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Fixed point of weak (β_B,ψ_B)-contractive mapping in C*- algebra valued partial metric spaces

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Abstract: In this paper, the concept of a weak (β_B, ψ_B)-contractive mapping has been introduced within the context of C^*-algebra valued partial metric space, and existence of fixed point is established by utilizing the C-class function. Further, several non-trivial examples are provided to show the validity of our results and as application of proved result an integral equation is solved.

Keywords: C^*-algebra valued partial metric space (C^*-AVPMS); weak contraction map; fixed point.
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1 Introduction

In 1922, Banach presented Banach contraction principal, which is now known as the FP theorem. Bakhtin extended the concept to b-metric spaces, providing an analogue of the Banach contraction principal in this broader context. G. J. Murphy [11] introduced C^*-algebra and operator theory. It is evident that when equipped with the norm topology, the collection L(H) composing all bounded linear operators on a Hilbert space H forms a C^*-algebra.
In 2014, Ma et al. [7] introduced to C^*-AVMS. This groundbreaking research paved the way for further exploration, as evidenced in subsequent works [1, 2, 4, 5, 6, 8, 9, 10, 14, 16], where additional FP results were achieved within the realm of C^*-algebra-valued metrics, including the more general domain of C^*-AVbMS. Moreover, inspired by the groundbreaking work of Samet et al., we explored contractive mappings in MS. Additionally, We have proposed a variant of (β, ψ) contractive mapping within the framework of a unital C^*-AVPMS. Chandok[3] further expanded this concept by inventing the class of C^*-AVPMSs, which led to the exploration of additional FP theorems.
In this paper, we introduce a contraction map in C^*-AVPMSs. In our investigation, we focused on the existence and uniqueness of FPs for self-operators defined in PMS belonging to the class of (β_B-ψ_B) contraction mappings. We discovered that this class encompasses a wide range of contraction-type operators and their FPs.

The paper's structure is as follows: The first section provides an introduction, the second section covers basic definitions in the preliminaries, the third section presents the mains results. In the fourth section, we demonstrate applications and illustrative examples.

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2 Preliminary

Consider a C^*-algebra denoted by $\mathbb{B}'$, which represents an algebra over the complex numbers field $\mathbb{C}$. In this context, we assume that $\mathbb{B}$ is a unital algebra, having a unit element denoted by e. The conjugate map $\mathbb{B}$ which is denoted by $* : \mathbb{B} \to \mathbb{B}$, that satisfies certain properties. Specifically, for all elements g and h in $\mathbb{B}$, we have $g* = g$ and $(gh)^* = h^* g^*$. Thus, the pair $(\mathbb{B}, *)$ is commonly referred to as a *-algebra.

The submultiplicative norm denoted by $\|.\|$. This norm satisfies the condition $\|g\| = \|g\|^*$ for all elements g in $\mathbb{B}$. The concept of C^*-algebra which is a Banach algebra that satisfies an extra property: $\|g\| = \|g^* g\|$. When considering a unital C^*-algebra $\mathbb{B}$, for any element u in $\mathbb{B}$, we have $u \preccurlyeq 1$ if and only if $\|u\| \leq 1$.

Definition 2.1. *[9] Consider a mapping $P : E \to E$ and $\beta : E \times E \to [0, \infty)$. If P as an β-admissible mapping when it satisfies the following condition:*

$$\beta(\aleph, \mathfrak{Q}) \geq 1 \textit{ implies that } \beta(P\aleph, P\mathfrak{Q}) \geq 1 \textit{ for all, } \aleph, \mathfrak{Q} \in E.$$

Definition 2.2. *[10] Let us consider a mapping $P : E \to E$ is said to be weak triangular β-admissible where $\beta : E \times E \to \mathbb{R}^+$, if it satisfies two conditions, it is a β-admissible mapping, and the condition:*

$$\beta(\aleph, P\mathfrak{Q}) \geq 1 \quad \textit{implies} \quad \beta(\aleph, P^2(\aleph)) \geq 1.$$

Definition 2.3. *[10] A map $P : E \to E$ is said to be weak triangular β-admissible, if there exist $\mathfrak{Q}_0 \in E$ such that*

$$\beta(\mathfrak{Q}_0, P\mathfrak{Q}_0) \geq 1.$$

If $\mathfrak{Q}_n = P^n \mathfrak{Q}_0$, then $\beta(\mathfrak{Q}_m, \mathfrak{Q}_n) \geq 1$, where $m < n$, for all $m, n \in \mathbb{N}$.

