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Efficient two-step with memory methods and their dynamics

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Abstract: In this work, a fourth-order without-memory method is proposed that has a self-accelerator parameter. This method doesn't need to compute a derivative function for solving nonlinear equations. We have approximated the self-accelerator parameter and have increased the convergence order to %50 without increase function evaluation. The efficiency index of the with-memory method sixth-order is equal to 1.81712. Which is higher than one-, two-, three-, and four-step optimal methods. The attraction basin of the proposed methods is compared by the famous Newton's method and Kung-Traub's method.

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1 Introduction

1.1 The overall purpose of the study

The problem of finding all zeros of a polynomial arises quite frequently in exercise and is of great importance in numerous branches of science and engineering. This work shows the with-memory methods for solving $f(x) = 0$ by numerical procedures. The Bisection method, an old-fashioned but productive method for encountering a zero of f(x), is an ideal introduction to numerical techniques. It starts with two values for x that bracket a root. It confines that they do fact bracket a zero because the function $f(x)$ changes signs at these two x-values. If $f(x)$ is continuous, then must be at least one root within the values. To find the roots of equations, iterative methods are usually used, and most of these methods are without-memory methods. The main weakness of repetitive methods without memory is their low convergence speed [23, 24, 25, 26, 27].

In this article, we will build a with-memory method that has a higher efficiency index with only three function evaluations than the proposed method of Geum and Kim [11] with five evaluations.

1.2 Basic methodology of your research

The Secant method is also one of the oldest iterative methods of memorization. The Secant method begins by encountering two points on the curve of $f(x)$, hopefully around the root we study. The approximate location of the equation root can be used to determine a few applications of bisection or a graph.

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The two points may both be on one flank of the root equation. Also, they could be on different sides. This method proposes as follows:

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} f(x_k),\ k = 1, 2, \ldots, \tag{1.1}$$

Due to the low efficiency of these methods, researchers have developed multi-step methods. Hence, In 1960, Ostrowski proposed a family of two-step method, which have an efficiency index higher than Newton's method, as follows: [18]:

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f'(x_k)},\ k = 0, 1, \ldots, \\ x_{k+1} = y_k - \frac{f(x_k)}{f'(x_k)} \frac{f(y_k)}{f(x_k) - 2f(y_k)}. \end{cases} \tag{1.2}$$

After Ostrowski, Jarratt also presented his two-step method as follows [12]:

$$\begin{cases} y_k = x_k - \frac{2}{3} \frac{f(x_k)}{f'(x_k)},\ k = 0, 1, \ldots, \\ x_{k+1} = x_k - \frac{1}{2} \frac{f(x_k)}{f'(x_k) - 3f'(y_k)}, \end{cases} \tag{1.3}$$

Furthermore, In 2011,Geum and Kim [11] proposed Steffensen-like optimal methods sixteenth-order as follows

$$\begin{cases} y_k = x_k + \theta f(x_k),\ g(x_k) = \frac{\theta f(x_k) f(y_k)}{f(x_k) - f(y_k)}, z_k = y_k + g(x_k),\ k = 0, 1, 2, \ldots, \\ K(x_k) = g(x_k) \frac{f(x_k) f(z_k)}{(f(x_k) - f(z_k))(f(y_k) - f(z_k))},\ s_k = z_k + H(x_k) + K(x_k), \\ q_k = z_k + K(x_k),\ T(x_k) = K(x_k) \frac{f(q_k)}{(f(x_k) - f(q_k))(f(y_k) - f(q_k))(f(z_k) - f(q_k))}, \\ H(x_k) = T(x_k)(f(x_k) f(y_k) + f(z_k)^2 - f(z_k) f(q_k)), \\ h1_k = (f(z_k)(f(z_k) - f(s_k)) + f(x_k) f(y_k)) f(z_k), \\ h2_k = f(q_k)(f(q_k) - f(s_k))(-f(x_k) - f(y_k) + f(q_k) + f(s_k)), \\ t1_k = f(x_k) f(y_k)(h1_k - h2_k) + f(z_k) f(q_k)(f(z_k) - f(q_k))(f(z_k) - f(s_k))(f(q_k) - f(s_k)), \\ t2_k = (f(x_k) - f(s_k))(f(y_k) - f(s_k))(f(z_k) - f(s_k))(f(q_k) - f(s_k)), \\ W(x_k) = T(x_k) f(s_k) \frac{t1_k}{t2_k},\ x_{k+1} = s_k + W(x_k). \end{cases} \tag{1.4}$$

This optimal method has a convergence order of 16 and an efficiency index of $16^{\frac{1}{5}} = 1.7411$.

