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Computational of eigenvalues and eigenfunctions of nonlinear differential equation by Sinc-Self-consistent method

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Abstract: We combine sinc method and self-consistent method to obtain eigenvalues and eigenfunctions of a class of nonlinear differential equations. Some properties of sinc method and self-consistent method are given. Numerical examples are given to demonstrate the validity of the present method. We compare these results with the existing results and show that the sinc-self-consistent method is suitable method to obtain eigenvalue and eigenfunction of equations on an infinite domain and produces the smallest eigenvalue with the most accuracy.

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1 Introduction

There are some papers in which non-linear eigenvalue differential equations are studied (see [5, 13]). These type of problem arise in physics, dynamical systems, electronic structure calculations, etc (see [11, 14, 21]). In this paper we consider the non-linear eigenvalue differential equation

$$\alpha F''(x) + \beta(x)F'(x) + \gamma(x)F(x) + Q(x)F^3(x) = EF(x), \qquad \int_a^b F^2(x)dx = 1 \tag{1.1}$$

on (a, b), $F(a)$=$F(b)$=0, where E and $F(x)$ are eigenvalues and corresponding eigenfunctions, respectively. Also $\beta(x)$, $\gamma(x)$ and $Q(x)$ are known functions. For a few well-known functions $\gamma(x)$, the equation (1.1) has an analytical solution[3, 4, 6, 7, 18]. But for some function $\gamma(x)$, this equation have not exact solutions. Several authors used different numerical methods such as homotopy analysis method [2], variational method [1, 12], NU method [19], fixed point method [22] to solve Eq. 1.1. In this paper, we will use the sinc-selfconsistent (SSCF) method to obtain the solutions of Eq. 1.1. Some examples with exact solutions are considered to show the accuracy of this technique. In Section 2, the preliminary concepts of the sinc function are given. We present a brief overview of the self-consistent method in Section 3. Sections 4 is devoted to the matrix form of Eq. 1.1 by the SSCF method. In section 5 we present the numerical solution

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of some examples by the SSCF method and compare them with the finite difference self-consistent method (FDSCF). Finally, a brief conclusion is presented in Section 6.

2 Sinc method

Sinc numerical method has been extensively used because of their exponential convergence rate, see e.g. [8, 9, 15, 20], etc. Sinc function properties are thoroughly discussed in [17]. The Sinc function is defined on the whole real line as:

$$Sinc(x) = \begin{cases} \frac{sin(\pi x)}{\pi x} & x \neq 0; \\ 1 & x = 0 \end{cases} \tag{2.1}$$

For $h > 0$, the translated Sinc functions with evenly spaced nodes are given as

$$S(k,h)(x) = Sinc\left(\frac{x - kh}{h}\right) \quad k = 0, \pm 1, \pm 2, ... \tag{2.2}$$

The basis functions on (a, b) are then taken to be the composite translated Sinc functions as

$$S_k(x) = S(k,h)o\phi(x) = Sinc\left(\frac{\phi(x) - kh}{h}\right) \quad k = 0, \pm 1, \pm 2, ... \tag{2.3}$$

To obtain the discrete system of Eq. 1.1, we use the derivatives of the composite translated Sinc function at the nodes x_i as follows [17]:

$$\delta_{ji}^{(0)} = S_j(x_i) = \begin{cases} 0 & i \neq j, \\ 1 & i = j. \end{cases} \tag{2.4}$$

$$\delta_{ji}^{(1)} = hS_j'(x_i) = \begin{cases} \frac{(-1)^{(i-j)}}{(i-j)h} & i \neq j, \\ 0 & i = j. \end{cases} \tag{2.5}$$

$$\delta_{ji}^{(2)} = h^2 S_j''(x_i) = \begin{cases} \frac{-2(-1)^{(i-j)}}{(i-j)^2 h^2} & i \neq j, \\ \frac{-\pi^2}{3h^2} & i = j. \end{cases} \tag{2.6}$$