Definition 2.4. *[8]Consider $E \neq \phi$ and let $\wp : E \times E \to \mathbb{R}^+$ satisfy the condition:*

1. *$\wp(\aleph, \aleph) = \wp(\mathfrak{Q}, \mathfrak{Q}) = \wp(\aleph, \mathfrak{Q})$ if and only if $\aleph = \mathfrak{Q}$;*
2. *$\wp(\aleph, \mathfrak{Q}) = \wp(\mathfrak{Q}, \aleph)$;*
3. *$\wp(\aleph, \aleph) \leq \wp(\aleph, \mathfrak{Q})$;*
4. *$\wp(\aleph, \mathfrak{Q}) \leq \wp(\aleph, \mathfrak{w}) + \wp(\mathfrak{w}, \mathfrak{Q}) - \wp(\mathfrak{w}, \mathfrak{w})$ for all $\aleph, \mathfrak{Q}, \mathfrak{w} \in E$.*

Then the pair $(E, \wp)$ are called PMS and $\wp$ is called a partial metric on E.

Definition 2.5. *[8] In PMS the following definitions are satisfying:*

1. *A sequence $\{\aleph_n\}$ converges to $\aleph$ if and only if*

$$\wp(\aleph, \aleph) = \lim_{n \to \infty} \wp(\aleph, \aleph_n).$$

2. *A sequence $\{\aleph_n\}$ is referred to as a Cauchy sequence if $\lim_{n,m \to \infty} \wp(\aleph_n, \aleph_m)$ exists and is finite.*
3. *A PMS $(E, \wp)$ is considered complete if every Cauchy sequence is convergent i.e,*

$$\wp(\aleph, \aleph) = \lim_{n,m \to \infty} \wp(\aleph_n, \aleph_m).$$

4. *Let us consider a mapping $P : E \to E$ is said to be continuous at $\mathfrak{Q}_0 \in E$ if for any given $\epsilon > 0$, there exists a $\delta > 0$ such that $P(B_\wp(\mathfrak{Q}_0, \delta)) \subseteq B_\wp(P\mathfrak{Q}_0, \epsilon)$ where $B_\wp$ is n open ball.*

Theorem 2.6. *[8] Let $(E, \wp)$ be a complete PMS. Consider a map $P : E \to E$ is said to be weak triangular β-admissible, w.r.t the altering distance function ψ and a continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ for all $\aleph, \mathfrak{Q} \in E$, the inequality*

$$\beta(\aleph, \mathfrak{Q})\psi(\wp(P\aleph, P\mathfrak{Q})) \leq \phi(M'(\aleph, \mathfrak{Q}))$$

holds, where

$$M'(\aleph, \mathfrak{Q}) = max\{\wp(\aleph, \mathfrak{Q}), \wp(\aleph, P\aleph), \wp(\mathfrak{Q}, P\mathfrak{Q})\}.$$

And $\psi(t) > \phi(t)$ for all $t > 0$, if there exists $\mathfrak{Q}_0 \in E$ such that

$$\beta(\mathfrak{Q}_0, P\mathfrak{Q}_0) \geq 1,$$

then the mapping P possesses a FP.