1.3 Major findings as a result of your analysis

Our goal in this work is to create an optimal method two-step without memory. Then to build memory methods that have fewer calls but a higher efficiency index. The new methods have a convergence order of 6 and an efficiency index of $6^{\frac{1}{3}} = .181712$. We have also discussed the self-accelerator parameter and its essential role in improving the degree of convergence, increasing the efficiency index, and increasing the basins of attraction.

1.4 A brief summary of your interpretations and conclusions

In the following Section 2 deals with modifying the two-step method without memory introduced by Geum and Kim [11], constructed by introducing one iterative parameter. This parameter is calculated to help Newton's interpolatory polynomial. The aim of this work has been presented by providing a with-memory method and improved order of convergence from 4 to 6 in Section 3. The comparisons of absolute errors and computational efficiencies are given in Section 4 to illustrate convergence behavior. The dynamical behavior of the new-family methods with and without memory is analyzed in Section 5. In section 6, we give the concluding remarks.

2 Without Memory Methods

We have used the two steps of these methods. And have created a without-memory-method self-accelerating parameter. Therefore, the new-methods have presented as follows:

$$\begin{cases} w_m = x_k + \gamma f(x_m),\ g(x_m) = \frac{\gamma f(x_m) f(w_m)}{f(x_m) - f(w_m)},\ \gamma \in \mathbb{R},\ m = 0, 1, 2, \dots, \\ y_m = w_m + g(x_m),\ K(x_m) = g(x_m) \frac{f(x_m) f(y_m)}{(f(x_m) - f(y_m))(f(w_m) - f(y_m))},\ x_{m+1} = y_m + K(x_m). \end{cases} \tag{2.1}$$

In the next theorem, we will show that the single-parameter method (2.1) has the degree of convergence is 4.

Theorem 2.1. *Allow $f : L \subseteq \mathbb{R} \to \mathbb{R}$ to be a sufficiently smooth function having continuous derivatives. If $f(x)$ has a simple root α in the open interval L and x_0 chosen in a sufficiently little neighborhood of α, then the method* (2.1) *is of local fourth-order convergence.*

$$e_{m+1} = (1 + \gamma f'(\alpha))^2 c_2 (2c_2^2 - c_3) e_m^4 + O(e_m^5), \tag{2.2}$$

Proof. : By using Taylor's expansion of $f(x)$ about α and taking into account that $f(\alpha) = 0$, we obtain

$$f(x_m) = f'(\alpha)(e_m + c_2 e_m^2 + c_3 e_m^3 + c_4 e_m^4 + O(e_m^5)). \tag{2.3}$$

Then, computing $e_{m,w} = w_m - \alpha$, we attain $w_m = x_m + \gamma f(x_m)$

$$w_m = \alpha + f'(\alpha)(1 + \gamma f'(\alpha)) e_m + \gamma f'(\alpha) c_2 e_m^2 + \gamma f'(\alpha) c_3 e_m^3 + \gamma f'(\alpha) c_4 e_m^4 + O(e_m^5), \tag{2.4}$$

and

$$\begin{aligned} f(w_m) =& f'(\alpha)(1 + \gamma f'(\alpha)) e_m + f'(\alpha)(1 + \gamma f'(\alpha)(3 + \gamma f'(\alpha))) c_2 e_m^2 + f'(\alpha) \\ & (2\gamma f'(\alpha)(1 + \gamma f'(\alpha)) c_2^2 + \gamma f'(\alpha) c_3 + (1 + \gamma f'(\alpha))^3 c_3) e_m^3 + f'(\alpha)(c_4 \\ & + \gamma f'(\alpha)(\gamma f'(\alpha) c_2^3 + (1 + \gamma f'(\alpha))(5 + 3\gamma f'(\alpha)) c_2 c_3 + (5 + \gamma f'(\alpha) \\ & (6 + \gamma f'(\alpha)(4 + \gamma f'(\alpha)))) c_4)) e_m^4 + O(e_m^5). \end{aligned} \tag{2.5}$$

Now by the Eqs. (2.3) and (2.5), we get that

$$\begin{aligned} g(x_m) = \frac{\gamma f(x_m) f(w_m)}{f(x_m) - f(w_m)} =& (-1 - f'(\alpha)\gamma) e_m + c_2 e_m^2 - (2 + \gamma f'(\alpha)(2 + \gamma \\ & f'(\alpha)))(c_2^2 - c_3) e_m^3 + ((4 + \gamma f'(\alpha)((5 + \gamma f'(\alpha) \\ & (3 + \gamma f'(\alpha)))) c_2^3 - (7 + \gamma f'(\alpha)(10 + \gamma f'(\alpha) \\ & (7 + 2\gamma f'(\alpha)))) c_2 c_3 + (3 + \gamma f'(\alpha)(5 + \gamma f'(\alpha) \\ & (4 + \gamma f'(\alpha)))) c_4) e_m^4 + O(e_m^5). \end{aligned} \tag{2.6}$$