Then for positive integer N and $h = \sqrt{\frac{\pi d}{\alpha M}}$ we have

$$F'(x_j) \approx \sum_{k=-N}^{N} F_k\Big(\phi' S_k'\Big)(x_j) = \sum_{k=-N}^{N} F_k \phi'(x_j)\delta_{kj}^{(1)} \tag{2.7}$$

$$\begin{aligned} F''(x_j) &\approx \sum_{k=-N}^{N} F_k \left(\phi'^2 S_k'' + \phi'' S_k'\right)(x_j) \\ &= \sum_{k=-N}^{N} F_k \left(\phi'^2 \delta_{kj}^{(2)} + \phi'' \delta_{kj}^{(1)}\right)(x_j). \end{aligned} \tag{2.8}$$

3 Self consistent field method

Let the non-linear eigenvalue problem:

$$A(X)X = \Lambda X \tag{3.1}$$

where $X \in R^{n\times 1}$, $X^T X = I$, $A(X) \in R^{n\times n}$ is a matrix that has a special structure and $\Lambda \in R$ is a diagonal matrix consisting of the smallest eigenvalues of $A(X)$. Some researches in [16, 23] investigated the convergence of Self consistent field iteration(SCF) which defined as follow to solve problem (3.1):

$$\begin{cases} \textit{Pick any initial guess } X^{(0)} \\ 1.\textit{For i} = 1,2,\ldots \textit{do} \\ 2.A^{(i)} = A(X^{(i-1)}); \\ 3.\textit{Compute } X^{(i)} \textit{such that} A^{(i)}X^{(i)} = X^{(i)}\Lambda^{(i)}, \textit{and } \Lambda^{(i)} \\ \textit{contains the smallest eigenvalues of } A^{(i)}; \\ 4.\textit{End do} \end{cases} \tag{3.2}$$

Yang et al. in [23] show that, the SCF iteration produces a sequence of approximate solutions that contains two convergent subsequences for some class of problems. They used the standard distance measure [10] between two columns $A, B \in \mathbb{R}^{n\times k}$ i.e., if $A^T A = B^T B = I_k$,

$$dist(A, B) = \|AA^T - BB^T\|_2$$

where for every matrix $H \in \mathbb{R}^{m\times n}$,

$$\|H\|_2 = \sup_{x\neq 0} \frac{\|Hx\|_2}{\|x\|_2}.$$

and obtained the following theorem:

Theorem 3.1. *Let $X^{(0)} \in \mathbb{R}^{n\times k}$ be the initial guess to the solution of the non-linear eigenvalue problem (3.1) that satisfies ${X^{(0)}}^T X^{(0)} = I_k$. If columns of $X^{(i)} \in \mathbb{R}^{n\times k}$ contain eigenvectors associated with the smallest k eigenvalues of $A(X^{(i-1)})$, as we would obtain when applying the SCF iteration to (3.1), and if the gap between the kth and the $k+1$st eigenvalues of $A(X^{(i)})$ is greater than or equal to $\delta > 0$ for all i, then*

$$\lim_{i\to\infty} dist^2(X^{i+2}, X^i) = 0.$$

4 Main results

In this section we obtain the matrix generated by the sinc method. By using the collocation method and relations (2.7) and (2.8) we have

$$\sum_{k=-N}^{N} F_k \left(\alpha {\phi'}^2 \delta_{kj}^{(2)} + \left(\alpha\phi'' + \beta\phi'\right)\delta_{kj}^{(1)}\right)(x_j) + \gamma(x_j)F_j + Q(x_j)F_j^3 = EF_j. \tag{4.1}$$

Let

$$p_{kj} = \left(\alpha {\phi'}^2 \delta_{kj}^{(2)} + \left(\alpha\phi'' + \beta\phi'\right)\delta_{kj}^{(1)}\right)(x_j).$$

Then, we can write system (4.1) as

$$(X + QW)\,F = EF \tag{4.2}$$

where

$$X = \begin{pmatrix} p_{-n,-n} + \gamma_{-n} & \dots & p_{-n,n} \\ \vdots & \ddots & \vdots \\ p_{n,-n} & \dots & p_{n,n} + \gamma_n \end{pmatrix}$$

$$Q = \begin{pmatrix} q_{-n} & 0 & \dots & 0 \\ 0 & q_{-n+1} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & q_n \end{pmatrix}$$

$$F = \begin{pmatrix} f_{-n} \\ \vdots \\ f_n \end{pmatrix}$$

and $W = (DiagF)^2$. Now we must solve the non-linear eigenvalue problem $H(F)F = EF$, where