Definition 2.7. *[9] Let $(E, \wp)$ be a PMS and β is a admissible mapping. Then E possess property (C) w.r.t β if for every sequence $\{\aleph_n\} \subset E$ with $\beta(\aleph_n, \aleph_{n+1}) \geq 1$, there exists $n_0 \in \mathbb{N}$ such that*

$$\beta(\aleph_m, \aleph_n) \geq 1 \ for \ all \ n > m \geq n_0, \ where \ n \in \mathbb{N}.$$

Definition 2.8. *[9] Let $(E, \wp)$ is a PMS. Let $P : E \to E$ be a mapping then P is a weak (β, ψ) contraction map if there exist two function β and ψ such that*

$$\beta(\aleph, \mathfrak{Q})\wp(P\aleph, P\mathfrak{Q}) \leq \psi(\max \wp(\aleph, \mathfrak{Q}), \wp(\aleph, P\aleph), \wp(\mathfrak{Q}, P\mathfrak{Q}))$$

and

$$\beta(\aleph, \mathfrak{Q})\wp(P\aleph, P\mathfrak{Q}) \leq \psi(\wp(\aleph, \mathfrak{Q}))$$

for all $\aleph, \mathfrak{Q} \in E$, where $\beta : E \times E \to [0, \infty)$ and $\psi \in \Psi$ then P is an (β, ψ)-contraction mapping.

Definition 2.9. *[9] Let $(E, \wp)$ is a PMS. Let $P : E \to E$ is a mapping then P is a weak (β, ψ) contraction mapping if it satisfying the following conditions:*

1. *The mapping P is a β-admissible;*
2. *There exists $\mathfrak{Q}_0 \in E$ such that $\beta(\mathfrak{Q}_0, P\mathfrak{Q}_0) \geq 1$;*
3. *The set E have property (C);*
4. *P is a continuous map on $(E, \wp^s)$, where $\wp^s = 2\wp(\aleph, \mathfrak{Q}) - \wp(\aleph, \aleph) - \wp(\mathfrak{Q}, \mathfrak{Q})$.*

Then the mapping P has a FP, or we can say that there exists $\mathfrak{Q}^* \in E$ such that $P\mathfrak{Q}^* = \mathfrak{Q}^*$.

Definition 2.10. *[9] Let $E \neq \phi$ and $\wp : E \times E \to \mathbb{B}$ is called a C^*-AVPMS on E if the following properties are satisfying:*

1. *$\aleph = \mathfrak{Q}$ if and only if $\wp(\aleph, \aleph) = \wp(\mathfrak{Q}, \mathfrak{Q}) = \wp(\aleph, \mathfrak{Q})$ and $\theta \preccurlyeq \wp(\aleph, \mathfrak{Q})$ for all $\aleph, \mathfrak{Q} \in E$;*
2. *$\wp(\aleph, \aleph) \preccurlyeq \wp(\aleph, \mathfrak{Q})$;*
3. *$\wp(\mathfrak{Q}, \aleph) = \wp(\aleph, \mathfrak{Q})$ for all $\aleph, \mathfrak{Q} \in E$;*
4. *$\wp(\aleph, \mathfrak{Q}) \preccurlyeq \wp(\aleph, w) + \wp(w, \mathfrak{Q}) - \wp(w, w)$ for all $\aleph, \mathfrak{Q}$ and $w \in E$.*

Then the pair (E, $\mathbb{B}$, $\wp$) is called C^*-AVPMS and $\wp$ is called a partial metric on E.

Definition 2.11. *[18] Let ψ_B represent the set of positive functions $\psi : \mathbb{B}^+ \to \mathbb{B}^+$ that satisfy the following conditions:*

1. *$\psi_B(a)$ is a continuous and non decreasing function;*
2. *$\psi_B(a) = 0$ if and only if $a = 0$;*

3. *For each $a \succ 0$, the series $\sum_{n=1}^{\infty} \psi_B^n(a)$ converges to a finite value, and $\lim_{n\to\infty} \psi_B^n(a) = 0$, where $\psi_B^n(a)$ denotes the nth iterate of ψ_B;*
4. *The series $\sum_{n=1}^{\infty} b^n \psi_B^n(a) < \infty$, for $a \succ 0$ and is increasing and continuous at 0, where $\mathbb{B}^+$ denote the set of elements $\aleph \in \mathbb{B}$ such that $\aleph \succcurlyeq 0_B$.*

Definition 2.12. *[7] Consider $E \neq \phi$. Let $\beta_B : E \times E \to (\mathbb{B}')^+$ be a function, and the operator P is β_B- admissible if for all $(\aleph, \mathfrak{Q}) \in E \times E$ with*

$$\beta_B(\aleph, \mathfrak{Q}) \succcurlyeq I_B \quad implies \quad \beta_B(P\aleph, P\mathfrak{Q}) \succcurlyeq I_B.$$

Where I_B is the unit of $\mathbb{B}$.