Using the second step of (2.3) and (2.6), we attain

$$\begin{aligned} y_m =& \alpha + (1+\gamma f'(\alpha))c_2 e_m^2 + (-(2+\gamma f'(\alpha)(2+\gamma f'(\alpha)))c_2^2 + (1+\gamma f'(\alpha)) \\ & (2+\gamma f'(\alpha))c_3)e_m^3 + ((4+\gamma f'(\alpha)(5+\gamma f'(\alpha)(3+\gamma f'(\alpha))))c_2^3 - (7+\gamma \\ & f'(\alpha)(10+\gamma f'(\alpha)(7+2\gamma f'(\alpha))))c_2 c_3 + (1+\gamma f'(\alpha))(3+\gamma f'(\alpha) \\ & (3+\gamma f'(\alpha)))c_4)e_m^4 + O(e_m^5). \end{aligned} \tag{2.7}$$

For $f(y_k)$, we also have

$$\begin{aligned} f(y_m) =& f'(\alpha)(1+\gamma f'(\alpha))c_2 e_m^2 + f'(\alpha)(-(2+\gamma f'(\alpha)(2+\gamma f'(\alpha)))c_2^2 + \\ & (1+\gamma f'(\alpha))(2+\gamma f'(\alpha))c_3)e_m^3 + f'(\alpha)((5+\gamma f'(\alpha)(7+\gamma f'(\alpha) \\ & (4+\gamma f'(\alpha))))c_2^3 - (7+\gamma f'(\alpha)(10+\gamma f'(\alpha)(7+2\gamma f'(\alpha))))c_2 c_3 + \\ & (1+\gamma f'(\alpha))(3+\gamma f'(\alpha)(3+\gamma f'(\alpha)))c_4)e_m^4 + O(e_m^5). \end{aligned} \tag{2.8}$$

Additionally, by using relations (2.3), (2.5), (2.6) and (2.8), we attain

$$\begin{aligned} K(x_m) =& g(x_m)\frac{f(x_m)f(y_m)}{(f(x_m)-f(y_m))(f(w_m)-f(y_m))} \\ =& -(1+\gamma f'(\alpha))c_2 e_m^2 + ((2+\gamma f'(\alpha)(2+\gamma f'(\alpha)))c_2^2 - (1+\gamma f'(\alpha)) \\ & (2+\gamma f'(\alpha))c_3)e_m^3 + (-(2+\gamma f'(\alpha)(1+\gamma f'(\alpha)(1+\gamma f'(\alpha))))c_2^3 \\ & + 2(3+\gamma f'(\alpha)(4+\gamma f'(\alpha)(3+\gamma f'(\alpha))))c_2 c_3 - (1+\gamma f'(\alpha)) \\ & (3+\gamma f'(\alpha)(3+\gamma f'(\alpha)))c_4)e_m^4 + O(e_m^5). \end{aligned} \tag{2.9}$$

By substituting (2.7) and (2.9) in (2.1), we obtain

$$e_{m+1} = (1+\gamma f'(\alpha))^2 c_2(2c_2^2 - c_3)e_m^4 + O(e_m^5), \tag{2.10}$$

and the proof of the theorem is complete. □

The error of equation (2.10) has a free parameter. This self-accelerating parameter has an important role in the improvement of the convergence order for the new families of methods, e. g. in (2.1).

3 New more efficient methods

This section concerns with extracting the novel method of with memory from (2.1) by using a self-accelerating parameter. Theorem (2.1) states that modified method (2.1) has order of convergence 4 if $\gamma \neq \frac{-1}{f'(\alpha)}$. It can be seen that if we set $\gamma = \frac{-1}{f'(\alpha)}$ then at least the coefficient of e_m^4 disappears. Since the exact value of the root α is not available, we have obtained an approximation of it using the available information. However, the following approximates are applied

$$\gamma_m = \frac{1}{f'(\alpha)} \approx \frac{-1}{N_3'(x_m)},\ m = 1, 2, 3, \ldots. \tag{3.1}$$

Let us define Newton's interpolating polynomials of the third degree as follows: $N_3(l) = N_3(l, x_m, x_{m-1}, w_{m-1}, y_{m-1})$. Directly, we can define the iterative method with memory as follows:

$$\begin{cases} w_0 = x_0 - \gamma_0 f(x_0), \gamma_m = \frac{1}{N_3'(x_m)},\ m = 1, 2, 3, \cdots, \\ g(x_m) = \frac{\gamma_k f(x_m) f(w_m)}{f(x_m) - f(w_m)},\ y_m = w_m + g(x_m),\ m = 0, 1, 2, \ldots, \\ ,\ K(x_m) = g(x_m) \frac{f(x_m) f(y_m)}{(f(x_{km}) - f(y_m))(f(w_m) - f(y_m))},\ x_{k+1} = y_m + K(x_m). \end{cases} \tag{3.2}$$

In the next section, we plan to confirm the convergence of method (3.2) using the idea and the R-order symbol. To get the convergence order of the with-memory method (3.2), we require the following technical lemma, like lemma 1 of [31].

