.93139708785 | 0.93139708785 |
| 0.7 | 0.90752950247 | 0.90753056386 | 0.90753056589 | 0.90753056589 |
| 0.8 | 0.88056289159 | 0.88056437617 | 0.88056436869 | 0.8805643687 |
| 0.9 | 0.850719137 | 0.85072185168 | 0.85072179291 | 0.85072179289 |
| 1.0 | 0.81823843751 | 0.81824308764 | 0.818242931 | 0.8182429307 |

Table 8: Numerical results with three other methods for Example 5.9.

| t | [19] | [34] | [20] |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 0.1 | 0.9980428424 | 0.9980428414 | 0.9981138095 |
| 0.2 | 0.9921894358 | 0.9921894348 | 0.9922758837 |
| 0.5 | 0.9519610938 | 0.9519611019 | 0.9520376245 |
| 1.0 | 0.8182429293 | 0.8182516669 | 0.8183047481 |

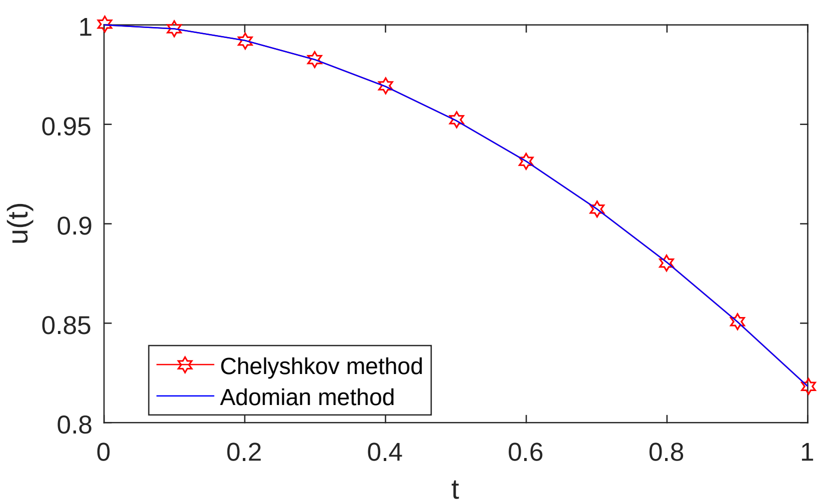


Figure 9: Graph of equation of Example 5.9 the Chelyshkov method and Adomian decomposition method [34].

$$u(0) = 1, \qquad u'(0) = 0, \quad t \in (0, 1],$$

The exact solution of this equation does not exist. The numerical results obtained by the Chelyshkov method for some values of $N$ are presented in Table 9. In order to compare with other methods, the results of four additional methods are presented in Table 10. It can be seen that the results from almost all methods match up to six decimal places. The series solutions using the Chelyshkov and Adomian decomposition methods [34] are given by (5.3) and (5.4), respectively. The graphs of these two polynomials are shown in Fig. 10.

$$\begin{aligned} u(t) \;&\cong\; \frac{31239955251}{99079191802150912}t^{10} - \frac{5395287835}{6192449487634432}t^{9} - \frac{10792481517}{1125899906842624}t^{8} \\ &- \frac{549191511}{562949953421312}t^{7} + \frac{167582094505}{1125899906842624}t^{6} - \frac{92541267}{562949953421312}t^{5} \\ &- \frac{34124568634975}{9007199254740992}t^{4} - \frac{2067}{549755813888}t^{3} - \frac{1263212036692281}{9007199254740992}t^{2} + 1 \end{aligned} \tag{5.3}$$

$$\begin{aligned} u(t) \;&\cong\; 1 - \frac{1}{6}\sin(1)t^{2} + \frac{1}{120}\sin(1)\cos(1)t^{4} + \sin(1)\left(\frac{\sin^{2}(1)}{3024} - \frac{\cos^{2}(1)}{5040}\right)t^{6} \\ &+ \sin(1)\cos(1)\left(-\frac{113}{3265920}(\sin(1))^{2} + \frac{\cos^{2}(1)}{362880}\right)t^{8} \\ &+ \sin(1)\left(\frac{1781}{898128000}(\sin(1)\cos(1))^{2} - \frac{\cos^{4}(1)}{39916800} - \frac{19}{23950080}\sin^{4}(1)t^{10}.\right. \end{aligned} \tag{5.4}$$

Table 9: Numerical results of Chelyshkov method for different values of $N$ for Example 5.10.