Definition 2.13. *[10] Let (E, $\mathbb{B}$, d'_B) be a C^*-AVbMS and $P : E \to E$ be a operator then P is a $\beta_B - \psi_B$ contractive mapping if there exist two functions $\beta_B : E \times E \to \mathbb{B}'$ and $\beta_B \in \psi_B$ such that*

$$\beta_B(\aleph, \mathfrak{Q})d'_B(P\aleph, P\mathfrak{Q}) \preccurlyeq \psi_B(d'_B(\aleph, \mathfrak{Q}))$$

for all $\aleph, \mathfrak{Q} \in E$.

Theorem 2.14. *[10] Suppose (E, $\mathbb{B}$, d'_B) be a C^*-AVbMS and $P : E \to E$ be an $\beta_B - \psi_B$ contraction mapping satisfy the following conditions:*

1. *P is a continuous.*
2. *There exist $\aleph_0 \in E$ such that $\beta_B(\aleph_0, P\aleph_0) \succcurlyeq I_B$.*
3. *P is a β_B-admissible.*

Then P has a FP in E.

Theorem 2.15. *[10] Suppose (E, $\mathbb{B}$, d'_B) be a C^*-AVbMS and $P : E \to E$, be a mapping satisfy*

$$\beta_B(\aleph, \mathfrak{Q})d'_B(P\aleph, P\mathfrak{Q}) \preccurlyeq \psi_B(d'_B(P\aleph, \aleph) + d'_B(P\mathfrak{Q}, \mathfrak{Q}))$$

for all $\aleph, \mathfrak{Q} \in E$, where $\beta_B : E \times E \to (B')^+$ and $\psi_B \in \Psi_B$, and the following condition satisfied:

1. *P is a β_B-admissible;*
2. *There exist $\aleph_0 \in E$ such that $\beta_B(\aleph_0, P\aleph_0) \succcurlyeq I_B$;*
3. *P is a continuous.*

Then P has a FP in E.

3 Main results

Definition 3.1. *Let (E, $\mathbb{B}$, $\wp_B$) is called C^*-AVPMS and $P : E \to E$ be a operator. We say that P is a weak (β_B, ψ_B) contractive mapping if there exist two function $\beta_B : E \times E \to \mathbb{B}^+$and $\psi_B \in \Psi_B$ such that*

$$\beta(\aleph, \mathfrak{Q})\wp_B(P\aleph, P\mathfrak{Q}) \leq \psi(\max \wp_B(\aleph, \mathfrak{Q}), \wp_B(\aleph, P\aleph), \wp_B(\mathfrak{Q}, P\mathfrak{Q})) \tag{3.1}$$

and

$$\beta(\aleph, \mathfrak{Q})\wp_B(P\aleph, P\mathfrak{Q}) \leq \psi(\wp_B(\aleph, \mathfrak{Q})),$$

for all $\aleph, \mathfrak{Q} \in E$, then P is an (β_B, ψ_B) contraction mapping.

Theorem 3.2. *Suppose $(E, \mathbb{B}, \wp_B)$ be a C^*-AVPMS and $P : E \to E$ be an weak $\beta_B - \psi_B$ contraction mapping satisfied the following conditions:*

1. *P is a continuous;*
2. *There exist $\aleph_0 \in E$ such that $\beta_B(\aleph_0, P\aleph_0) \succcurlyeq I_B$;*
3. *E has the property (C) w.r.t β;*
4. *P is a weak β_B-admissible.*

Then P has a FP in E.