$$H(F) = X + Q(DiagF)^2.$$

For this end, let $W^0 = 0$ and use the following algorithm until converged to solve problem (4.2):
1. For $j = 1, 2, ...$ do
2. $H^{(j)} = X + QW^{(j-1)}$
3. Compute $Y^{(j)}$ such that $H^{(j)}Y^{(j)} = E^{(j)}Y^{(j)}$, and $E^{(j)}$ contains the smallest eigenvalues of $H^{(i)}$;
4. $F^{(j)} = \frac{Y^{(j)}}{\|Y^{(j)}\|_2}$
5. $W^{(j)} = (DiagF^{(j)})^2$
6. End do
So, we can obtain the eigenvalues and eigenfunctions of Eq. 1.1.

5 Numerical results

In this section, we obtain eigenvalues and eigenfunctions of Eq. 1.1 for various functions $\gamma(x)$. We denote the eigenvalues of Eq. 1.1 with E_i. Moreover, we report the CPU time for our method.

Example 5.1. *In [12], the even eigenvalues of Eq. 1.1 on $(-\infty, +\infty)$ for $\beta(x) = 0$ and $\gamma(x)=\frac{1}{2}x^2$ are obtained by the variational method as*

$$E_n = -\alpha b^2 - \alpha(n + \frac{1}{2})(a^2 - \frac{1}{2\alpha a^2}) + \frac{Qa}{h_n^2} I_n. \tag{5.1}$$

Where $h_n = 2^n\sqrt{\pi}n!$, $I_n = \int_{-\infty}^{\infty} H_n^4(x)e^{-2x^2}dx$, $H_n(x)$ are Hermit polynomials [4], a is a positive root of

$$\frac{1}{2\alpha} = -a^4 + \frac{QI_n}{2\alpha(2n+1)h_n^2}a^3,$$

and the eigenfunctions are in the form $F_n(x) = \sqrt{\frac{a}{h_n}}H_n(ax)e^{-\frac{a^2x^2}{2}}$. In this example, we set $\alpha = -\frac{1}{2}$, $Q = 1$, $N = 30$ and $h = \sqrt{\frac{\pi}{N}}$. Let $i = 10$ be the number of iterations. Table 1 represents the even eigenvalues obtained from SSCF and results of reference [12]. In figure (1), we show the convergence of

this method through variation of smallest eigenvalue as a function of the number of iterations for $Q = 1, 2, 3$. We observe that the convergence of our method is better full filled when the non-linear parameter $|Q|$ is smaller. Also, figure (2), shows the variation of the minimum eigenvalue as a function of Q. This figure shows that the method work better, for smaller values of the non-linear coefficient $|Q|$.

Table 1: Comparison of the eigenvalue of example 5.1 obtained by SSCF and Ref [12] .

Eigenvalues	SSCF	Ref [12]
E_0	0.8699440500	0.8726179080
E_2	2.652621066	2.753164076
E_4	4.610725009	4.709682021
E_6	6.591042186	6.685128136
CPU time(s)	36.9	–

Example 5.2. *Consider Eq. 1.1 with α=$-\frac{1}{2}$, $Q(x) = Q \leq 0$ and $\beta(x) = \gamma(x) = 0$ on $(-1, 1)$. By [7], we obtain the solution of this equation in the form*

$$F(x) = C.cn(\lambda(x - x_0), k)$$

where, λ and x_0 are arbitrary constants and k and C are determined as follows:

$$k^2 = \frac{\lambda^2 - 2E}{2\lambda^2}, C^2 = -\frac{\lambda^2 - 2E}{2Q}$$

and then

$$k = \frac{C\sqrt{-Q}}{\lambda}.$$

Since for odd functions $F(0) = 0$ and for even functions $F'(0) = 0$, we have $x_0 = 0$ and $x_0 = 3EllipticK(k)$, respectively. By using the constants x_0, even and odd solutions of equations can be obtained

$$F_{even} = C.cn(\lambda x, k), F_{odd} = C\sqrt{1 - k^2}\frac{sn(\lambda x, k)}{dn(\lambda x, k)}$$