Lemma 3.1. *If* $\gamma_m = \frac{-1}{N_3'(x_m)}$, *then:*

$$(1 + \gamma_m f'(\alpha)) \sim c_4 e_{m-1} e_{m-1,w} e_{m-1,y}. \tag{3.3}$$

Theorem 3.2. *If the initial estimation* x_0 *is in close vicinity to the single root* α *of the real and suitably smooth function* $f(x) = 0$, *technique (3.2) will possess a convergence rate of at least 6.*

Proof. Assume the series x_m is a sequence of approximations made by an iterative (IM) method. If this series converges to the root α, we maintain the equation $f(x) = 0$ with R-order, $O_r((IM), \alpha) \geq r$:

$$e_{m+1} \sim D_{m,r} e_m^r,\ e_m = x_m - \alpha. \tag{3.4}$$

Where $D_{m,r}$ tends to the constant asymptotic error D_m of the iterative method (IM) when $m \to \infty$. Therefore

$$e_{m+1} \sim D_{m,r} e_m^r = D_{m,r}(D_{m-1,r} e_{m-1}^r)^r = D_{m,r} D_{m-1,r} e_{m-1}^{r^2}. \tag{3.5}$$

If we assume that the minimum R-order duplicate sequences w_m and y_m are equal to p, and q, respectively, then we have:

$$e_{m,w} \sim D_{m,r} e_m^p = D_{m,r}(D_{m-1,r} e_{m-1}^r)^p = D_{m,r} D_{m-1,r} e_{m-1}^{rp}. \tag{3.6}$$

And

$$e_{m,y} \sim D_{m,r} e_k^q = D_{m,r}(D_{m-1,r} e_{m-1}^r)^q = D_{m,r} D_{m-1,r} e_{m-1}^{rq}. \tag{3.7}$$

Currently,employing Lemma (3.1) and R-orders we will maintain

$$(1 + \gamma_m f'(\alpha)) \sim c_4 e_{m-1} e_{m-1,w} e_{m-1,y} \sim c_4 D_{m-1,p} e_{m-1}^p D_{m-1,q} e_{m-1}^q e_{m-1} = c_4 D_{m-1,p} D_{m-1,q} e_{m-1}^{1+p+q}. \tag{3.8}$$

By Theorem (2.1), we can write

$$e_{m,w} \sim (1 + \gamma_m f'(\alpha)) e_m, \tag{3.9}$$

$$e_{m,y} \sim (1 + \gamma_m f'(\alpha)) e_m^2, \tag{3.10}$$

$$e_{m+1} \sim (1 + \gamma_m f'(\alpha))^2 e_m^4, \tag{3.11}$$

Thus,regarding the relations (3.8), (3.9), (3.10), and (3.11),we get:

$$e_{m,w} \sim (1 + \gamma_m f'(\alpha)) e_m \sim c_4 D_{m-1,p} D_{m-1,q} D_{m-1,r} e_{m-1}^{(1+p+q)+r}, \tag{3.12}$$

$$e_{m,y} \sim (1 + \gamma_m f'(\alpha)) e_m^2 \sim c_4 D_{m-1,p} D_{m-1,q} D_{m-1,r} e_{m-1}^{(1+p+q)+2r}, \tag{3.13}$$

$$e_{m+1} \sim (1+\gamma_k f'(\alpha))^2 e_m^4 c_4 D_{m-1,p} D_{m-1,q} D_{m-1,r} e_{m-1}^{2(1+p+q)+4r}, \tag{3.14}$$

Combining (3.9)-(3.12), (3.10)-(3.13), and (3.11)-(3.14), we conclude

$$e_{m,w} \sim e_{m-1}^{(1+p+q)+r}, \tag{3.15}$$

$$e_{m,y} \sim e_{m-1}^{(1+p+q)+2r}, \tag{3.16}$$

and

$$e_{m+1} \sim e_{m-1}^{2(1+p+q)+4r}. \tag{3.17}$$

Accordingly, by resembling exponents of e k-1 emerging in three pairs of relations ((3.6), (3.15)), ((3.7), (3.16)) and ((3.5), (3.17)), considering the power equations,we will eventually arrive the system of the following three unknown equations:

$$\begin{cases} rp - r - (p+q+1) = 0, \\ rq - 2r - (p+q+1) = 0, \\ r^2 - 4r - 2(p+q+1) = 0. \end{cases} \tag{3.18}$$

Afterwards, solving the exceeding system of equations, we get $r = 6$, $q = 3$, $p = 2$. So, the proof of Theorem 3.2 finish. □