Proof. Let $\aleph_0 \in E$ such that $\beta_B(\aleph_0, P\aleph_0) \succcurlyeq I_B$. Define the sequence $\{\aleph_n\}$ in E by

$$\aleph_{n+1} = P\aleph_n \quad for \quad all \quad n \in \mathbb{N}.$$

Let us consider, $\aleph_{n+1} = \aleph_n$ for some $n \in \mathbb{N}$, then $\aleph_n$ is a FP for P. On the other hands assume that $\aleph_{n+1} \neq \aleph_n$ for all $n \in \mathbb{N}$. Since P is weak β admissible, we get,

$$\beta_B(\aleph_0, \aleph_1) = \beta_B(\aleph_0, P\aleph_0) \succcurlyeq I_B,$$

implies that,

$$\beta_B(P\aleph_0, P\aleph_1) = \beta_B(\aleph_1, \aleph_2) \succcurlyeq I_B.$$

By induction $\beta_B(\aleph_n, \aleph_{n+1}) \succcurlyeq I_B$ for all $n \in \mathbb{N}$. Put $\aleph = \aleph_n$ and $\mathfrak{Q} = \mathfrak{Q}_n$ and using above, we obtain

$$\begin{aligned}\wp_B(\aleph_n, \aleph_{n+1}) &= \wp_B(P\aleph_{n-1}, \aleph_n) \preccurlyeq \beta_B(\aleph_{n-1}, \aleph_n)\wp_B(P\aleph_{n-1}, \aleph_n)\\ &\preccurlyeq \psi(\max\{\wp_B(\aleph_{n-1}, \aleph_n), \wp_B(\aleph_{n-1}, P\aleph_{n-1}), \wp_B(\aleph_n, P\aleph_n)\})\\ &= \psi\big(\max\{\wp_B(\aleph_{n-1}, \aleph_n), \wp_B(\aleph_n, \aleph_{n+1}\}\big).\end{aligned}$$

If,

$$\max\{\wp_B(\aleph_{n-1}, \aleph_n), \wp_B(\aleph_n, \aleph_{n+1})\} = \wp_B(\aleph_n, \aleph_{n+1}).$$

From Equation (3.1), we get

$$\wp_B(\aleph_n, \aleph_{n+1}) \preccurlyeq \psi(\wp_B(\aleph_n, \aleph_{n+1})) < \wp_B(\aleph_n, \aleph_{n+1}).$$

We obtain a contraction, therefore

$$\max\{\wp_B(\aleph_{n-1}, \aleph_n), \wp_B(\aleph_n, \aleph_{n+1})\} = \wp_B(\aleph_{n-1}, \aleph_n),$$

and hence

$$\wp_B(\aleph_n, \aleph_{n+1}) \preccurlyeq \psi\wp_B(\aleph_{n-1}, \aleph_n) \quad for \quad all \quad n \in \mathbb{N}. \tag{3.2}$$

By induction $\wp_B(\aleph_n, \aleph_{n+1}) \preccurlyeq \psi^n\wp_B(\aleph_{n-1}, \aleph_n)$ for all $n \in \mathbb{N}$.
Since E has the property (C) w.r.t β if for each sequence $\{\aleph_n\} \subset E$ such that $\alpha(\aleph_n, \aleph_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $\beta(\aleph_m, \aleph_n) \geq 1$ for all $n > m \geq n_0$. Above Equation (3.2) hold for

$n = m$. proved for $m \geq 1, p \geq 1$, it follows

$$\begin{aligned}
\wp_B(\aleph_m, \aleph_{m+p}) &\preccurlyeq \wp_B(\aleph_m, \aleph_{m+1}) + \wp_B(\aleph_{m+1}, \aleph_{m+p}) + \wp_B(\aleph_{m+1}, \aleph_{m+1}) \\
&\preccurlyeq \wp_B(\aleph_m, \aleph_{m+1}) + \wp_B(P\aleph_m, Pr\aleph_{m+p-1}) \\
&\preccurlyeq \wp_B(\aleph_m, \aleph_{m+1}) + \beta_B(\aleph_m, \aleph_{m+p-1})\wp_B(P\aleph_m, P\aleph_{m+p-1}) \\
&\preccurlyeq \wp_B(\aleph_m, \aleph_{m+1}) + \psi(\max\{\wp_B(\aleph_m, \aleph_{m+p-1}), \wp_B(\aleph_m, \aleph_{m+1}), \wp_B(\aleph_{m+p-1}, \aleph_{m+p})\}) \\
&\preccurlyeq \psi^m \wp_B(\aleph_0, \aleph_1) + \psi^{m+1}\wp_B(\aleph_0, \aleph_1) + \ldots + \psi^{m+p-2}\wp_B(\aleph_0, \aleph_1) + \psi^{m+p-1}\wp_B(\aleph_0, \aleph_1) \\
&= \sum_{k=1}^{p-1} \psi^{m+k-1}\wp_B(\aleph_0, \aleph_1) + \psi^{m+p-1}\wp_B(\aleph_0, \aleph_1) \\
\wp_B(\aleph_m, \aleph_{m+p}) &\preccurlyeq \sum_{k=1}^{p-1} \psi^{m+k-1}\wp_B(\aleph_0, \aleph_1) + \psi^{m+p-1}\wp_B(\aleph_0, \aleph_1).
\end{aligned}$$