Now by using $F(\pm 1) = 0$ and $\int_{-1}^{1} F^2 = 1$, we have

$$E_n = \frac{(1 - 2K^2)(EllipticK(k))^2 n^2}{2}$$

where n is the number of eigenvalues and k is the solution of the equation

$$-\frac{2EllipticK(k)}{Q}(EllipticE(k) - (1 - k^2)EllipticK(k)) = \frac{1}{n^2}$$

Let $N = 100$ be the number of nods and $i = 10$ be the number of iterations. Table (2), represents the eigenvalues obtained by SSCF, FDSCF and reference [18] for $Q = -1$ as well as the absolute errors of SSCF and FDSCF methods with $Er_S = |E_{Ref[18]} - E_{SSCF}|$ and $Er_F = |E_{Ref[18]} - E_{FDSCF}|$, respectively. In figure (3), we show the convergence of SSCF and FDSCF methods through variation of the smallest eigenvalue and error as a function of the number of nodes for $Q = -1$. Also, figure (4) shows the convergence of SSCF and FDSCF methods through variation of the error as a function of number of eigenvalues for $Q = -1$. This figure shows that the FDSCF work betterfor 7 lowest eigenvalues. However,

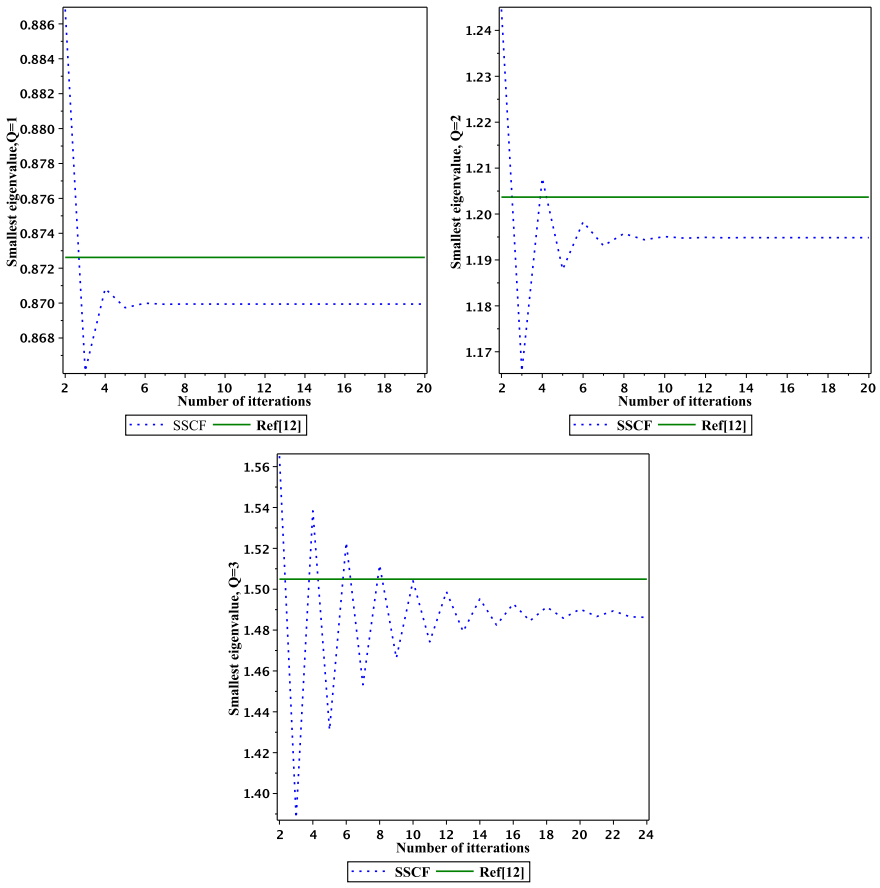

Figure 1: Figures show the variation of the smallest eigenvalue as a function of the number of iterations for $Q = 1, 2, 3$ (example 5.1).