Remark 3.3. *As was observed, method (3.2) with the convergence order of six has the same number of evaluations of an optimal two-step-method, i. e. three. The efficiency index of new methods is equal to* $EI = 6^{1/3} = 1.8171206$. *Since the efficiency index of the Geum-Kim method is* $EI = 16^{1/5} = 1.7411011$, *so the proposed method, which has only two steps, has higher efficiency.*

4 Numerical examples

This section shows the practical-performance of the new methods of memorization. We have solved seven nonlinear equations given in Table 1. Table 1 gives each nonlinear equation, the exact-root, and the initial root approximation. For more information on the initial approximation of the root of any nonlinear equation, see Reference [32]. Considering Table 2 is shown that the efficiency index of the new methods is higher than all the optimal iterative methods. It is also higher than the methods mentioned in [3, 8]. The error $|x_k - \alpha|$ of the approximation to the sought zeros, produced by the different-methods at the first four iterations. The symbol $m(-n)$ stands for $m \times 10^{-n}$. This table also include, for each test function, the initial estimation values and the last value of the computational order of convergence r_c [19] computed by the expression

$$r_c = \frac{\log|f(x_n)/f(x_{n-1})|}{\log|f(x_{n-1})/f(x_{n-2})|}. \tag{4.1}$$

5 Basins of Attraction

In this section, we consider the dynamical behaviours of the without-memory iterative methods in the complex plane. This gives useful details about the stability and reliability of the iterative methods. Here, we analogize the stability of the proposed methods with other methods. For the comparison, we

Table 1: The test functions

Nonlinear function	Root	Initial guess
$f_1(x)=\sin(5x)e^x-2$	$\alpha\approx 1.36$	$x_0=1$
$f_2(x)=1+\frac{1}{x^4}-\frac{1}{x}-x^2$	$\alpha=1$	$x_0=1.4$
$f_3(x)=(x-2)(x^{10}+x+2)e^{-5x}$	$\alpha=2$	$x_0=2.3$
$f_4(x)=e^{x^3-x}-\cos(x^2-1)+x^3+1$	$\alpha=-1$	$x_0=-1.3$
$f_5(x)=\frac{-5x^2}{2}+x^4+x^5+\frac{1}{1+x^2}$	$\alpha=1$	$x_0=1.3$
$f_6(x)=\log(1+x^2)+e^{-3x+x^2}\sin(x)$	$\alpha=0$	$x_0=0.3$
$f_7(x)=x^3+4x^2-10$	$\alpha\approx 1.3652$	$x_0=1$

Table 2: Numerical results for the test functions $f_i(x), i=1,2,3,\ldots,7$ the proposed method (2.1) and (3.2)

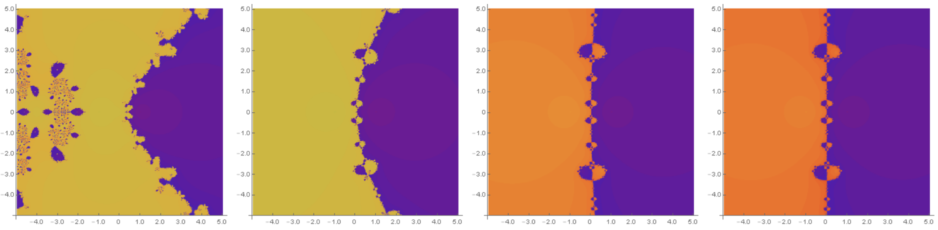
function	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	$\lvert x_4-\alpha\rvert$	COC	EI
$f_1(x)$, $\gamma_0=0.1$	0.36000(0)	0.11794(1)	0.49823(0)	14307(1)	4.00000	1.5874
$f_2(x)$, $\beta_0=0.1$	0.40000(0)	0.53415(−2)	0.42057(−9)	0.16519(−36)	4.00000	1.5874
$f_3(x)$, $\beta_0=0.1$	0.30000(0)	0.11033(−1)	0.14842(−9)	0.10882(−40)	4.00000	1.5874
$f_4(x)$, $\beta_0=0.1$	0.30000(0)	0.13902(−2)	0.42688(−11)	0.37455(−45)	4.00000	1.5874
$f_5(x)$, $\beta_0=0.1$	0.30000(0)	0.80751(−1)	0.27506(−2)	0.10557(−7)	4.00000	1.5874
$f_6(x)$, $\beta_0=0.1$	0.30000(0)	0.68577(−2)	0.13234(−7)	0.19796(−30)	4.00000	1.5874
$f_7(x)$, $\beta_0=0.1$	0.36520(0)	0.72203(−1)	0.62274(−4)	0.30013(−4)	4.00000	1.5874
$f_1(x)$, $\gamma_0=0.1$	0.11794(1)	0.48066(0)	0.45889(0)	45888(0)	6.00000	1.81712
$f_2(x)$, $\beta_0=0.1$	0.53415(−2)	0.24470(−12)	0.21083(−73)	0.84555(−440)	6.00000	1.81712
$f_3(x)$, $\beta_0=0.1$	0.11033(−1)	0.12805(−14)	0.67358(−92)	0.15068(−555)	6.00000	1.81712
$f_4(x)$, $\beta_0=0.1$	0.13902(−2)	0.79305(−17)	0.71512(−104)	0.38758(−626)	6.00000	1.81712
$f_5(x)$, $\beta_0=0.1$	0.10808(1)	0.10001(1)	0.10000(1)	0.10000(1)	6.00000	1.81712
$f_6(x)$, $\beta_0=0.1$	0.68577(−2)	0.24313(−14)	0.26199(−85)	0.39945(−511)	6.00000	1.81712
$f_7(x)$, $\beta_0=0.1$	0.72203(−1)	0.30019(−4)	0.30013(−4)	0.30013(−4)	6.00000	1.81712