Thus, $\{\aleph_n\}$ is a Cauchy sequences in E. Since (E, $\mathbb{B}$, $\wp_B$) is complete, there exist $r \in E$ such that $r_n \to r$ as $n \to \infty$, from continuity of P, it follows that $r_{n+1} = Pr_n$ as $n \to \infty$.
We consider the one condition (H_A) : For all $x, y \in E$, there exists $z \in E$ such that

$$\beta_B(x, z) \succcurlyeq I_B \; and \; \beta_B(y, z) \succcurlyeq I_B.$$

To proved the uniqueness of the limit. Suppose that x and y are two FP of P. From (H_A), there exist $z \in E$ then

$$\beta_B(x, z) \succcurlyeq I_B \; and \; \beta_B(y, z) \succcurlyeq I_B.$$

Since P is β_B-admissible, we get

$$\beta_B(x, P^n z) \succcurlyeq I_B \; and \; \beta_B(y, P^n z) \succcurlyeq I_B \; for \; all \; n \in \mathbb{N}.$$

Now,

$$\begin{aligned}
\wp_B(x, P^n z) &= \wp_B(Px, P(P^{n-1}z)) \\
&\preccurlyeq \beta_B(x, P^{n-1}z)\wp_B(Px, P(P^{n-1}z)) \\
&\preccurlyeq \psi_B^n(\wp_B(x, z)) \; for \; all \; n \in \mathbb{N} \\
&\to 0_B \; as \; n \to +\infty.
\end{aligned}$$

Thus $P^n z = x$. Similarly $P^n z = y$ as $n \to \infty$. So, $x = y$. □

4 Application

We derive an outcome of existence for a specific integral equation of the following form:

$$a(\aleph) = \int_E \mathcal{S}(\aleph, \theta, a(\theta))\, d\theta + \varpi(\aleph), \quad \aleph, \theta \in E, \tag{4.1}$$

where E be a measurable set, $\mathcal{S} : E^2 \times \mathbb{R} \to \mathbb{R}$.
Let $\mathcal{G} = L^2(E)$, $L(\mathcal{G}) = \mathcal{A}$. Define $d_B : \mathbb{B} \times \mathbb{B} \to \mathcal{A}$ by

$$d_B(\varpi, k) = \pi_{(|\varpi - k|^p + I)}, \; where \; \mathbb{B} = L^\infty(E),$$

for all $\varpi, k, I \in \mathbb{B}$, $p \geq 1$, and $\|\omega\| = k < 1$.
Let $\pi_v : \mathcal{G} \to \mathcal{G}$ are the operator defined by:

$$\pi_v(\phi) = v \cdot \phi.$$

Let us consider that there exists a continuous function $\beta : E^2 \to \mathbb{R}$ and $0 < k < 1$ such that

$$|\mathcal{S}(\aleph, \theta, a(\theta)) - \mathcal{S}(\aleph, \theta, b(\theta))| \leq k|\beta(\aleph, \theta)| \left(|a(\theta) - b(\theta)| + I - k^{-1}I\right), \tag{4.2}$$

for all $\aleph, \theta \in E$ and $a, b \in \mathbb{B}$ and

$$\sup_{\aleph \in E} \int_E |\beta(\aleph, \theta)| \, d\theta \leq 1.$$

Then, the integral Equation (4.1) has a unique solution.