| t | $N = 5$ | $N = 6$ | $N = 8$ | $N = 10$ |
|---|---|---|---|---|
| 0 | 1.0 | 1.0 | 1.0 | 1.0 |
| 0.1 | 0.998597928750458 | 0.998597931213983 | 0.99859793191456 | 0.998597931913943 |
| 0.2 | 0.994396347152585 | 0.994396292637564 | 0.994396282794174 | 0.994396282794428 |
| 0.3 | 0.987408908801763 | 0.98740879138231 | 0.987408770918898 | 0.987408770919056 |
| 0.4 | 0.977658519544914 | 0.97765845385318 | 0.977658434107493 | 0.977658434107925 |
| 0.5 | 0.965177727179492 | 0.965177897265481 | 0.965177883160555 | 0.965177883160695 |
| 0.6 | 0.95000911115249 | 0.9500096589203 | 0.950009641476366 | 0.950009641485341 |
| 0.7 | 0.932205672259426 | 0.932206583238428 | 0.932206555776594 | 0.932206555805307 |
| 0.8 | 0.91183122234335 | 0.911832266552764 | 0.911832258735383 | 0.911832258655652 |
| 0.9 | 0.888960773993836 | 0.888961559659205 | 0.888961660161799 | 0.888961659401403 |
| 1.0 | 0.863680930245984 | 0.863681128126019 | 0.863681439241949 | 0.863681436847002 |

Table 10: Numerical results with four other methods for Example 5.10.

| t | [9] | [34] | [19] | [20] |
|---|---|---|---|---|
| 0.1 | 0.9985979273 | 0.9985979358 | 0.9986051425 | 0.9985979304 |
| 0.2 | 0.9943962648 | 0.9943962733 | 0.9944062706 | 0.9943962682 |
| 0.5 | 0.9651777801 | 0.9651777886 | 0.9651881683 | 0.9651777836 |
| 1.0 | 0.8636811238 | 0.8636811027 | 0.8636881301 | 0.8636811288 |

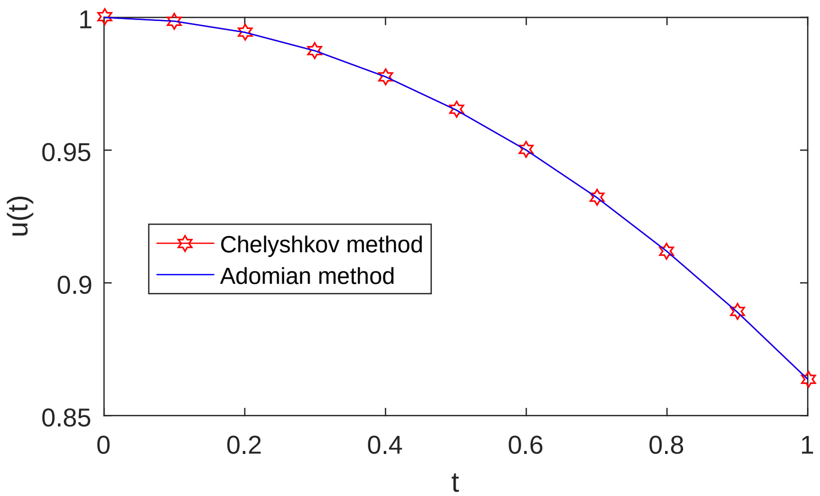


Figure 10: Graph of equation of Example 5.10 the Chelyshkov method and Adomian decomposition method [34].

# 6 Conclusion

A numerical method based on Chelyshkov polynomials for solving linear and nonlinear Lane–Emden equations has been investigated and proposed. It has been observed that when an analytical solution is available and has a polynomial form, the capability of the proposed method is confirmed. Additionally, when the Lane–Emden equation has a non-polynomial analytical solution, the numerical results obtained using the proposed method demonstrate good accuracy. In cases where the nonlinear equation lacks an exact solution, the results from the Chelyshkov method show good agreement with other numerical methods mentioned in the literature. In ten examples, it has been demonstrated that accurate numerical results can be achieved with relatively small values of $N$. Furthermore, the results are not only presented in tabular form but also approximated as a polynomial solution.
This study has demonstrated the effectiveness of the Chelyshkov collocation method in solving Lane-Emden equations with high accuracy and reduced computational costs. However, there remain several avenues for future research. One potential direction is the extension of this method to more complex differential equations arising in other fields, such as fluid dynamics, thermodynamics, and quantum mechanics. Additionally, exploring the application of Chelyshkov polynomials in higher-dimensional problems or systems of partial differential equations could further enhance the versatility of the proposed method.

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