employ the iterative methods to the complex polynomial of orders five,four,three and two, $p_{k-1}(z)=z^k-1, k=2,3,4,5$. We take a square $D=[-5,5]\times[-5,5]\subset C$ of 400×400 grid points and lay on a colour to each point $z\in D$, according to the roots corresponding to which the technique starting from z converges. The roots of the polynomial are represented by the white dots. We catch the point z—where the methods diverge from a root with the tolerance 10^{-6} and a maximum iteration 25—as black, and these black points are considered as divergent points. In the basins of attraction of each iterative method, a brighter colour area indicates that the iterative method converges to the root in the minimum number of iterations and a darker region denotes that the method requires more iterations to converge towards the root. Some significant results concerning the dynamic performances of the iterative methods have been obtained in [4, 16]. Figures (1), (2), (3), (4), (5), (6), (7), and (8) show that the basins of attraction with-memory method are higher than the without-memory-method. And, the accelerator parameter play a decisive role in increasing the absorption domain of a repetitive-method. Also, the smaller the size of the self-accelerator parameter, the greater the stability savings.

In the continuation of the discussion, we have compared the attraction basin of the proposed method

Table 3: Comparison efficiency index of the proposed methods by other methods

Without-memory method	EF	p	EI	With-memory method	EF	p	EI
BMM [3]	3	4.00000	1.58740	CCGTM [7]	3	4.23607	1.61803
TM [33]	3	4.00000	1.58740	CPJM [5]	3	5.00000	1.70998
CRTVM [9]	3	4.00000	1.58740	CPJM [5]	3	4.56155	1.65846
TKM [34]	4	8.00000	1.68179	CPJM [5]	4	4.79129	1.68584
CHMTM [8]	4	8.00000	1.68179	ZZZMM [44]	2	3.00000	1.73205
GKM [11]	5	16.00000	1.74110	Secant (1.1)	2	1.61803	1.61803
JM [12]	3	4.00000	1.58740	ASKSM [1]	4	8.35890	1.70035
KTM [14]	3	4.00000	1.58740	MWBM [17]	2	2.73205	1.65289
LLM [13]	3	4.00000	1.58740	ASKSM [1]	4	8.79583	1.72214
ZCTM [40]	3	4.00000	1.58740	TM [36]	2	2.41421	1.55377
MM [15]	3	4.00000	1.58740	PIDM [20]	3	4.23607	1.61803
OM [18]	3	4.00000	1.58740	PIDM [20]	3	4.44949	1.64476
ZLHM [42]	3	4.00000	1.58740	ASKSM [1]	4	10.902	1.78227
PIDM [20]	3	4.00000	1.58740	WFM [37]	3	4.23607	1.61803
RWBM [21]	3	4.00000	1.58740	WFM [37]	3	4.44949	1.64476
SSSLM [22]	5	16.00000	1.74110	WM [38]	3	4.44949	1.64476
SMJM [35]	5	16.00000	1.74110	WM [38]	3	4.23607	1.61803
ZOCM [41]	3	4.00000	1.58740	WM [38]	3	4.64575	1.66860
ZOCM [41]	4	8.00000	1.68179	ZZLM [43]	2	3.00000	1.73205
ZLHM [42]	2	2.00000	1.41421	ZZLM [43]	3	4.23607	1.61803
ZLHM [42]	4	8.00000	1.68179	ZZLM [43]	3	4.74483	1.68038
ZLHM [42]	5	16.00000	1.74110	TM (3.2)	3	6.00000	1.81712