To find the solution, define $P : \mathbb{B} \to \mathbb{B}$ by

$$P_a(\aleph) = \int_E \mathcal{S}(\aleph, \theta, a(\theta)) \, d\theta + h(\aleph), \quad for \ all \ \ \aleph, \theta \in E.$$

Let $\omega = kI$, then $\omega \in \mathcal{A}$. For any $v \in \mathcal{G}$ and $p \geq 1$, we have

$$\begin{aligned}
\|d_B(P_a, P_b)\| &= \sup_{\|v\|=1} \left(\pi_{|P_a - P_b|^p + I}.v, v\right) \\
&= \sup_{\|v\|=1} \int_E \left[\left|\int_E \mathcal{S}(\aleph, \theta, a(\theta)) - \mathcal{S}(\aleph, \theta, b(\theta)) \, d\theta\right|^p\right] v(\aleph) v(\overline{\aleph}) \, d\aleph + \sup_{\|v\|=1} \int_E v(\aleph) v(\overline{\aleph}) \, d\aleph I \\
&\leq \sup_{\|v\|=1} \int_E \left[\left|\int_E \mathcal{S}(\aleph, \theta, a(\theta)) - \mathcal{S}(\aleph, \theta, b(\theta)) \, d\theta\right|\right]^p |v(\aleph)|^2 \, d\aleph + \sup_{\|v\|=1} \int_E |v(\aleph)|^2 \, d\aleph I \\
&\leq \sup_{\|v\|=1} \int_E \left[\int_E \left|\kappa\beta(\aleph, \theta)(a(\theta) - b(\theta) + I - \kappa^{-1}I)\right| \, d\theta\right]^p |v(\aleph)|^2 \, d\aleph + I \\
&\leq \kappa^p \sup_{\|v\|=1} \int_E \left[\int_E |\beta(\aleph, \theta)| \, d\theta\right]^p |v(\aleph)|^2 \, d\aleph \|a - b\|_\infty^p \\
&\leq \kappa \sup_{\|v\|=1} \int_E \int_E |\beta(\aleph, \theta)| \, d\theta \sup_{\|v\|=1} \int_E |v(\aleph)|^2 \, d\aleph \|a - b\|_\infty^p \\
&\leq k\|a - b\|_\infty^p \\
&= \|\omega\| \|d_B(a, b)\|.
\end{aligned}$$

Therefore the Fredholm integral Equation (4.1) possesses a singular solution, satisfying that P has a unique FP.

Now, to illustrate the above result, we are giving the following example:

Example 4.1. *Let $E = [0, 1]$, $B = L^\infty(E)$, and $\mathcal{G} = L^2(E)$. Define $d_B : B \times B \to \mathcal{L}(\mathcal{G})$ by:*

$$d_B(\varpi, k) = \pi_{|\varpi - k|^2 + I},$$

where $\pi_v : \mathcal{G} \to \mathcal{G}$ are the operator defined by:

$$\pi_v(\phi) = v \cdot \phi.$$

Consequently, the triple $(B, \mathcal{A}, d_B)$ forms a complete C^*-AVPMS. Let's examine a function $\beta : E^2 \to \mathbb{R}$ given by $\beta(\aleph, \theta) = 1$ for all $\aleph, \theta \in E$. This leads to the following result:

$$\sup_{\aleph \in E} \int_E |\beta(\aleph, \theta)| \, d\theta \leq 1.$$

Now, we defined $\mathcal{S} : E^2 \times \mathbb{R} \to \mathbb{R}$ by $\mathcal{S}(\aleph, \theta, a(\theta)) = (\aleph - \theta)a(\theta)$. Let P be a operator on $\mathbb{B}$ by:

$$P_a(\aleph) = \int_E \mathcal{S}(\aleph, \theta, a(\theta))\, d\theta, \quad for\ all\ \aleph, \theta \in E.$$

Note that $\|d_B(P_a, P_b)\| \leq \|\omega\|\|d_B(a, b)\|$ holds true for all $a, b \in B$, where $\omega = kI$ for any $k \in [0, \frac{1}{2}]$. Consequently, all the conditions are satisfied, leading to the existence of a unique solution $a(\aleph)$ satisfying $P_a = a$ for Equation (4.1).