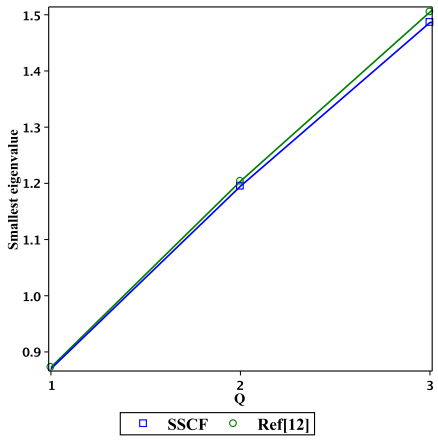

Figure 2: Variation of the smallest eigenvalue as a function of Q(example 5.1).

the error is fixed for all obtained eigenvalues. Therefore, in the situations that the few lowest eigenvalues are required, both of he presented methods can be used. But if we need the whole eigenvalue spectrum or at least a large portion of the eigenvalues, the SSCF is more reliable because we can more generally know the errors of higher index eigenvalues. Figure (5), also shows the variation of the smallest eigenvalue and the error of SSCF and FDSCF as a function of Q. This figure shows that the method work better, for smaller values of the non-linear coefficient $|Q|$. We observe that the error of our method is smaller when the non-linear parameter $|Q|$ is smaller. However this is true for smaller eigenvalues. Also, we see that for larger eigenvalues, we have not a rule of thumb for larger value of the eigenvalues.

Table 2: Comparison of eigenvalues of example 5.2 obtained through SSCF, FDSCF and Ref[18], for $Q = -1$.

Eigenvalues	SSCF	FDSCF	Ref [18]	Er_S	Er_F
E_0	0.4626016170	0.462459047	0.462579418	2.21990e-05	1.20371e-04
E_1	4.443995283	4.442553417	4.179929550	0.264065733	0.262623867
E_2	10.60665214	10.59860958	10.35117007	0.25548207	0.24743951
E_3	19.24142796	19.21512218	18.98801387	0.25341409	0.22710831
E_4	30.34402278	30.28031329	30.091750	0.25227278	0.18856329
E_5	43.91437108	43.78267728	43.662690	0.25168108	0.11998728
E_6	59.95245468	59.70867902	59.70093840	0.25151628	0.00774062
E_7	78.45806592	78.04253889	78.20653790	0.25152802	0.16399901
CPU time(s)	161	113	–	–	–

Example 5.3. *Consider Eq. 1.1 with* α=-1, $\beta(x) = 0$ *and* $\gamma(x) = 0.452\cos(\pi(1-x))$ *on* $(-1, 1)$. *In [1], the eigenvalues are obtained by using the discretized Euler-Lagrange variational method. Let* $N = 100$ *and* $i = 10$. *Table (3) represents the smallest eigenvalue obtained from reference [1] as well as SSCF and FDSCF methods for* $Q(x) = 0.5...2$. *This table shows that both of the methods work well in this situation.*

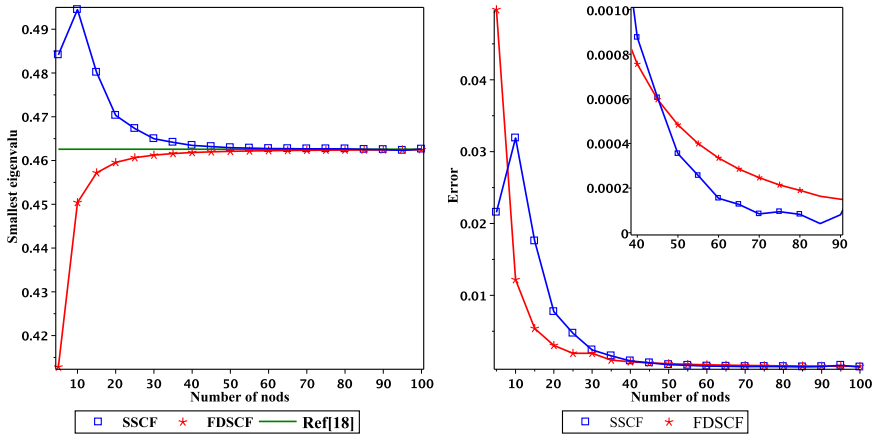

Figure 3: Left figure shows the variation of the smallest eigenvalue as a function of the number of nods and right figure shows the variation of the error as a function of the number of nods for $Q = -1$ (example 5.2).