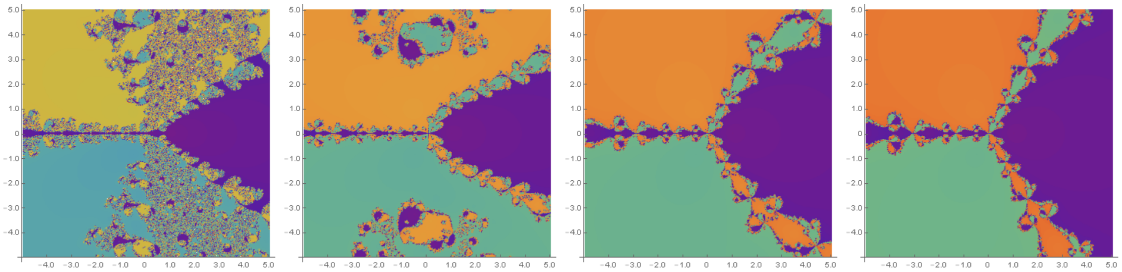
(a) TM6, $\gamma = 1$ (b) TM6 , $\gamma = 0.1$ (c) TM6 , $\gamma = 0.01$ (d) TM6, $\gamma = 0.001$

Figure 1: Basins of attraction for sixth-order methods for $p_1(z)$

with one-step methods Newton(NM) and Steffenson(SM)[29], two-step methods Kung-Traub(KTM)[14] and Weerakoon-Fernando(WFM) [39], and three-step Chun-Lee's methods (CLM) [6].

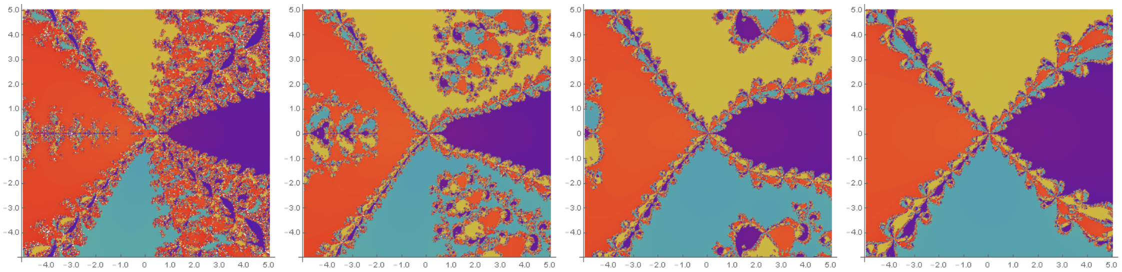
5.1 Advantages and limitations of the proposed work

The key to finding the roots of non-linear equations that do not have exact roots is to choose an appropriate initial approximation. Improper selection of the convergence property of the iterative method is lost. The main advantage of memory-based methods is higher efficiency using less evaluation. The main advantage of the proposed method is higher efficiency with lower evaluation.



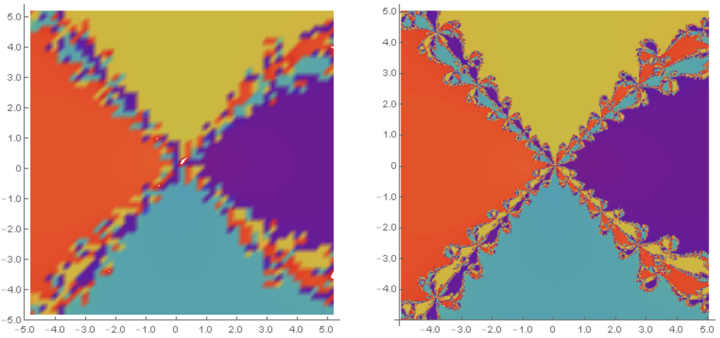
(a) TM6, $\gamma = 1$ (b) TM6, $\gamma = 0.1$ (c) TM6, $\gamma = 0.01$ (d) TM6, $\gamma = 0.001$

Figure 2: Basins of attraction for sixth-order methods for $p_2(z)$



(a) TM6, $\gamma = 1$ (b) TM6, $\gamma = 0.1$ (c) TM61, $\gamma = 0.01$ (d) TM6, $\gamma = 0.001$