References

[1] Alsulami, H. H., Agarwal, R. P., Karapınar, E., and Khojasteh, F. (2016). A short note on C^*-valued contraction mappings. Journal of Inequalities and Applications, 2016(1), 50.

[2] Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundamenta mathematicae, 3(1), 133-181.

[3] Chandok, S., Kumar, D., and Park, C. (2019). C^*-algebra-valued partial metric space and fixed point theorems. Proceedings-Mathematical Sciences, 129, 1-9.

[4] Kadelburg, Z., and Radenović, S. (2016). Fixed point results in C^*-algebra-valued metric spaces are direct consequences of their standard metric counterparts. Fixed point theory and applications, 2016(1), 53.

[5] Kamran, T., Postolache, M., Ghiura, A., Batul, S., and Ali, R. (2016). The Banach contraction principle in C^*-algebra-valued b-metric spaces with application. Fixed Point Theory and Applications, 2016(1), 10.

[6] Klin-eam, C., and Kaskasem, P. (2016). Fixed Point Theorems for Cyclic Contractions in C -Algebra-Valued b-Metric Spaces. Journal of Function Spaces, 2016(1).

[7] Ma, Z., and Jiang, L. (2015). C^*-algebra-valued b-metric spaces and related fixed point theorems. Fixed Point Theory and Applications, 2015(1), 222.

[8] Mani, G., Gnanaprakasam, A. J., Ege, O., Aloqaily, A., and Mlaiki, N. (2023). Fixed Point Results in C^*-Algebra-Valued Partial b-Metric Spaces with Related Application. Mathematics, 11(5), 1158.

[9] Massit, H., and Rossafi, M. Fixed point theorem for (ϕ, F)-contraction on C-algebra valued partial metric spaces. Italian Journal of Pure and Applied Mathematics, 439.

[10] Mlaiki, N., Asim, M., and Imdad, M. (2020). C^*-algebra valued partial b-metric spaces and fixed point results with an application. Mathematics, 8(8), 1381.

[11] Murphy, G. J. (2014). C^*-algebras and operator theory. Academic press.

[12] Mustafa, R., Omran, S., and Nguyen, Q. N. (2021). Fixed point theory using ψ contractive mapping In C^*-Algebra valued b-metric space. Mathematics, 9(1), 92.

[13] Narzary, S., Das, D., Singh, Y. M., Khan, M. S., and Sessa, S. (2023). C^*-Algebra-Valued Partial Modular Metric Spaces and Some Fixed Point Results. Symmetry, 15(6), 1135.

[14] Paul, M., Sarkar, K., and Tiwary, K. (2023). Some unique common fixed point results on C^*-algebra valued metric space using C-class function. Annals of Mathematics and Computer Science, 16, 1-20.

[15] Saboori, M. H., Hassani, M., Allahyari, R., and Mehrabinezhad, M. (2020). Fixed point theorems in C^*-algebra-valued $b_{v(s)}$-metric spaces with application and numerical methods. Journal of Mathematical Extension, 15.

[16] Shagari, M. S., Alotaibi, T., Mohamed, O. K. S., Mustafa, A. O., and Bakery, A. A. (2023). On existence results of Volterra-type integral equations via C-algebra-valued F-contractions. AIMS Mathematics, 8(1), 1154-1171.

[17] Shagari, M. S., Imam, A. T., Danbaba, U. A., Yahaya, J., Oni, M. O., and Tijjani, A. A. (2023). Existence of fixed points via C^*-algebra-valued simulation functions with applications. The Journal of Analysis, 31(2), 1201-1221.

[18] Shahi, P., Kaur, J., and Bhatia, S. S. (2016). On Fixed Points of Generalized $(\alpha - \psi)$ Contractive Type Mappings in Partial Metric Spaces. International Journal of Analysis and Applications, 12(1), 38-48.