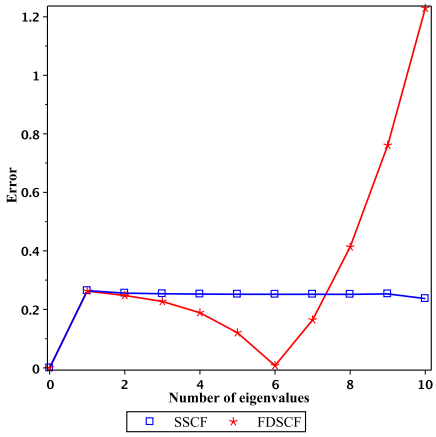

Figure 4: Variation of the error as a function of number of eigenvalues for $Q = -1$ (example 5.2).

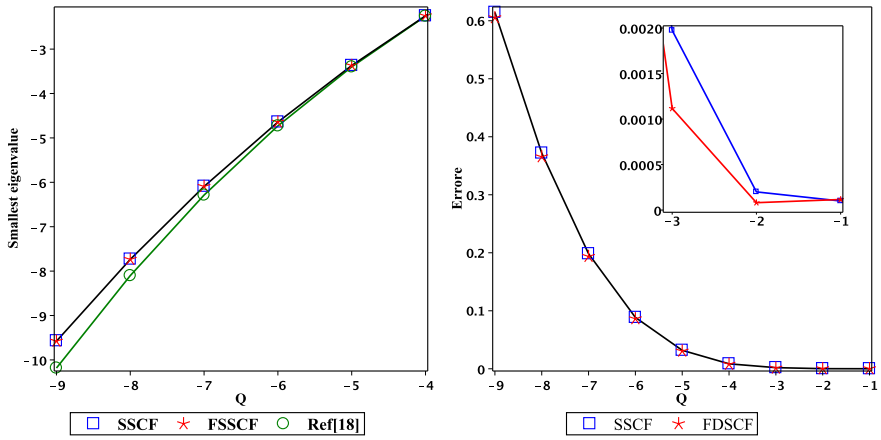

Figure 5: Left figure shows the variation of the smallest eigenvalue as a function of Q and right figure shows the variation of the error as a function of Q(example 5.2).

However, the CPU time of the SSCF is greater that the FDSCF method. In figure (6), we show the variation of the smallest eigenvalue as a function of Q. Again, we see that both of the presented methods work excellent in this example.

Table 3: Comparison of the smallest eigenvalue of example 5.3 obtained through SSCF, FDSCF and Ref [1].

Q	SSCF	FDSCF	Ref [1]	CPU time$_{SSCF}$	CPU time$_{FDSCF}$
0.5	2.616897689	2.616948710	2.616951848	172	108
1	2.990597503	2.990592936	2.99059549	172	106
1.5	3.360892363	3.359893114	3.35989571	172	112
2	3.726258011	3.725158240	3.725158948	174	109

6 Conclusion

In this paper, the SSCF method is applied to a class of non-linear eigenvalue differential equation with homogeneous boundary conditions. The eigenvalues obtained through this method are compared with some other references and exact eigenvalues. Three examples are examined to demonstrate the effectiveness of the proposed method. Based on numerical experiments, we conclude that, the method work better, for smaller values of the non-linear coefficient $|Q|$ in the equation 1.1. Also we see that the results for smallest eigenvalue have enough accuracy. But accuracy is lesser satisfactory for large eigenvalues. We see the CPU time of the SSCF is greater that the FDSCF method but the SSCF is more reliable because we can more generally know the errors of higher index eigenvalues. However, the difference in the CPU time is not so large that we are not able to solve realistic physics and engineering problems.

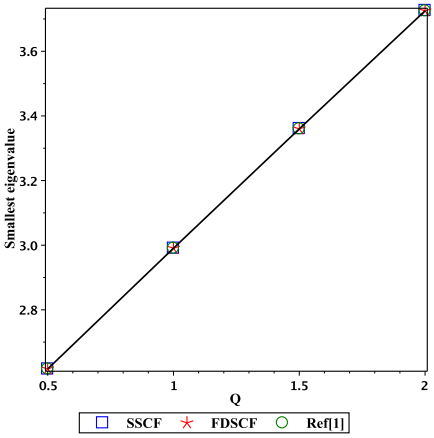

Figure 6: Variation of the smallest eigenvalue as a function of the number of Q (example 5.3).

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