Figure 3: Basins of attraction for sixth-order methods for $p_3(z)$

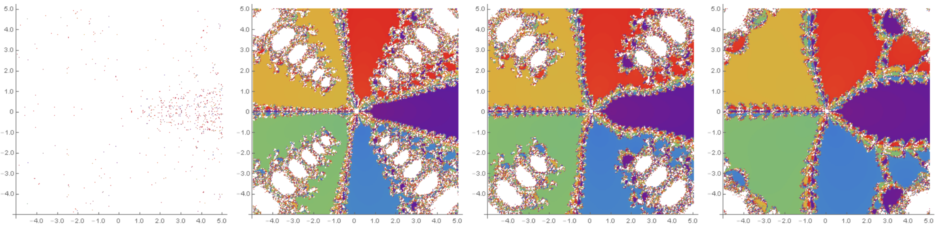


(a) TM6 (3.2), $\gamma = 0.001, n = 50$ (b) TM6 (3.2), $\gamma = 0.001, n = 400$

Figure 4: Basins of attraction for sixth-order methods for $p_3(z)$

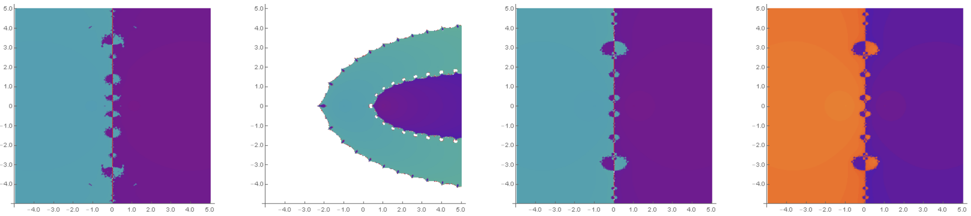
6 Final remark

As observed, the proposed method (3.2) has a convergence order of six with only three evaluations function. We saw the correctness of this issue theoretically in Theorem (3.2) of Section 3. In Section 4, we



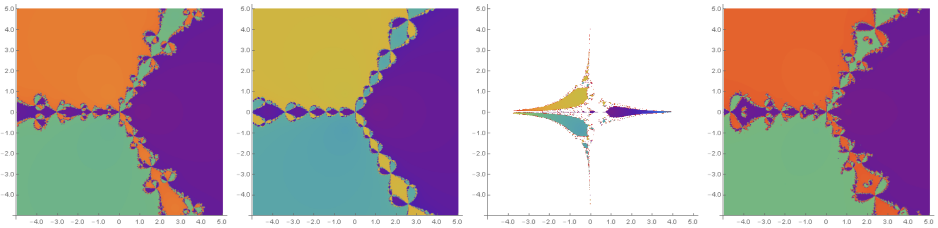
(a) TM6, $\gamma = 1$ (b) TM6, $\gamma = 0.1$ (c) TM6, $\gamma = 0.01$ (d) TM6, $\gamma = 0.001$

Figure 5: Basins of attraction for sixth-order methods for $p_4(z)$

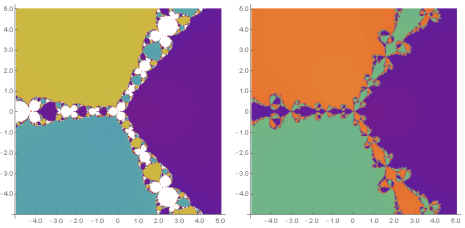


(a) CLM (b) SM (c) KTM (d) TM6

Figure 6: Basins of attraction for sixth-order methods for $p_1(z)$



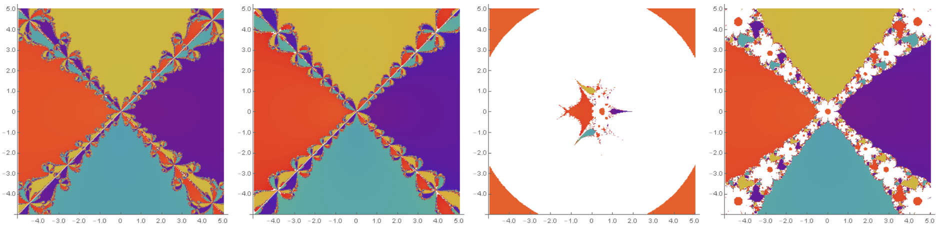
(a) KTM (b) NM (c) SM (d) CLM



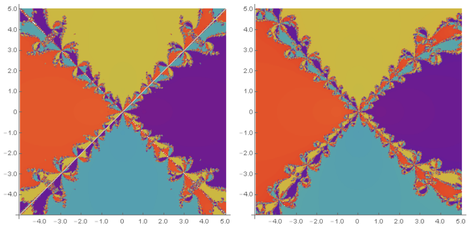
(e) WFM (f) TM6

Figure 7: Basins of attraction for sixth-order methods for $p_2(z)$

have solved seven nonlinear equations and, we have shown a 50% improvement in the degree of convergence in Table 2. Also, Table 3 shows the efficiency index of the proposed method is higher than other methods proposed reference [2, 10, 35]. Section 5 shows the dynamic behaviors of the proposed method



(a) KTM (b) NM (c) SM (d) WFM



(e) CLM (f) TM6

Figure 8: Basins of attraction for sixth-order methods for $p_3(z)$

for solving four nonlinear equations. As one has seen, by setting the value of the self-accelerator parameter γ equal to 0.001, we will have a more stable convergence region. According to Figures (6), (7), and (8), it is concluded that the attraction basin of the proposed methods is competitive with the methods (NM,SM,CLM,WFM,KTM) and has an even larger absorption area than other methods. It also does not have the problem of calculating the derivative of a function.

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